Momentum-Space Josephson Effects

Junpeng Hou,¹ Xi-Wang Luo,¹ Kuei Sun,¹ Thomas Bersano,² Vandna Gokhroo,² Sean Mossman,² Peter Engels,² and Chuanwei Zhang¹,*

¹Department of Physics, The University of Texas at Dallas, Richardson, Texas 75080-3021, USA
²Department of Physics and Astronomy, Washington State University, Pullman, Washington 99164, USA

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The Josephson effect is a prominent phenomenon of quantum supercurrents that has been widely studied in superconductors and superfluids. Typical Josephson junctions consist of two real-space superconductors (superfluids) coupled through a weak tunneling barrier. Here we propose a momentum-space Josephson junction in a spin-orbit coupled Bose-Einstein condensate, where states with two different momenta are coupled through Raman-assisted tunneling. We show that Josephson currents can be induced not only by applying the equivalent of “voltages,” but also by tuning tunneling phases. Such tunneling-phase-driven Josephson junctions in momentum space are characterized through both full mean field analysis and a concise two-level model, demonstrating the important role of interactions between atoms. Our scheme provides a platform for experimentally realizing momentum-space Josephson junctions and exploring their applications in quantum-mechanical circuits.

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Introduction.—The Josephson effect [1,2] is an intriguing quantum phenomenon of supercurrents across a device known as a Josephson junction (JJ). A typical JJ consists of two macroscopic quantum systems [e.g., superconductors, superfluids, or Bose-Einstein condensates (BECs)] that are separated in real or spin space and weakly coupled by quantum tunneling through a thin barrier [Fig. 1(a)] or by Rabi coupling between different spins. Because of quantum tunneling of particles across the junction, JJs have found important applications in quantum-mechanical circuits, such as SQUIDs [3,4], superconducting qubits [5–8], and precision measurements [3]. In experiments, JJs have been widely realized in solid state superconductors [9,10], superfluid helium [11–14], and recently, in ultracold atomic gases [15–26], where oscillating supercurrents were generated by applying a voltage drop (or its equivalent) across JJs while maintaining a constant weak coupling (i.e., ac Josephson effect [27]).

While JJs have been well studied in real space, a natural and important question is whether Josephson effects can also be observed in momentum space. In this Letter, we address this question and propose a scheme for realizing momentum-space JJs (MSJJs). In analogy to bosonic JJs in a real-space double well [22,23], a MSJJ may be realized with a momentum-space double-well dispersion [see Fig. 1(a)], which is an essential property of spin-orbit coupled systems [28,29]. Spin-orbit coupling (SOC) is ubiquitous in solid state materials and has recently been realized experimentally in ultracold atomic gases [29–41]. In the presence of SOC, condensates at distinct band minima can be considered as two distinct independent quantum systems. However, unlike quantum tunneling between two wells in real space, two BECs at distinct momenta are not directly coupled.

Here we propose a MSJJ facilitated by a tunable interwell coupling in a spin-orbit coupled BEC [42,43], where the coupling is generated by an additional pair of counterpropagating Raman lasers. Such Raman-assisted tunneling between two momentum states changes both the atomic spin and momentum, and thus couples the condensates at the two band minima. The SOC strength dictates the height of the insulating barrier while the Raman detuning serves as an effective voltage between the two band minima. Suddenly changing the detuning (i.e., applying a voltage) induces a coherent oscillation of the BECs between the two band minima (i.e., supercurrent

FIG. 1. (a) Illustration of a conventional JJ for real-space superconductors (top) versus MSJJ (bottom), where the double-well band dispersion is generated using a spin-orbit coupled BEC. (b),(c) Experimental setup for realizing a MSJJ. Two pairs of Raman lasers realize SOC (blue) and weak coupling (red) between two band minima, respectively.
oscillations), similar to traditional ac Josephson effects in superconductors. More interestingly, the phase of the
Raman-assisted tunneling between BECs at the two band
minima is highly tunable [44], in contrast to real tunneling
coefficients for real-space JJs in superconductors [9,10] and
double-well BECs [22,23]. We show that a sudden change
of the tunneling phase (while keeping the effective voltage
unchanged) can also induce Josephson effects of super-
currents, a phenomenon that we name as “tunneling-phase-
driven JJ.” We focus on this new type of Josephson effect
and study its properties through both full mean-field
mechanical circuits.

