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Exotic superfluidity in spin-orbit coupled Bose-Einstein condensates

QIZHONG ZHU[1], CHUANWEI ZHANG[2] and BIAO WU[1(a)]

[1] *International Center for Quantum Materials, Peking University - 100871, Beijing, China*
[2] *Department of Physics and Astronomy, Washington State University - Pullman, WA, 99164 USA*

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Abstract – We study the superfluidity of a spin-orbit coupled Bose-Einstein condensate (BEC) by computing its Bogoliubov excitations, which are found to have two branches: one is gapless and phonon-like at long wavelength; the other is typically gapped. These excitations imply superfluidity that has two new features: i) due to the absence of the Galilean invariance, one can no longer define the critical velocity of superfluidity independent of the reference frame; ii) the superfluidity depends not only on whether the speed of the BEC exceeds a critical value, but also on *cross-helicity* that is defined as the direction of the cross-product of the spin and the kinetic momentum of the BEC.

Superfluidity was first discovered in 1938 and has fascinated physicists ever since. This interesting phenomenon was explained by Landau [1], whose theory has been very successful in explaining many important properties of superfluids. However, Landau's theory of superfluidity may be facing challenges brought by the recent experimental realization of artificial gauge fields for ultracold bosonic atoms [2–5]. When the artificial gauge field is non-Abelian [6–8], it is effectively spin-orbit coupling (SOC). SOC has played a crucial role in many exotic phenomena such as topological insulators [9]. However, in superfluids, the SOC is generally absent and its effects have remained largely unexplored. Note that this issue is not limited to ultracold atoms since the Bose-Einstein condensation with SOC also exists for excitons in semiconductors [10,11].

There have been some theoretical works, where many interesting properties of spin-orbit coupled Bose-Einstein condensates (BECs) are explored [11–23]. For instance, it was shown in ref. [11] that SOC can lead to unconventional Bose-Einstein condensation with the breaking of time-reversal symmetry. Later, a stripe phase that breaks rotational symmetry was found [12,13]. In this letter we concentrate on the superfluidity of the spin-orbit coupled BEC.

To put our study into perspective, we briefly review Landau's theory for a superfluid without SOC. Consider such a superfluid flowing in a tube. With the Galilean transformation, Landau found that the excitation of this flowing superfluid is related to the excitation of a motionless superfluid as [1]

$$\varepsilon_v(\mathbf{p}) = \varepsilon_0(\mathbf{p}) + \mathbf{p}\cdot\mathbf{v}, \tag{1}$$

where $\varepsilon_0(\mathbf{p})$ is the excitation for a stationary superfluid and $\mathbf{p}$ is the momentum of the excitation. For phonon excitation $\varepsilon_0(\mathbf{p}) = c|\mathbf{p}|$, the excitation $\varepsilon_v(\mathbf{p})$ can be negative only when $|\mathbf{v}| > c$. Therefore, the speed of sound c is the critical velocity beyond which the flowing superfluid loses its superfluidity and suffers viscosity. We switch to a different reference frame, where the superfluid is at rest while the tube is moving. It is apparent to many that these two reference frames are equivalent so that the superfluid will be dragged along only when the tube speed exceeds the speed of sound c. However, this equivalence is based on that the superfluid is invariant under the Galilean transformation. As SOC breaks the Galilean invariance of the system [24], we find that these two reference frames are no longer equivalent as shown in fig. 1: the critical speed for scenario (a) is different from the one for scenario (b). For easy reference, the critical speed for (a) is hereafter called the critical flowing speed and the one for (b) the critical dragging speed.

