

Linear Programming: Simplex Algorithm

A function of several variables, $f(X)$ is said to be linear if it satisfies the two conditions: (i) $f(X + Y) = f(X) + f(Y)$ and (ii) $f(\alpha X) = \alpha f(X)$, where X and Y are vectors of dimension n and α is a scalar. The first of these is called additivity and the second, proportionality. All linear functions are of the form: $\sum_{j=1}^n c_j x_j$ where c_j are constants. A linear program is a problem of maximizing or minimizing a linear function of several variables whose values are restricted (or constrained) to satisfy relations each of which is of the type:

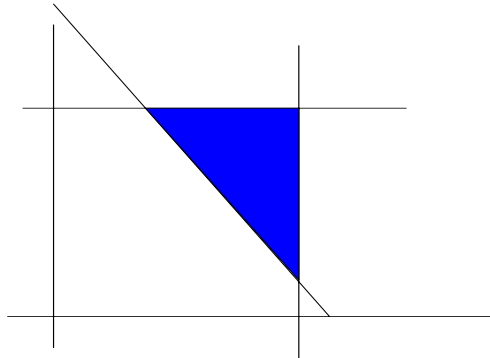
some linear function \forall a number

where \forall is one of: $\leq, \geq, =$.

Example I:

$$\begin{aligned} \max & 40x_1 + 36x_2 \\ & x_1 \leq 8 \\ & x_2 \leq 10 \\ & 5x_1 + 3x_2 \geq 45 \\ & x_1 \geq 0; x_2 \geq 0 \end{aligned}$$

When we have a problem in two variables, we can solve the problem graphically as shown below.



The essential steps in this process are: (i) draw and mark the axes clearly; (ii) for each constraint, draw the corresponding equation first, and if the constraint is an inequality (and only in this case), then show the direction of the inequality; (iii) mark the feasible region if it is nonempty and indicate it is empty, if that is the case; (iv) draw constant (objective) value contours indicating the direction in which we need to go to find optimal solutions; if the problem is unbounded, indicate that it is so; if not, show precisely the location(s) of optimal solution(s).

When there are more than two variables (and there is no way you can see to reduce the number to two without mutilating the problem), then we cannot use the graphical method to solve the problem. For solving such problems, we

have a method called the **simplex algorithm** that produces optimal solutions, indicates infeasibility or shows that the problem is unbounded, whichever is the case.

Ideally, we would like our algorithms to terminate (correctly) and do so in as few steps as possible. Although the simplex algorithm is **theoretically inefficient** (in some sense), it works very well practically, and until recently, it was the most widely used algorithm.

Now we are ready to describe the simplex algorithm to solve linear programs, and we begin by considering an “easy” example first to illustrate the logic. Consider the following example:

Example II:

$$\begin{aligned} \min & 40x_1 + 36x_2 \\ & x_1 \leq 8 \\ & x_2 \leq 10 \\ & 5x_1 + 3x_2 \leq 45 \\ & x_1 \geq 0; x_2 \geq 0 \end{aligned}$$

We now outline the steps involved in solving this problem:

Step 1: Convert each inequality to an equation by **introducing new variables called slack variables**. These newly introduced variables (as well as the old ones will be required to be nonnegative).

For example the first constraint becomes $x_1 + s_1 = 8$. Doing this to the entire problem results in:

$$\begin{aligned} \max & -40x_1 - 36x_2 \\ & x_1 + s_1 = 8 \\ & x_2 + s_2 = 10 \\ & 5x_1 + 3x_2 + s_3 = 45 \\ & x_1 \geq 0; x_2 \geq 0; s_1 \geq 0; s_2 \geq 0; s_3 \geq 0 \end{aligned}$$

Why can't we use the same slack more than once?