Experimental setup and theoretical modeling.—We con-
sider a BEC confined in an elongated trap. Two internal
states $|\uparrow\rangle$ and $|\downarrow\rangle$ are coupled by two counterpropagating
Raman lasers with Rabi frequencies $\Omega_a$ and $\Omega_b$, forming an
effective one-dimensional (1D) SOC dispersion relation
along the $x$ direction [see Figs. 1(b) and 1(c)]. Hereafter we
choose recoil momentum $h k_R$ and recoil energy $E_R = h^2 k_R^2/2m$ for the Raman lasers as the units of momentum and
energy. Consequently, we have length and time in units of
$1/k_R$ and $\hbar/E_R$. The 1D SOC displays a double-well
band dispersion in momentum space with two band minima
located at $\pm k_L = \pm \sqrt{1 - (\Omega/4)^2}$, where $\Omega$ is the Raman
coupling strength [46]. The tunneling between BECs at
$\pm k_L$ requires simultaneous change of spin and momentum,
which can be realized using another independent pair of
Raman lasers $\Omega_a'$ and $\Omega_b'$ incident at an angle $\theta_L = \arccos (1 - k_L)$ to the $x$ axis [Fig. 1(b)]. The frequencies
of the pair $(a', b')$ are shifted from those of the pair $(a, b)$
by $\Delta' \approx 100$ MHz so that the interference between them is
negligible. The frequency difference between $a'$ and $b'$
should match that between $a$ and $b$ to generate a time-
indepedent coupling.

Since only the $x$ direction is relevant for the SOC
dynamics, the other two directions can be integrated out,
yielding an effective 1D system. The dynamics of the
system can be described by the GPE

$$\frac{i}{\hbar} \frac{\partial}{\partial t} \psi = \left[ H_0 + \frac{1}{2} a_0^2 x^2 + \frac{g}{2} |\psi|^2 \right] \psi \quad (1)$$

under the mean-field approximation, where $\psi = (\psi_1, \psi_2)^T$
is the two component condensate wave function normalized
by the average particle number density $n = \int dx \psi^* \psi$, $\omega_s$
represents trapping frequency of harmonic trap. For a
typical $^{87}$Rb BEC, the effective density interaction $ng \sim$
0.1 with $\sim 10^4$ atoms (see “Experimental consideration.”
section) and the spin interaction is negligible. The Raman
coupling does not affect atomic interactions. The single
particle Hamiltonian can be written as [47,49]

$$H_0 = \left( \frac{\Omega_0}{2} + e^{i\phi_L} \Omega_L e^{2ik_L x} \right) \left( p_x + 1 \right)^2 + \frac{\delta}{2} \right), \quad (2)$$

where $\Omega_L$ is the coupling strength generated by the
tunneling lasers, $\phi_L$ is the relative phase between the
two Raman couplings, and $\delta$ is the detuning.

The ground state of the BEC is obtained from the
imaginary time evolution of the GPE [47,50] using a
time-split-operator method, resulting in the phase diagram
shown in Fig. 2(a) in the $\Omega_L - \delta$ plane, where the color
represents spin polarization $\langle \sigma_z \rangle$. For weak $\Omega_L$, interactions
lock the condensate to one momentum minimum, yielding
a plane-wave phase at large detunings. There is a first-order
phase transition [black line in the inset of Fig 2(a)] when $\delta$
crosses 0. With increasing $\Omega_L$, the single-particle coupling
dominates over the interaction; hence, the ground state is in
a striplike phase with a real-space density modulation
[Fig. 2(b)], and $\langle \sigma_z \rangle$ varies continuously and smoothly with respect to $\delta$ [white line in the inset of Fig. 2(a)]. While a
supersolid stripe phase is defined through spontaneous
breaking of both continuous translational and gauge sym-
metries [51–54], here continuous translational symmetry is
synthetically broken by the periodic potential $e^{2ik_L x}$.
Nevertheless, the ground state is the superposition of
two band minima, similar to an authentic stripe phase
induced by interactions.