(a) E-mail: wubiao@pku.edu.cn

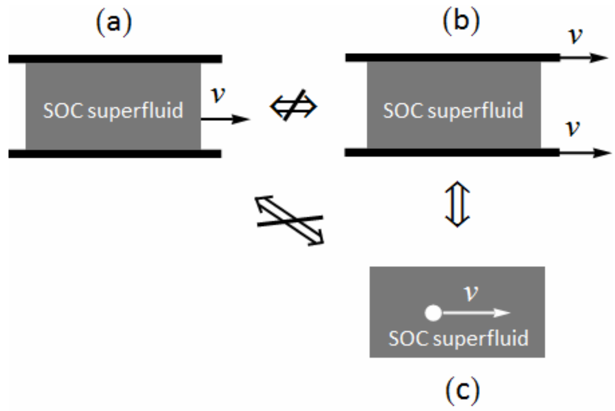

Fig. 1: (a) A superfluid with spin orbit-coupling moves while the tube is at rest. (b) The superfluid is dragged by a tube moving at speed v. (c) An impurity moves at v in the SOC superfluid. The reference frame is the lab. The two-way arrow indicates the equivalence between different scenarios and the arrow with a bar indicates the non-equivalence.

Our study of the superfluidity for a BEC with SOC is based on the computation of the elementary excitations using the Bogoliubov equation. We calculate how the elementary excitations change with the flow speed and manage to derive from these excitations the critical speeds for the two different scenarios shown in fig. 1(a), (b). We find that there are two branches of elementary excitations for a BEC with SOC: the lower branch is phonon-like at long wavelengths and the upper branch is generally gapped. Careful analysis of these excitations indicates that the critical flowing velocity for a BEC with SOC (fig. 1(a)) is non-zero while the critical dragging speed is zero (fig. 1(b)). This shows that critical velocity depends on the reference frame for a BEC with SOC and, probably, for any superfluid that has no Galilean invariance.

In addition, we find that the properties of a flow of BEC with SOC are also related to its spin direction. We characterize this spin direction with cross-helicity, which is defined as the cross-product of the spin and the kinetic momentum of the flow. A BEC flow with Rashba SOC is always unstable if its cross-helicity is negative.

Model: We consider a BEC with pseudospin 1/2 and the Rashba SOC. The system can be described by the Hamiltonian [12,15,25,26]

$$\begin{aligned}\mathcal{H} = \int \mathrm{d}\mathbf{r} \Bigg\{ &\sum_{\sigma=1,2} \Psi_\sigma^* \left(-\frac{\hbar^2\nabla^2}{2m} + V(\mathbf{r})\right)\Psi_\sigma \\ &+\gamma\left[\Psi_1^*(i\hat{p}_x+\hat{p}_y)\Psi_2 + \Psi_2^*(-i\hat{p}_x+\hat{p}_y)\Psi_1\right] \\ &+\frac{C_1}{2}\left(|\Psi_1|^4+|\Psi_2|^4\right) + C_2|\Psi_1|^2|\Psi_2|^2 \Bigg\}, \end{aligned} \tag{2}$$

where γ is the SOC constant, C_1 and C_2 are interaction strengths between the same and different pseudospin states, respectively.

We focus on the homogeneous case $V(\mathbf{r})=0$ despite that the BEC usually resides in a harmonic trap in experiments. The primary reason is that the superfluidity can be discussed more clearly in the homogeneous case, and be compared directly with the conventional superfluidity of a spinless bosonic system. In addition, the results in the homogeneous case can be adopted to understand the superfluidity in more complicated situations with the local density approximation. We also limit ourselves to the case $C_1>C_2$, where the system is stable against phase separation [12,15]. In the following discussion, for simplicity, we set $\hbar=m=1$ and ignore the non-essential z-direction, treating the system as two-dimensional. This does not impair the validity of our model. In the following we assume the BEC moves in the y-direction, and the critical velocity is found to be not influenced by the excitation in the z-direction.

The Gross-Pitaevskii equation obtained from the Hamiltonian (2) has plane-wave solutions

$$\phi_{\mathbf{k}} = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}e^{i\theta_{\mathbf{k}}}\\ -1\end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}-i\mu(\mathbf{k})t}, \tag{3}$$

where $\tan\theta_{\mathbf{k}}=k_x/k_y$, $\mu(\mathbf{k})=|\mathbf{k}|^2/2-\gamma|\mathbf{k}|+(C_1+C_2)/2$. The solution $\phi_{\mathbf{k}}$ is the ground state of the system when $|\mathbf{k}|=\gamma$. There are another set of plane-wave solutions, which have higher energies and are not relevant to our discussion.

