CS 7301 Assignment #1 FALL 2008 R. Chandrasekaran

I. Consider the following LP:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$		rhs
0	1	0	α	1	0	3	0	=	β
0	0	1	-2	2	Δ	-1	0	=	2
1	0	0	0	-1	2	1	0	=	3
0	0	0	δ	3	γ	ξ	1	=	0

What are the necessary and sufficient conditions on $\alpha, \beta, \gamma, \delta, \xi$,and Δ for:

(i) the problem to be feasible.

(ii) for the problem to be unbounded.

(iii) for the problem to have one or more optimal solutions.

Solution:

(i) If $\beta \geq 0$ then $x_2 = \beta; x_3 = 2; x_1 = 3; x_4 = x_5 = x_6 = x_7 = 0$ is a feasible solution. If $\beta < 0$, $\alpha < 0$ is a necessary condition for feasibility. It is also sufficient since $x_4 = \frac{\beta}{\alpha}; x_1 = 3; x_3 = 2 + \frac{2\beta}{\alpha}; x_2 = x_5 = x_6 = x_7 = 0$ is a feasible solution. So the the required conditions are either $\beta \geq 0$ or $\beta < 0$ and $\alpha < 0$.

(ii) All conditions for feasibility must be there. In addition, we must have a solution to $Ay = 0; y \geq 0; cy < 0$. If $\alpha > 0$, then in such a solution, we must have $y_2 = y_4 = y_5 = y_7 = 0$ from the first equation. This in turn inplies $y_1 = y_6 = 0$ from the third equation and $y_3 = 0$ from the second equation. Hence it is not possible to have an unbounded case if $\alpha > 0$. If $\alpha = 0, \beta \geq 0, \delta < 0$ this canonical form indicates unboundedness. If $\alpha < 0, \delta < 0$, also we have unboundedness by increasing variable x_4 indefinitely. The only remaining possibility is $\alpha < 0, \delta \geq 0$. This is a very complicated case to verify – there are three subcases.

(iii) For this (i) must be true and (ii) must be not true.

II. Show that the LP: [min $cx : x \geq 0; Ax = b$] is unbounded iff it is feasible and $\exists\, y \geq 0$ satisfying $Ay = 0; cy < 0$.

Solution: Let x^0 satisfy the relations: $x^0 \geq 0; Ax^0 = b$ and let y^0 satisfy $y^0 \geq 0; Ay^0 = 0; cy^0 < 0$. Consider $x(\theta) = x^0 + \theta y^0$ for $\theta \geq 0$. It is clear that $x(\theta) \geq 0; Ax(\theta) = b$ for all $\theta \geq 0$. Moreover, $cx(\theta) = cx^0 + \theta cy^0 \rightarrow -\infty$ as $\theta \rightarrow \infty$. Hence if there is a vector y^0 satisfying the relations mentioend, the problem is unbounded if it is feasible. Conversely suppose the problem is unbounded. Applying the simplex method we end up with a basis

$$\hat{B} = \begin{array}{|c|c|} \hline B & 0 \\ \hline c_B & 1 \\ \hline \end{array}$$

which satisfies the relations

$$\begin{aligned} \overline{A}x &= \overline{b} \\ \overline{c}x - z &= -z^0 \end{aligned}$$

where $\overline{b} \geq 0; \overline{c} = c - c_B B^{-1}A; \overline{A} = B^{-1}A$. There is an index s such that $\overline{c}_s < 0; \overline{A}_{.s} \leq 0$. Consider the vector $y \geq 0$ defined as follows:

$$\begin{aligned} y_s &= 1 \\ y_i &= -\overline{a}_{i,s} \geq 0 \qquad \text{for } i \text{ basic} \\ y_i &= 0 \qquad \text{for } i \neq s; i \text{ nonbasic} \end{aligned}$$

This vector satisfies $y \geq 0; \overline{A}y = 0; \overline{c}y < 0$. Hence it also satisfies the relations $Ay = B\overline{A}y = 0$. Moreover, $cy = \overline{c}y + c_B B^{-1}Ay = \overline{c}y < 0$. This is the required vector.

III. Solve the LP: $[\min \sum_{j=1}^{n} jx_j : \sum_{j=1}^{i} x_j \geq i; 1 \leq i \leq n; x \geq 0]$.

Solution: The dual of this problem is given by

$$\begin{aligned} y &\geq 0 \\ \sum_{i=n-k+1}^{n} y_i &\leq n - k + 1; 1 \leq k \leq n \end{aligned}$$

$$\max \sum_{k=1}^{n} ky_k$$

It is easy to verify that $\{x_1 = n; x_j = 0; 2 \leq j \leq n\}$ and $\{y_n = 1; y_i = -; 1 \leq i \leq n-1\}$ provide feasible solutions with equal value to the two problems and hence are optimal.

IV. State whether the following statements are true or false giving reasons or counterexamples in each case:

(i) If the problem $[\min cx : x \geq 0; Ax = b]$ has an optimal solution then so does $[\min cx : x \geq 0; Ax = b' = b + \sum_j A_{.j}]$.

Since dual to both problems is the same as far as feasibility is concerned, and since the first primal has an optimal solution, it follows that both duals are feasible. Since the first primal has a feasible solution say x^0 it is easy to verify that $x^0 + e$ is a feasible solution to the second primal where e is the vector all of whose components are equal to 1.Hence the statemnt is true.

(iii) If a variable leaves the basis then it can not reenter at the next iteration.

True since the value of $\overline{c}_j > 0$ for this variable at the enxt step.

(iv) The value of the objective function changes iff the step is nondegenerate.

The change is value equals $\frac{\overline{b}_r \overline{c}_s}{\overline{a}_{r,s}}$ and all except the right hand side is nonzero. Hence if the right hand side is also nonzero, the value changes. Hence the statement is true.

(v) The regular simplex algorithm (not using any modification for degeneracy) will terminate in finitely many steps if there are no ties at any step for the variable leaving the basis.

In this case the regular simplex behaves much the same as the lexicographic method which has a finite termination. Hence the statement is true.

V. The following LP:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$		rhs
3	-3	2	8	1	0	0	0	=	2
4	6	-4	-4	0	1	0	0	=	3
5	-2	1	3	0	0	1	0	=	5
-3	-1	-5	-4	0	0	0	1	=	0

was solved using the simplex method and after several iterations the following incomplete information is available at some iteration.

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$		rhs
?	?	?	?	**1**	**0**	**−2**	?	=	?
?	?	?	?	**0**	**1**	**4**	?	=	?
?	?	?	?	**0**	**0**	**1**	?	=	?
?	?	?	?	0	0	?	?	=	?

Can you complete the table?

Solution Method: The bold 3×3 matrix represents the inverse of the current basis since it is in place of an identity matrix in the first table. By multiplying the first three rows by the bold matrix, we can identify the new set of basic variables. This identifies c_B and hence we can calculate $\overline{c} = c - c_B B^{-1} A$ and also $\overline{b} = B^{-1} b$ and all this lets us proceed further.