CS 6363 → Fall '02 → Chap 24.4 (25.5)

Difference constraints & Shortest Paths

- Linear Programming: optimize a linear function subject to a set of linear constraints
  - recall LP, 2-D LP ↦ $O(n)$ time
  - introduce SOLF, 2-D SOLF ↦ $O((m+n)\log(m+n)+m^2)$ time.

- Special cases of LP:
  - 2-D LP
  - special case that can be reduced to finding SPs from single source

Linear programming:

- General form:
  - Given an $m \times n$ matrix A,
    m vector b
    n vector c
  - Want vector x of n elements that maximizes objective function $\underbrace{\sum_{i=1}^{n} c_i x_i}_{cx}$ subject to the m constraints given by $Ax \le b$

- Simplex (not polynomial)
- Polynomial time solutions known.
- Knowing that a problem can be cast as a polynomial sized LP ⇒ poly time algorithm for the problem.

- Many special cases of LP for which faster solutions exist.

  E.g.: SSSP ↦ special case of LP

  Other problems that can be cast as LP:
  - single pair shortest-path
  - maximum flow

- Sometimes we don't care about objective function
  ↦ want feasible solution → any!
  ↳ vector x s.t. $Ax \le b$
  or determine if there is no feasible solution

- Focus on feasibility problem
  ↓
  Systems of difference constraints

- each row of the LP matrix A contains one 1, one −1, rest are 0 ⇒ $Ax \le b$: set of m difference constr. with n unknown

  $x_j - x_i \le b_k$, $1 \le i, j \le n$, $1 \le k \le m$

E.g.: find $X = (x_1, x_2, x_3, x_4, x_5)$ s.t.

~~1 −1 0 0 0~~

→ take from book!

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \le \begin{pmatrix} 0 \\ -1 \\ 1 \\ 5 \\ 4 \\ -1 \\ -3 \\ -3 \end{pmatrix}$$

⟺ find unknown x_i, $i = 1, \dots, 5$ s.t. following DCs satisfied:

$$x_1 - x_2 \le 0$$
$$x_1 - x_5 \le -1$$
$$x_2 - x_5 \le 1$$
$$x_3 - x_1 \le 5$$
$$x_4 - x_1 \le 4$$
$$x_4 - x_3 \le -1$$
$$x_5 - x_3 \le -3$$
$$x_5 - x_4 \le -3$$

$\xmapsto{\text{sol.}}$ $X = (-5, -3, 0, -1, -4)$

$X' = (0, 2, 5, 4, 1)$

$(x'^{j}_{i} = x^{j}_{i} + 5)$

Lemma: Let $x = (x_1, x_2, \ldots, x_n)$ be a sol. of $Ax \le b$ of difference constraints, and let d be any constant. Then $x + d = (x_1 + d, x_2 + d, \ldots, x_n + d)$ is a sol. of $Ax \le$

Proof: .... (simple)

Applications: x_i → starting / ending time of a job

- product assembly : start { no latter than / no earlier than

Constraint Graphs

- interpret systems of difference constraints in a graph-theoretic framework.
  - A can be viewed as transpose of an incidence matrix for a graph with n vertices and m edges.

    ($m \times n \not\mapsto n \times m$) → No!

    - each $v_i \in G(V, E)$, $i = \overline{1, n}$ ↦ one of the n unknown, x_i
    - each directed edge ↦ one of the m inequalities
- Given $Ax \le b$ ↦ $G = (V, E)$ is a weighted, directed graph