The plane-wave solution $\phi_{\mathbf{k}}$ represents a BEC flow with the velocity $\mathbf{v}=\mathbf{k}-\gamma\hat{\mathbf{k}}$($\hat{\mathbf{k}}=\mathbf{k}/|\mathbf{k}|$). This velocity is essentially the kinetic momentum of the BEC, which is different from the conjugate momentum $\mathbf{k}$ due to the presence of SOC. For example, the ground state has a non-zero conjugate momentum $\mathbf{k}$, but its kinetic momentum (or velocity) is zero. Besides its velocity, the BEC flow described by $\phi_{\mathbf{k}}$ has another feature, the direction of the spin, which has to be included for a complete description of the flow. For example, the flow at $\mathbf{k}=3\gamma/2\hat{y}$ has the same velocity $\gamma/2\hat{y}$ as the flow at $\mathbf{k}=-\gamma/2\hat{y}$. However, they have different spin directions. With the Rashba-type SOC, the spin direction of the eigenstate is always perpendicular to the velocity and has only two choices. As a result, it is sufficient and also very convenient to use $w=\mathrm{sign}[(\hat{\mathbf{v}}\times\hat{\sigma})\cdot\hat{z}]$ to denote the spin direction. This variable w, which is either 1 or -1, is the cross-helicity mentioned in the introduction. Note that the usual helicity in literature is defined as the inner product of the spin and the momentum.

Critical velocities: We study first the scenario depicted in fig. 1(a), where the BEC flows with a given velocity. Since the system is not invariant under the Galilean transformation, we cannot use eq. (1) to find the excitations for the flowing BEC from the excitation of a stationary BEC. We have to compute the excitations directly. This can be done by computing the elementary excitations of the state $\phi_{\mathbf{k}}$ with the Bogoliubov equation for different values of $\mathbf{k}$.

Without loss of generality, we choose $\mathbf{k}=k\hat{y}$ with $k>0$. Following the standard procedure of linearizing the

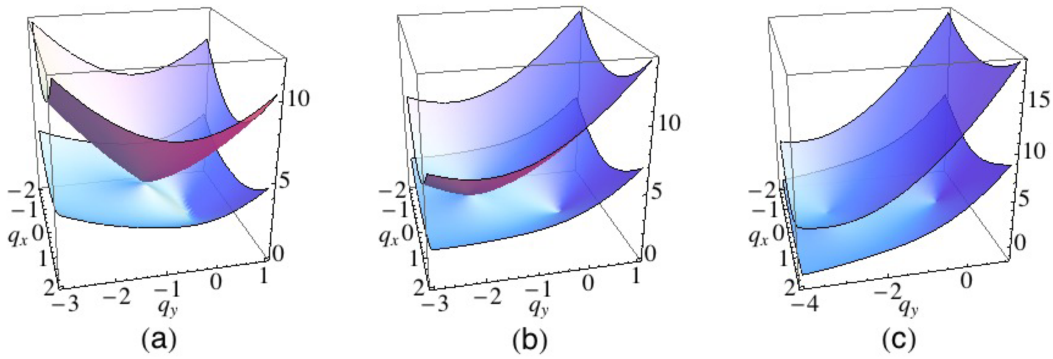

Fig. 2: (Color online) Elementary excitations of a BEC flow with the SOC. (a) $k=1$; (b) $k=2.5$; (c) $k=4$. $C_1=10$, $C_2=4$, $\gamma=1$.