Step 2: Convert (if necessary) the problem to one of minimization (this step is done solely to avoid confusion). In our example, we want to maximize $-40x_1 - 36x_2$ which is the same as minimizing $(40x_1 + 36x_2)$; to make this into another equation, we let $z = 40x_1 + 36x_2$, or equivalently $-z + 40x_1 + 36x_2 = 0$

Gathering all the equations and rewriting them in a form convenient for later use, we have:

x_1	x_2	s_1	s_2	s_3	$-z$	=	<i>RHS</i>
1		1	0	0	0		8
	1	0	1	0	0		10
5	3	0	0	1	0		45
40	36	0	0	0	1		0

$x_j \geq 0; s_i \geq 0$

Such a system of equations is called a (Gauss-Jordan)-canonical form of equations. The variables s_1, s_2, s_3 , and $-z$ are called the **basic variables with respect to this canonical form** or, conversely, this is the canonical form with respect to this set of basic variables. The remaining variables are called **nonbasic variables**. **The solution obtained by setting the nonbasic variables equal to zero is called the current basic solution**. If it is nonnegative (except possibly in the z component), then it is also **feasible** and is called the **current basic feasible solution**. **The simplex algorithm requires a starting canonical form whose current basic solution is feasible**. We will show how this can always be achieved later by using an artificial problem from whose solution we will be able to find the solution of the original problem. Note that the last equation implies $z = (40x_1 + 36x_2) \geq 0$ for all values of x_i that are nonnegative. Our current solution has $z = 0$, and hence it is optimal; thus an optimal solution is $z = 0; s_1 = 8; s_2 = 10; s_3 = 45; x_1 = 0; x_2 = 0$. We were fortunate that we did not have to do any calculations in this example. In general, we will not be so fortunate, and we will now describe what to do in these cases by an illustration.

Example III:

$$\begin{aligned} \max & 3x_1 + 2x_2 \\ & -x_1 + 2x_2 \leq 4 \\ & 3x_1 + 2x_2 \leq 14 \\ & x_1 - x_2 \leq 13 \\ & x_1 \geq 0; x_2 \geq 0 \end{aligned}$$

After introducing slack variables, converting the problem to one of minimization, and introducing the variable $z = -3x_1 - 2x_2$, we get:

x_1	x_2	s_1	s_2	s_3	$-z$		<i>RHS</i>
-1	2	1	0	0	0		4
3	2	0	1	0	0	=	14
1	-1	0	0	1	0		13
-3	-2	0	0	0	1		0
$x_j \geq 0; s_i \geq 0$							

The current basic solution is $z = 0; s_1 = 4; s_2 = 14; s_3 = 13; x_1 = 0; x_2 = 0$. It is feasible since all x_j and s_i are nonnegative. This is equivalent to all numbers in the RHS being nonnegative except possibly the bottom one. It is not optimal since increasing either x_1 or x_2 will decrease z (and hence improve) the solution. We choose to increase x_1 keeping all other nonbasic variables (which in this case is only x_2) at zero value. This corresponds to the one-at-a-time process which is the simplex algorithm. The more we increase this variable, the better the value of z , and hence “better” is our solution. However, as we increase this variable, s_2 and s_3 decrease, and at some point, one of these reaches the value zero (while the other is still nonnegative), and we cannot

increase this nonbasic variable any further if we wish to preserve feasibility at all times. In our example, this value for x_1 is $\frac{14}{3}$. At this point, we have an improved solution: $z = 14$, $s_1 = 4 + \frac{14}{3} = \frac{26}{3}$, $x_1 = \frac{14}{3}$, $s_3 = 13 - \frac{14}{3} = \frac{25}{3}$, $x_2 = 0$; $s_2 = 0$. Please note that the variable that became zero (from a positive value) is s_2 . Now we would like a new canonical form in which the role of x_1 and s_2 are interchanged. For this purpose, we use the equation in which s_2 “appears” (it appears only once) and use this to “solve” for x_1 and substitute this expression in all other equations for x_1 . Doing this results in the system:

x_1	x_2	s_1	s_2	s_3	$-z$		<i>RHS</i>
0	$\frac{8}{3}$	1	$\frac{1}{3}$	0	0	=	$\frac{26}{3}$
1	$\frac{4}{3}$	0	$\frac{1}{3}$	0	0		14
0	$-\frac{5}{3}$	0	$-\frac{1}{3}$	1	0		$\frac{25}{3}$
0	0	0	1	0	1		14

$x_j \geq 0; s_i \geq 0$

At this point, we know that the current basic solution [$-z = 14$, $s_1 = \frac{26}{3}$, $x_1 = \frac{14}{3}$, $s_3 = \frac{25}{3}$, $x_2 = 0$; $s_2 = 0$] is optimal. This is one kind of termination in which we find one or more optimal solutions. Not all problems have this kind of termination. For example, consider:

$$\begin{aligned}
 &\max 5x_1 + 3x_2 \\
 &-2x_1 + x_2 \leq 8 \\
 &-x_1 + 5x_2 \leq 10 \\
 &x_2 \leq 15 \\
 &x_1 \geq 0; x_2 \geq 0
 \end{aligned}$$

