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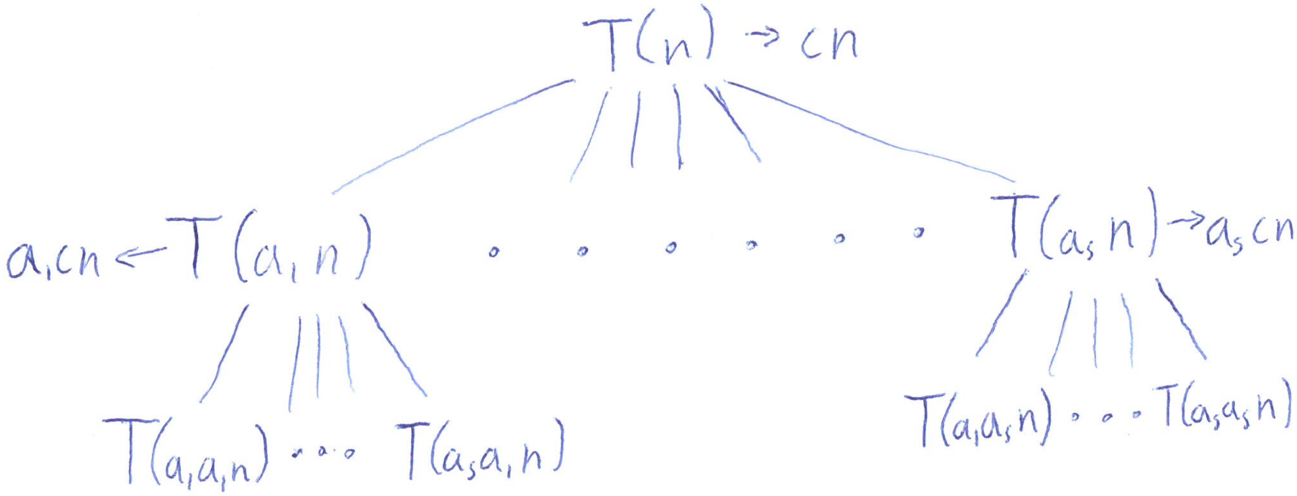
Let $T(n) = T(a_1 n) + T(a_2 n) + \dots + T(a_s n) + cn$

$$= cn + \sum_{i=1}^{s} T(a_i n)$$

with $0 < a_1 \le a_2 \le \dots \le a_s < 1$

and $\sum_{i=1}^{s} a_i < 1$ (Let us call $\sum_{i=1}^{s} a_i = A$, so $A < 1$)

Recursive Tree Diagram:



Let $k_1, k_2, \dots, k_z$ be the input sizes to $T()$ for some level of the recursion tree.
The cost incurred by this level is $\sum_{i=1}^{z} k_i \cdot c = K$
Let us also assume that no base case is reached in this level or the next so that we obtain an upperbound of the cost incurred by the next level.
The input sizes of the next level are thus:

$$a_1 k_1, a_2 k_1, \dots, a_s k_1, a_1 k_2, \dots a_s k_2, \dots, a_s k_z$$

Thus the total cost of the level is:

$$\sum_{i=1}^{z} \sum_{j=1}^{s} a_j k_i \cdot c = \sum_{i=1}^{z} (k_i \cdot c) \sum_{j=1}^{s} a_j = KA$$

The result is that if the cost incurred by some level is K, the max cost incurred by the following level is AK

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For a finite input size n we should expect a finite tree height h

$$T(n) = cn + \sum_{i=1}^{s} T(a_i n) \leq \sum_{i=0}^{h} cnA^i \leq \sum_{i=0}^{\infty} cnA^i = cn \sum_{i=0}^{\infty} A^i = \frac{cn}{1-A}$$

$$\Rightarrow T(n) = O(n) \checkmark$$