Gross-Pitaevskii: equation [27,28], we have the following Bogoliubov equation:

$$\mathcal{L}\begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \varepsilon \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}, \tag{4}$$

where

$$\mathcal{L} = \begin{pmatrix} H_k^+ & b_{12} & -\frac{1}{2}C_1 & -\frac{1}{2}C_2 \\ b_{21} & H_k^+ & -\frac{1}{2}C_2 & -\frac{1}{2}C_1 \\ \frac{1}{2}C_1 & \frac{1}{2}C_2 & H_k^- & b_{34} \\ \frac{1}{2}C_2 & \frac{1}{2}C_1 & b_{43} & H_k^- \end{pmatrix} \tag{5}$$

with $H_k^\pm = \pm\frac{q_x^2+(q_y\pm k)^2}{2} \pm A$, $A = \frac{C_1}{2} - \frac{k^2}{2} + \gamma k$, $b_{12} = -\gamma(iq_x+q_y+k) + \frac{C_2}{2}$, $b_{21} = \gamma(iq_x - q_y - k) + \frac{C_2}{2}$, $b_{34} = \gamma(iq_x - q_y + k) - \frac{C_2}{2}$, and $b_{43} = -\gamma(iq_x + q_y - k) - \frac{C_2}{2}$. As usual, there are two groups of eigenvalues and only the ones whose corresponding eigenvectors satisfy $|u_i|^2 - |v_i|^2 = 1$ $(i=1,2)$ are physical.

For comparison, we consider the case without SOC. This is to put $\gamma=0$ and reduce our system to a two-component BEC system, which is well studied in the literature [29]. For this case, $\mathcal{L}$ can be diagonalized analytically and there are two branches of excitations,

$$\varepsilon_\pm(\mathbf{q}) = q_y k + \sqrt{\frac{C_1 \pm C_2}{2}\mathbf{q}^2 + \frac{\mathbf{q}^4}{4}}\,. \tag{6}$$

These results show that the system at the ground state ($k=0$) has two different speeds of sound, $\sqrt{(C_1+C_2)/2}$ and $\sqrt{(C_1-C_2)/2}$. Since the excitation ε_- becomes negative only when $k > \sqrt{(C_1-C_2)/2}$, the critical flowing velocity in this case is $\sqrt{(C_1-C_2)/2}$. When $C_2=0$, these two branches of excitations merge into one and the critical velocity is $\sqrt{C_1/2}$, which is well known and was confirmed in a BEC experiment [30].

In general, there are no simple analytical results. We have numerically diagonalized $\mathcal{L}$ to obtain the elementary excitations. We find that part of the excitations are imaginary for BEC flows with $|\mathbf{k}| < \gamma$. This means that all the flows with $|\mathbf{k}|<\gamma$ are dynamically unstable and therefore do not have superfluidity. For other flows with $k \geqslant \gamma$, the excitations are always real and they are plotted in fig. 2. One immediately notices that the excitations have two branches, which are in contact with each other at a single point. A closer examination shows that the upper branch is gapped in most of the cases while the lower branch has phonon-like spectrum at large wavelength. These features are more apparent in fig. 3, where only the excitations along the x-axis and y-axis are plotted.

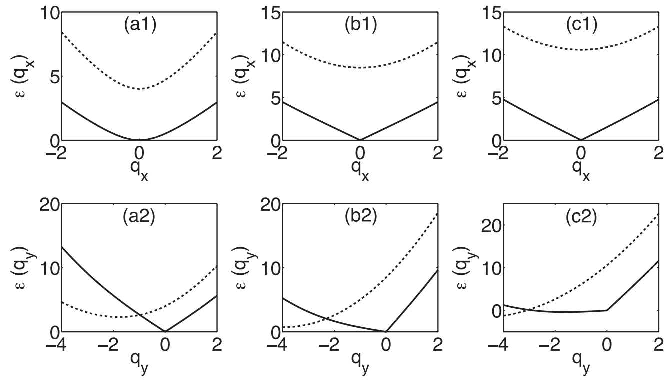

Fig. 3: Excitations along the x-axis (the first row) and y-axis (the second row) at different values of k. (a1), (a2): $k=1$; (b1), (b2): $k=3$; (c1), (c2): $k=4$. $C_1=10$, $C_2=4$, $\gamma=1$.