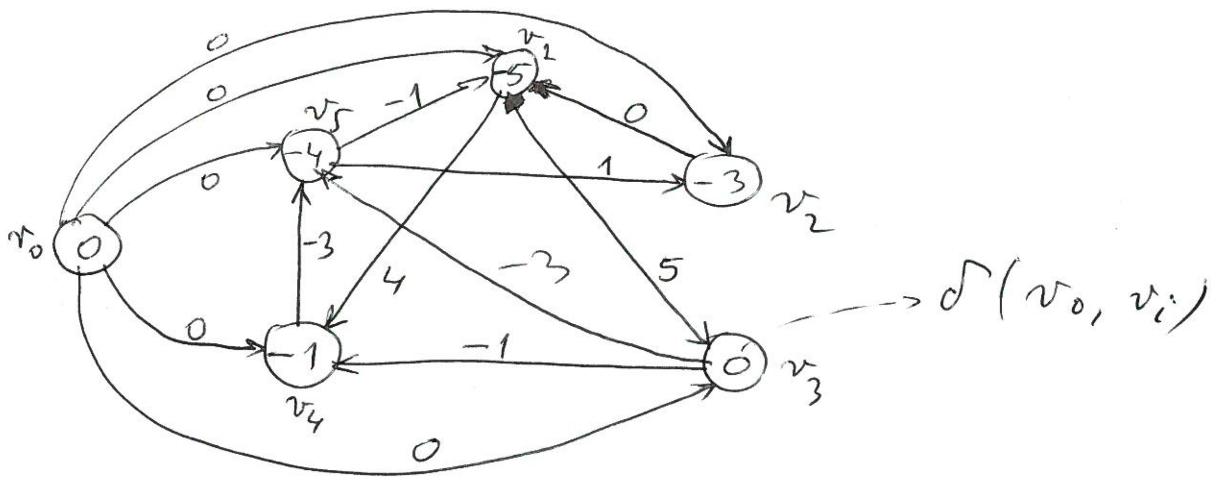
  → constraint graph :

  $V = \{v_0, v_1, \ldots, v_n\}$

  $E = \{(v_i, v_j) : x_j - x_i \le b_k \text{ is a constraint}\} \cup \{(v_0, v_1), \ldots, (v_0, v_n)\}$

Vertex v_0 is used to assure every v_i, $i=\overline{1,n}$ can be reached from it.

- if (v_i, v_j) corresponds to $x_j - x_i \le b_k$, then $\boxed{w(v_i, v_j) = b_k}$
- the weights for (v_0, v_i), $i=\overline{1,n}$ are 0.



- We can find a sol. to a system of DC by finding SP weights in constraint graph (CG)

Th: Given system $Ax \le b$ of DC, let $G=(V,E)$ be the CG.

If G has no negative-weight cycles then

$x = (\delta(v_0,v_1), \delta(v_0,v_2), \dots, \delta(v_0,v_n))$ is a feasible solution for the system

If G has a negative-weight cycle then no feasible sol

Proof: Consider no negative-weight cycle.

Take any edge $(v_i, v_j) \in E$.

By triangle inequality: $\delta(v_0, v_j) \le \delta(v_0, v_i) + w(v_i v_j)$

- let $x_i = \delta(v_0, v_i)$, $x_j = \delta(v_0, v_j)$

$\Rightarrow x_j - x_i \le w(v_i v_j)$

Consider negative-weight cycle c exits in G:

WLOG, $c = \langle v_1, v_2, \ldots, v_k \rangle$, $v_1 = v_k$

(v_0 cannot be on c since it has no incomming edge).

$\Downarrow$

$$x_2 - x_1 \le w(v_1 v_2)$$
$$x_3 - x_2 \le w(v_2, v_3)$$
$$\vdots$$
$$x_k - x_{k-1} \le w(v_{k-1}, v_k)$$
$$\cancel{x_1 - x_k \le w(v_k, v_1)}$$

If sol. for x ~~that~~ satisfying all inequalit

$\Downarrow$

Σ is also satisfi

$$\Sigma \quad 0 \le w(c)$$

but c is negative-weight cycle $\rightarrow w(c) < 0$

$\Rightarrow 0 \le w(c) < 0$ impossible (contradiction

Solving systems of DC

- use Bellman-Ford algorithm
  - any negative-weight cycle is reachable from v_0
    (edges $(v_0, v_i) \in G$, $i = \overline{1, n}$)
  - if B-F returns TRUE, SP weights give feasible sol
  - if B-F — " — FALSE, no feasible sol.

- DC system with $\left.\begin{array}{l} m \text{ constraints} \\ n \text{ unknowns} \end{array}\right\} \to$ graph $\left\{\begin{array}{l} n+1 \text{ vertices} \\ n+m \text{ edges} \end{array}\right.$

  $\Rightarrow O(n^2 + nm)$ time