The additional Raman lasers $\Omega_L'$ couple not only the two
band minima, but also other states from both lower and

- FIG. 2. (a) Ground state phase diagram, where $\Omega = 2.7, \phi_L = 0$ and $ng = 0.07$. The inset shows the first-order phase transition for small $\Omega_L$. Black, dark gray, light gray, and white lines correspond to $\Omega_L = 0.01, 0.1, 0.2,$ and 0.5, respectively. (b) Real
space density modulation for the ground state with parameters
$\delta = 0.054$ and $\Omega_L = 0.015$ as denoted by the black cross in (a).
(c) Illustration of induced couplings between six most relevant
momentum states.
upper bands. The six most relevant momentum states \( \psi_i \) are shown in Fig. 2(c). Expanding the wave function \( \psi = \sum_{i=1}^{6} C_i \psi_i \) in this six-state basis, we obtain a \( 6 \times 6 \) effective Hamiltonian \([47]\). The direct coupling between the two band minima at 2 and 5 is \( -V_0 e^{\pi i \delta_L} \) with \( V_0 = \frac{1}{2} \Omega_L (1 + k_L) \), while the couplings with other neighboring high-energy states are \( -\sqrt{(1 - k_L/2)} e^{\pi i \delta_L} \) and \( \frac{1}{2} \sqrt{1 - k_L^2} e^{\pi i \delta_L} \), which approach 0 when \( k_L \to 1 \), leaving \( V_0 \) as the dominant tunneling term. We focus on the region \( \Omega_L \ll \Omega \) to avoid significant modification of the original SOC band dispersion and also for the observation of Josephson effects with weak tunneling.

**Tunneling-phase-driven MSJJ.**—In an ac JJ, a suddenly applied voltage can induce an oscillation of supercurrents between two superconductors. In our system, BECs at the two band minima marked 2 and 5 are considered as two superfluids and the detuning between them corresponds to a voltage. A sudden change of \( \delta \) induces an oscillation of the BEC between the two minima, yielding a MSJJ whose properties are described in the Supplemental Material \([47]\). Here we focus on the relative phase \( \phi_L \) for the tunneling element between 2 and 5, which is highly tunable in experiments \([44]\). In contrast, such tunneling is a real number for a real space JJ between two superconductors or double-well BECs. A sudden change of the phase \( \phi_L \) (keeping \( \delta \) constant) can induce a different type of Josephson effect, i.e., a tunneling-phase-driven JJ.

In Figs. 3(a)–3(c) we show dynamics from simulations of the GPE with a sudden change of the phase \( \phi_L \) from an initial \( \phi_{L0} \) to \( \phi_{LF} = 0 \). In panel (a) we plot the population \( P_i(t) \) at each momentum state for \( \phi_{L0} = 0.4\pi \). Clearly only the states 2 and 5 at the two band minima are largely populated while all other states can be neglected due to their small initial populations, weak coupling to states 2 and 5, and high energies. Panel (b) shows the relative phase between BECs in states 2 and 5. For \( \phi_{L0} = 0.4\pi \) (blue solid line), the phase varies through \([0, 0.2\pi]\), representing a Josephson type of oscillation, while for \( \phi_{L0} = 0.3\pi \) (yellow dashed line), the phase oscillates in a small range, showing a plasma oscillation. The polarization \( \langle \sigma_\phi \rangle \) exhibits sinusoidal oscillations for both cases [panel (c)].

Because the population of the BEC stays mainly at the two band minima 2 and 5, we can neglect the other states to derive an effective two-level model, yielding an equation of motion \([46,47]\)

\[
\dot{\psi}_2 = \left( C_2 \right) = \left( \begin{array}{c} H_0^{\text{eff}} + H_1^{\text{eff}} \\ \end{array} \right) \left( C_2 \right),
\]