In fig. 2(c) and fig. 3(c2), we notice that some of the excitations in the upper branch are negative, indicating that the underlying BEC flow is thermodynamically unstable and has no superfluidity. In fact, our numerical computation shows that there exists a critical value k_c: when $k>k_c$ either part of the upper branch of excitations or part of the lower branch or both become negative. This means that the flows described by the plane-wave solution $\phi_{\mathbf{k},-}$ with $|\mathbf{k}|>k_c$ suffer Landau instability and have no superfluidity. Combined with the fact that the flows with $|\mathbf{k}|<\gamma$ are dynamically unstable, we can conclude that only the flows with $\gamma \leqslant |\mathbf{k}| \leqslant k_c$ have superfluidity. Physically, these super-flows have speeds smaller than $v_c = k_c - \gamma$ and

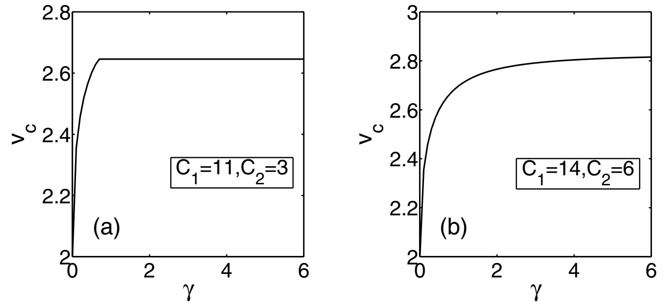

Fig. 4: Critical flowing velocity v_c (solid line) of a BEC as a function of the SOC parameter γ. (a) $C_1 > 3C_2$; (b) $C_1 < 3C_2$.

cross-helicity $w = 1$. We have plotted how the critical flowing velocities vary with the SOC parameter γ in fig. 4.

We find two different asymptotic behaviors for the critical velocity when γ gets large. As shown in fig. 4, the critical flowing velocity becomes a constant, $v_c = \sqrt{(C_1 + C_2)/2}$, beyond a threshold value in the parameter regime $C_1 > 3C_2$ while it approaches an asymptotic value $\sqrt{C_1 - C_2}$ in the regime $C_1 < 3C_2$.

We turn to another reference frame illustrated in fig. 1(b), where the BEC can be viewed as being dragged by a moving tube. To simplify the discussion, we replace the moving tube with a macroscopic impurity moving inside the BEC as shown in fig. 1(c). Correspondingly, the question as to whether the BEC will be dragged along by the moving tube is replaced by an equivalent question: will the impurity experience any viscosity? Suppose that the moving impurity generates an excitation in the BEC. According to the conservations of both momentum and energy, we should have

$$m_0 \mathbf{v}_i = m_0 \mathbf{v}_f + \mathbf{q}, \tag{7}$$

$$\frac{m_0 \mathbf{v}_i^2}{2} = \frac{m_0 \mathbf{v}_f^2}{2} + \varepsilon_0(\mathbf{q}), \tag{8}$$

where m_0 is the mass of the impurity, $\mathbf{v}_i$ and $\mathbf{v}_f$ are the initial and final velocities of the impurity, respectively, and $\varepsilon_0(\mathbf{q})$ is the excitation of the BEC at $k = \gamma$. If the excitations were purely phonons, *i.e.*, $\varepsilon_0(\mathbf{q}) = c|\mathbf{q}|$, these two conservations would not be satisfied simultaneously when $v \approx |\mathbf{v}_i| \approx |\mathbf{v}_f| < c$. This means that the impurity could not generate phonons in the superfluid and would not experience any viscosity when its speed is smaller than the sound speed. This is in fact nothing but the Cerenkov radiation [31,32], where a charged particle radiates only when its speed exceeds the speed of light in the medium. For our BEC system, the elementary excitations $\varepsilon_0(\mathbf{q})$ are not purely phonons. In this case, the critical dragging velocity derived from eqs. (7), (8) is given by

$$v_c = \left| \frac{\varepsilon_0(\mathbf{q})}{|\mathbf{q}|} \right|_{min}. \tag{9}$$

For the special case $\gamma = 0$, we have

$$\varepsilon_0^{\pm}(\mathbf{q}) = \sqrt{\frac{C_1 \pm C_2}{2}\mathbf{q}^2 + \frac{\mathbf{q}^4}{4}}. \tag{10}$$

From eq. (9), we obtain the critical dragging speed $\sqrt{\frac{C_1 - C_2}{2}}$, which is the same as the critical flowing speed.