Using slack variables, this reduces to:

x_1	x_2	s_1	s_2	s_3	$-z$		<i>RHS</i>
-2	1	1	0	0	0	=	8
-1	5	0	1	0	0		10
0	1	0	0	1	0		15
-5	-3	0	0	0	1		0

$x_i \geq 0; s_j \geq 0$

Increasing variable x_1 improves the solution, and there is no limit to its increase since no basic variable decreases as it is increased. Hence, we can increase z without limit. Hence the problem is said to be unbounded, and there is no optimal solution in this case, and we terminate the algorithm.

Now we consider another example to illustrate yet another feature.

Example IV:

$$\begin{aligned}
&\max x_1 + 2x_2 + 3x_3 + 4x_4 \\
&-2x_1 + 2x_2 + x_3 = 4 \\
&3x_1 + x_2 + x_4 = 6 \\
&x_j \geq 0; j = 1, 2, 3, 4
\end{aligned}$$

Although this is not quite in canonical form corresponding to a feasible basic solution, to render it so is quite easy. The given system is:

x_1	x_2	x_3	x_4	$-z$	=	<i>RHS</i>
-2	2	1	0	0		4
3	1	0	1	0		6
-1	-2	-3	-4	1		0

$x_j \geq 0; j = 1, 2, 3, 4$

Substituting for x_3 and x_4 in the z equation, we get the desired canonical form and can start the simplex algorithm. Such problems are said to be in almost canonical form.

x_1	x_2	x_3	x_4	$-z$	=	<i>RHS</i>
-2	2	1	0	0		4
3	1	0	1	0		6
5	8	0	0	1		36

$x_j \geq 0; j = 1, 2, 3, 4$

Phase I:

Consider the example discussed at the outset:

Example I:

$$\begin{aligned}
&\min 40x_1 + 36x_2 \\
&x_1 \leq 8 \\
&x_2 \leq 10 \\
&5x_1 + 3x_2 \geq 45 \\
&x_1 \geq 0; x_2 \geq 0
\end{aligned}$$

This is almost like example II and setting up yields:

x_1	x_2	s_1	s_2	s_3	$-z$	=	<i>RHS</i>
1		1	0	0	0		8
	1	0	1	0	0		10
5	3	0	0	-1	0		45
40	36	0	0	0	1		0

$x_j \geq 0; s_i \geq 0$

or

x_1	x_2	s_1	s_2	s_3	$-z$
1		1	0	0	0
	1	0	1	0	0
-5	-3	0	0	1	0
40	36	0	0	0	1

$$=$$

RHS
8
10
-45
0

$$x_j \geq 0; s_i \geq 0$$

Thus, we do not have a feasible canonical form. We need a basic variable in the last equation. What we propose to do is to convert the problem to an almost canonical form that is feasible for a related but artificial problem; the solution of this artificial problem will lead us to one of two conclusions: (i) the original problem has an optimal canonical form, in which case one will be produced at optimality to the artificial problem, or (ii) the original problem does not have an optimal solution. In case (ii) we will, in some cases, be able to ascertain whether the original problem is feasible or not and, in some other cases, conclude that it (the original problem) is either infeasible or it is unbounded. In either case, it does not have an optimal solution and we would stop. The related artificial problem for our example is:

x_1	x_2	s_1	s_2	s_3	v_1	$-z$	$-w$
1	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0
5	3	0	0	-1	1	0	0
40	36	0	0	0	0	1	0
0	0	0	0	0	1	0	1

$$=$$

RHS
8
10
45
0
0

$$x_j \geq 0; s_i \geq 0; v_i \geq 0$$

This is in almost canonical form and the objective function for this phase is $w = v_1$. We needed only one artificial variable v_1 in this case since only one constraint did not have a basic variable. The corresponding canonical form is given below by eliminating the variable v_1 from the last row.

x_1	x_2	s_1	s_2	s_3	v_1	$-z$	$-w$
1	0	1	0	0	0	0	0
0	1	0	1	0	0	0	0
5	3	0	0	-1	1	0	0
40	36	0	0	0	0	1	0
-5	-3	0	0	1	0	0	1

$$=$$

RHS
8
10
45
0
-45

$$x_j \geq 0; s_i \geq 0; v_i \geq 0$$

x_1	x_2	s_1	s_2	s_3	v_1	$-z$	$-w$	<i>RHS</i>
1	0	1	0	0	0	0	0	8
0	1	0	1	0	0	0	0	10
0	3	-5	0	-1	1	0	0	5
0	36	-40	0	0	0	1	0	-320
0	-3	5	0	1	0	0	1	-5

