All pairs SPs for sparse graphs $\mapsto$ Johnson's algorithm

- Floyd-Warshall algorithm $\mapsto O(V^3)$
  - dynamic programming

  $\downarrow$ $V \cdot E$ if $E = \Theta(V$

- Run a SSSP algorithm for each vertex

  (a) Dijkstra (≥ 0 weights) $\Rightarrow O(V(V \log V + E))$

  $= O(V^2 \log V + VE)$

  (b) Bellman-Ford (no restriction on weights)

  $\Rightarrow O(V \cdot (V \cdot E)) = O(V^2 \cdot E)$

- Can do better for general weights, sparse G
  - Johnson's algorithm

    (a) reweight the graph so $w \mapsto \hat{w}$, with $\hat{w} \geq 0$

    (b) run Dijkstra's algorithm V times

    $\Rightarrow O(T_r + V^2 \log V + VE)$

Reweighting G

(1) For all pairs $u, v \in V$, a path $u \overset{p}{\rightsquigarrow} v$ is shortest using $w()$ iff is shortest using $\hat{w}()$.

(2) For all $(u,v) \in E$, $\hat{w}(u,v) \geq 0$.

(1) Preserving SPs by reweighting

Let $h : V \mapsto \mathbb{R}$. For each $(u,v) \in E$, define

$$\hat{w}(u,v) = w(u,v) + h(u) - h(v)$$

Let p be any path from v_0 to v_K

$$p = \langle v_0, v_1, \ldots, v_K \rangle$$

Then $w(p) = \delta(v_0, v_K) \iff \hat{w}(p) = \hat{\delta}(v_0, v_K)$

Also, G has negative weight cycle with $w()$ iff it has with $\hat{w}()$.

<u>Proof:</u> $\hat{w}(p) = \sum_{i=1}^{K} \hat{w}(v_{i-1}, v_i) = \sum_{i=1}^{K} (w(v_{i-1}, v_i) + h(v_{i-1}) -$

$$= \sum_{i=1}^{K} w(v_{i-1}, v_i) \cancel{h(v_0) + h(v_K} + \underline{h(v_0) - h(v_K)}$$

Cycle c, $w(c) < 0$: $c = \langle v_0, v_1, \ldots, v_K \rangle$, $v_0 = v$

$$\hat{w}(c) = w(c) + h(v_0) - h(v_K) = w(c)$$

$$\Rightarrow \hat{w}(c) < 0.$$

(2) Want $\hat{w}(u,v) \geq 0$ for all $(u,v) \in E$

Let $G' = (V', E')$ with $V' = V \cup \{s\}$,

$E' = E \cup \{(s,v) : v \in V\}$

Set $w(s,v) = 0$, for all $v \in V$.

Assign $h(v) = \delta(s,v)$, for all $v \in V'$ (V)

$\Rightarrow h(v) \leq h(u) + w(u,v)$ for all $(u,v) \in E'$ $(E,$

$$\Rightarrow \hat{w}(u,v) = w(u,v) + \delta(s,u) - \delta(s,v) =$$

$$= w(u,v) + h(u) - h(v) \geq 0$$

Can find $\delta(s,v)$ for all $v \in V'$ using Bellman-Ford $\Rightarrow O(VE)$

$\Rightarrow T_r = O(VE)$

$\Rightarrow T = O(V^2 \log V + VE)$