When $\gamma \neq 0$, the excitations $\varepsilon_0(\mathbf{q})$ also share two branches. Along the x-axis, these two branches are

$$\varepsilon_0^{\pm}(q_x) = \sqrt{s_1 + s_2 q_x^2 + \frac{q_x^4}{4} \pm \sqrt{t_1 + t_2 q_x^2 + t_3 q_x^4 + \gamma^2 q_x^6}}, \tag{11}$$

where $s_1 = 2\gamma^4 + \gamma^2 (C_1 - C_2)$, $s_2 = 2\gamma^2 + \frac{1}{2}C_1$, $t_1 = s_1^2$, $t_2 = 2 s_1 s_2$, and $t_3 = 2s_1 + \left(\gamma^2 + C_2/2\right)^2$. Along the y-axis, the excitations of the ground state are

$$\varepsilon_0^{-}(q_y) = \sqrt{\frac{C_1 + C_2}{2} q_y^2 + \frac{q_y^4}{4}}, \tag{12}$$

$$\varepsilon_0^{+}(q_y) = 2\gamma q_y + \sqrt{2s_1 + \left(s_2 - \frac{C_2}{2}\right) q_y^2 + \frac{q_y^4}{4}}. \tag{13}$$

When $\gamma > 0$, the upper branch $\varepsilon_0^{+}(q_x)$ is always parabolic at small q_x with a gap $\sqrt{2s_1}$. When expanded to the second order of q_x, the lower branch has the following form:

$$\varepsilon_0^{-}(q_x) \approx q_x^2 \sqrt{\frac{C_1 + C_2}{8\gamma^2}}. \tag{14}$$

This shows that $\varepsilon_0^{-}(q_x)$ is parabolic at long wavelengths instead of linear as usually expected for a boson system. This agrees with the results in ref. [17]. This parabolic excitation has a far-reaching consequence: according to eq. (9), the critical dragging velocity v_c is zero, very different from the critical flowing velocity for a BEC moving in a tube. This shows that the critical velocity for a BEC with SOC is not independent of the reference frame, in stark contrast with a homogeneous superfluid without SOC. This surprising finding of course has the root in the fact that the BEC described by the SOC Hamiltonian (2) is not invariant under the Galilean transformation [24].

Rashba and Dresselhaus SOC: We have also investigated the superfluidity with the general form of SOC, which is a mixture of Rashba and Dresselhaus coupling. Mathematically, this SOC term has the form $\alpha\sigma_x p_y - \beta\sigma_y p_x$. The essential physics is the same: the critical flowing speed is different from the critical dragging speed, and therefore the critical velocity depends on the choice of the reference frame. However, the details do differ when $\alpha \neq \beta$. First, the critical dragging speed is no longer zero. Without loss of generality, we let $\alpha > \beta$. The slope of the excitation spectrum for the ground state along the y-axis is $v_y = \sqrt{\sqrt{2\alpha^2 (C_1 - C_2 + 2\alpha^2)} + 2\alpha^2 + \frac{C_1 - C_2}{2}} - 2\alpha$, and the slope along the x-axis, $v_x = \sqrt{\left(1 - \frac{\beta^2}{\alpha^2}\right) \frac{C_1 + C_2}{2}}$. The critical dragging velocity is the smaller of the above two slopes, which are both non-zero. Secondly, the BEC with negative cross-helicity can also be stable; as a result, there is a different critical flowing speed for either cross-helicity. For the Rashba type ($\alpha = \beta$), the critical flowing speed for negative cross-helicity is always zero since the BEC is always dynamically unstable for negative cross-helicity.