=

x_1	x_2	s_1	s_2	s_3	v_1	$-z$	$-w$	<i>RHS</i>
1	0	1	0	0	0	0	0	8
0	0	$\frac{5}{3}$	1	$\frac{1}{3}$	$-\frac{1}{3}$	0	0	$\frac{25}{3}$
0	1	$-\frac{5}{3}$	0	$-\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{5}{3}$
0	0	20	0	12	-12	1	0	-380
0	0	0	0	0	0	0	1	0

Since the coefficients in the last row are nonnegative and the value of $w = 0$, we are at the end of a successful Phase I. We have a feasible canonical form to our original problem as indicated below after removing all traces of artificiality (v_i , and w):

x_1	x_2	s_1	s_2	s_3	$-z$	<i>RHS</i>
1	0	1	0	0	0	8
0	0	$\frac{5}{3}$	1	$\frac{1}{3}$	0	$\frac{25}{3}$
0	1	$-\frac{5}{3}$	0	$-\frac{1}{3}$	0	$\frac{5}{3}$
0	0	20	0	12	1	-380

We now continue the the simplex algorithm on the original problem till we either find an optimal solution or unboundedness. If at the end of Phase I the value of w was not zero, the original problem is infeasible.

Unrestricted Variables:

Note that the algorithm assumes that all variables are required to be nonnegative. Consider the example:

Example V:

$$\begin{aligned}
 &\max x_1 - 2x_2 + 3x_3 \\
 &x_1 + x_2 + x_3 \leq 7 \\
 &x_1 - x_2 + x_3 \geq 2 \\
 &3x_1 - x_2 - 2x_3 = -5 \\
 &x_1 \geq 0; x_2 \geq 0; x_3 \text{ unrestricted in sign}
 \end{aligned}$$

There are two methods to resolve this; the selection of the method to use depends on the circumstance under consideration.

Method I: Using any equation (other than the objective function equation) that has a nonzero coefficient for the unrestricted variable, solve for that variable and eliminate it from all the remaining equations by substitution. The result is one less equation in one less variable. For example, in the above system if we

use the third equation to solve for x_3 and eliminate it from all other constraints we get:

$$\begin{aligned}
 x_3 &= \frac{3}{2}x_1 - \frac{1}{2}x_2 + \frac{5}{2} \\
 x_1 + x_2 + \frac{3}{2}x_1 - \frac{1}{2}x_2 + \frac{5}{2} &\leq 7 \\
 x_1 - x_2 + \frac{3}{2}x_1 - \frac{1}{2}x_2 + \frac{5}{2} &\geq 2 \\
 x_1 &\geq 0; x_2 \geq 0 \\
 \max x_1 - 2x_2 + 3\left(\frac{3}{2}x_1 - \frac{1}{2}x_2 + \frac{5}{2}\right)
 \end{aligned}$$

and upon simplification this yields:

$$\begin{aligned}
 \frac{5}{2}x_1 + \frac{1}{2}x_2 &\leq \frac{9}{2} \\
 \frac{5}{2}x_1 - \frac{3}{2}x_2 &\geq -\frac{1}{2} \\
 \max \frac{11}{2}x_1 - \frac{7}{2}x_2 + \frac{15}{2} \\
 x_1 &\geq 0; x_2 \geq 0
 \end{aligned}$$

When the solution of this problem is found, back-substitute for the variable that was eliminated. Why can we not do this with all variables?

Method II: We replace the variable that is unrestricted by two whose difference it is. For example, we replace x_3 by $P_3 - N_3$ to get:

$$\begin{aligned}
 \max x_1 - 2x_2 + 3(P_3 - N_3) \\
 x_1 + x_2 + P_3 - N_3 &\leq 7 \\
 x_1 - x_2 + P_3 - N_3 &\geq 2 \\
 3x_1 - x_2 - 2(P_3 - N_3) &= -5 \\
 x_1 \geq 0; x_2 \geq 0; P_3 \geq 0; N_3 \geq 0
 \end{aligned}$$