where \( H_0^{\text{eff}} = \left( -k_L \delta_i^2 - V_0 e^{\pi i \delta_L} \right) \) is the effective single-particle Hamiltonian, and \( H_1^{\text{eff}} = 2 g_G \langle |C_2|^2 |C_i|^2 \rangle \) is the effective interaction term obtained through a variational approximation of the GPE. Generally, \( g_G \) depends on \( |C_2|^2 |C_i|^2 \) but is approximately a constant when the interaction strength is weak compared to \( E_R \), yielding \( g_G = ng(1 - k_L^2) \). Note that the coupling phase \( \phi_L \) in Eq. (3) can be incorporated into the relative phase between \( C_2 \) and \( C_5 \) through a simple phase transformation; therefore, the quench of \( \phi_L \) is mathematically equivalent to a quench of the relative phase between condensates at two minima (2, 5), although the latter is experimentally impractical.

When the coupling \( V_0 \) is strong, the dynamics of the BECs are governed by single particle physics, yielding a linear Rabi oscillation with period \( T = \pi / \omega \), where the Rabi frequency \( \omega = \sqrt{(k_L \delta_i^2)^2 + |V_0|^2} \). Such a simple formula for the period does not apply when the tunneling \( V_0 \) is comparable to or weaker than the interparticle interactions, although the two-level model still agrees reasonably well with the GPE simulations, as shown in Fig. 3(d). We see that the period is similar for interacting and single-particle cases for a large coupling \( \Omega_L = 0.025 \), but shows strong deviations [see the sharp peak for the solid red line in Fig. 3(d)] from the single particle curve for \( \Omega_L = 0.015 \). For a very large detuning \( \delta \) (i.e., voltage), all \( T \) collapse to the same line as the single particle case, as expected.

In the two-level approximation, we can choose the normalization \( |C_2|^2 + |C_5|^2 = 1 \), and recast the equation of motion, Eq. (3), as \([47]\)

\[
\partial_z z = -\sqrt{1 - z^2} \sin (\phi - \phi_{LF}),
\]
FIG. 4. (a) Classical trajectories in the $z$-$\phi$ plane for $0 < \phi_{L0} \leq \pi$, with initial value of $z$ at 0.434. (b) Same as (a) but generated through the GPE simulation. Parameters are $n_g = 0.07$, $\delta = 0.008$ (corresponding to initial polarization 0.434), $\Omega = 2.7$, and $\Omega_L = 0.015$. The three colors correspond to $\phi_{L0} = 0.2$ (blue), 0.4 (orange), 0.8 (green), respectively. The arrows denote the direction of each trajectory.

\[ \partial_t \phi = \frac{g_0 z}{V_0} + \frac{z}{\sqrt{1-z^2}} \cos(\phi - \phi_{L0}) + \frac{k_1 \delta}{2V_0}, \]  
(5)

using the population difference $z = (N_2 - N_5)/N$ and relative phase $\phi = \theta_2 - \theta_4$, where $N_i$ and $\theta_i$ are defined through $C_2 = \sqrt{N_2 e^{i\theta_2}}$ and $C_5 = \sqrt{N_5 e^{i\theta_5}}$. These two classical equations characterize the essential dynamics of MSJJ.

Figure 4(a) shows how the initial value $\phi_{L0}$ affects the dynamics. For a relatively small $\phi_{L0}$, the classical trajectory is a closed loop around a fixed point with a small amplitude of $z$ and a confined range of phase change $\Delta \phi$, showing a plasma oscillation [17]. With increasing $\phi_{L0}$, the amplitudes for both $\phi$ and $z$ increase. Beyond a critical $\phi_{L0}$, $\phi$ varies through $[0, 2\pi]$, showing a Josephson oscillation. The system returns to plasma oscillation around another fixed point when $\phi_{L0}$ exceeds another critical point. These classical trajectories from the two-level model agree with those from the GPE simulations in Fig. 4(b). Note that the trajectories around two fixed points have opposite directions. In the single-particle case, these two fixed points correspond to two opposite Zeeman fields for spin precession of the Rabi oscillation [47].