Experimental observation: Spin-orbit coupled BECs have been realized recently by three different groups [3–5] through coupling ultracold ^{87}Rb atoms with laser fields. The strength of the SOC in the experiments can be tuned by changing the directions of the lasers [3–5] or through the fast modulation of the laser intensities [33]. The interaction between atoms can be adjusted by varying the confinement potential, the atom number or through the Feshbach resonance [34]. For the scenario in fig. 1(b), one can use a blue-detuned laser to mimic the impurity for the measurement of the critical dragging speed similar to the experiment reported in ref. [30]. For the scenario in fig. 1(a), there are two possible experimental setups for measuring the critical flowing speed. In the first one, one generates a dipole oscillation similar to the experiment in ref. [5] but with a blue-detuned laser in the middle of the trap. The second one is more complicated: At first, one generates a moving BEC with a gravitomagnetic trap [35], then uses Bragg spectroscopy [36,37] to measure the excitations of the moving BEC, from which the superfluidity can be inferred. For the typical atomic density of 10^{14}–10^{15} cm^{-3} achievable in current experiments [30], and the experimental setup in ref. [3], the critical flowing velocity is 0.2–0.6 mm/s, while the critical dragging velocity is still very small, about 10^{-3}–10^{-2} mm/s. To further magnify the difference between the two critical velocities, one can use the Feshbach resonance to tune the s-wave scattering length.

Perspective: This is not the first BEC system, where there are two different critical velocities of superfluidity. The BEC in an optical lattice [38], a superfluid with its density periodically modulated, is another system of two different critical velocities. Supersolid helium may also be regarded as a periodic superfluid [39].

We consider first the case when the BEC is locked with the lattice and they move together as shown in fig. 5(a), (b), (d). Since this system is invariant under the Galilean transformation, the two scenarios depicted in fig. 5(a), (b) are equivalent and Landau's argument is still applicable. The caveat is that the critical velocity is always zero no matter how the elementary excitation of the superfluid looks. The key reason is that the momentum $\mathbf{p}$ of the excitation is not well defined due to the presence of the lattice: two momenta which differ by a reciprocal lattice vector are equivalent. This result has been verified in extensive computations for Cerenkov radiation in a periodic medium [31]. Recently, a similar numerical calculation was done specifically for a moving defect in a BEC in an optical lattice and zero critical velocity is found [40].

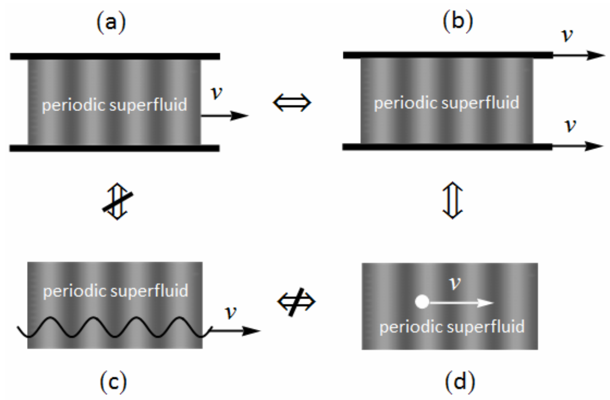

Fig. 5: In (a), (b), (d), the periodic superfluid and the lattice are locked in and move together with respect to the obstacle. In (c), the lattice is gradually accelerated to speed v. The reference frame is the lab. The scenario in (c) is not equivalent to the scenarios in (a), (b), (d) as there is relative motion between the lattice and the fluid in (c).

The presence of a lattice does put a new twist into the system. One can gradually accelerate the lattice to a certain velocity and see how the superfluid changes (fig. 5(c)). There exists a critical velocity for the lattice beyond which the system loses its superfluidity. This critical velocity for lattice is different from the critical velocity of a moving defect and is called trawler critical velocity in ref. [41]. This type of critical velocity was considered theoretically in ref. [27] and demonstrated experimentally in ref. [42].

Note that a more detailed version of the above discussion for the superfluidity of a periodic superfluid can be found in ref. [41]. However, an error was made in ref. [41]: a non-zero critical velocity for the scenarios shown in fig. 5(a), (b), (d) was predicted.

With BEC, we now have many types of superfluids, which are different from the conventional spinless and homogeneous superfluid helium. We expect more of this kind of new superfluids to appear in the future, which will surely extend and enrich our understanding of superfluidity.

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