Strong interaction between atoms can dramatically change the BEC dynamics and lead to a self-trapping effect [17,22], where the oscillation amplitude of $z$ is strongly suppressed. We consider a symmetric oscillation with $\delta = 0$. For a weak interaction of $n_g = 0.07$, the oscillation of $\langle \sigma_z \rangle$ shows a perfect sinusoidal pattern (blue line), as seen by the blue line Fig. 5(a) obtained from the GPE simulation. When the interaction is doubled $n_g = 0.14$, the oscillation amplitude is reduced and the average $\langle \sigma_z \rangle$ in one period changes from 0 to a finite value (orange line). For a larger but still practicable interaction of $n_g = 0.35$, the oscillatory behavior disappears and the condensate is locked at the initial band minimum because of strong density interaction. Such nonlinear self-trapping effects can also be captured in the classical trajectories in the two-level model [Fig. 5(b)]. With increasing $n_g$, the initial plasma oscillation with a large variation of $z$ becomes the self-trapped Josephson oscillation with a small $z$ change.

Experimental consideration.—The periodic density modulation for the stripelike ground state can be measured using Bragg scattering, similar to the recent experiments for observing supersolid stripe phases [55]. Consider a $^{87}$Rb BEC confined in a quasi-1D harmonic trap. The Raman lasers for generating SOC are incident at $45^\circ$ with the $x$ axis, yielding an effective wave vector $k_R = (2\pi/\sqrt{2})$ with $\lambda = 784$ nm. The corresponding recoil energy $E_R = 2\pi \hbar \times 1.8$ kHz; thus, the time and length units are $\hbar/E_R = 0.088$ ms and $2\pi/k_R = 1109$ $\mu$m, respectively. The Raman coupling strength for SOC $\Omega = 2.7 E_R$; thus, $k_L = 0.738 k_R$ and the second pair of Raman lasers should be incident at an angle $\theta_L = 58.6^\circ$ with respect to the $x$ axis. The $s$-wave scattering length of $^{87}$Rb is $a_s = 100.86 a_0$, where $a_0$ is the Bohr radius. Considering a particle number $N_0^3$ to $10^6$ and typical trapping frequencies $\omega_z \sim 2\pi \times 5$ Hz and $\omega_x = \omega_y \sim 2\pi \times 75$ Hz, one has the average particle density $n \sim 10^{13}$ to $10^{14}$ $cm^{-3}$ under Thomas-Fermi approximation [45]. The effective interaction strength can be evaluated through $n_g = 4\pi \hbar^2 a_s n/m \sim 0.07$ to 0.48$E_R$, resulting in the time period $T \sim 10$ ms for tunneling-phase-driven Josephson oscillations [Fig. 3(d)].

Discussion and conclusion.—Our two major proposed concepts, a momentum-space $JJ$ and a tunneling-phase-driven $JJ$, may also be realized in other physical systems where a double-well band dispersion with two almost degenerate local band minima can be generated to ensure the long life time of the BEC at different momenta [56]. For instance, the double-well band dispersion may be realized in optical superlattices with Raman assisted tunneling [57], where two momentum minima can be coupled with additional Raman transitions. The double-well band dispersion can be generalized to triple-well or even more multiple-degenerate momentum states, and the coupling between...
neighboring minima may form a momentum-space optical lattice [56], which can be considered as a Josephson junction array [20] in momentum space. The linear momentum discussed here can be generalized to orbital angular momentum (OAM), and an OAM-space JJ may be realized for a BEC on a ring utilizing recent proposals for spin-OAM coupling [58–60] for cold atoms. The discreteness of OAM states may induce interesting Josephson effects that are different from those in continuous real or momentum space. Finally, although absent in solid-state superconductors, the proposed tunneling-phase-induced JJ may be realized in real-space optical superlattices with Raman assisted tunneling [57], where the phase for the Raman tunneling may also be tuned.

In conclusion, we propose a new category of Josephson effects in momentum space, which can be built in a spin-orbit coupled BEC. In addition to traditional voltage-driven Josephson effects, we introduce quenching of the tunneling phase as a novel driving mechanism. Our work may motivate further experimental and theoretical works for studying MSJJ and provides a platform for exploring their applications in building novel quantum mechanical circuits.

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*chuanwei.zhang@utdallas.edu


See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.120.120401 for derivation of system Hamiltonian, 2-mode model, voltage-driven MSJJ, and numerical recipes for GPE simulations, which includes Ref. [48].


