NP Completeness

Classes P / NP

Decision problems only (Yes/No)

P	NP
Polynomial time (Model: Deterministic Turing Machine)	Nondeterministic Poly time (Model: Nondeterm TM)

NP

Problem, Input $= w \in \Sigma^*$
Yes/No? If an addition "guess" g is supplied, then in polynomial time, we can verify that w is an Yes instance of problem.

Yes	No
$\exists$ guess g such that verification algorithm outputs "Yes" in determ poly time	$\forall g$, verification alg always outputs "No"

Equivalence between Decision / Optimization problems

(with respect to polynomial time)

1. Clique problem: Given $G=(V,E)$, find a clique of max cardinality. [$S \subseteq V$ is a clique if the induced subgraph of S is a complete graph]

Decision (clique): Given $G=(V,E)$, integer k, does G have a clique of cardinality k?

If clique (decision) has a polynomial time algorithm, ($\in P$) then we can find a max clique in poly time.

$k \leftarrow 1$ while $\langle G, k \rangle$ is ~~an~~ Yes instance of clique do $k{+}{+}$

for $u \in V$ do
    if $\langle G - \{u\}, k \rangle$ is a Yes instance of clique do
        $G \leftarrow G - \{u\}$

Output G

2. <u>Subset Sum problem:</u> (Knapsack) $\left[\begin{array}{l} x_i > 0 \quad i=1..n \\ t > 0 \end{array}\right.$

→ Given $x_1, x_2 \ldots x_n$ and a target sum t,

find a subset of $x_1 \ldots x_n$ whose sum is as close as possible to t, without going over t.

<u>Decision:</u> Given $x_1 \ldots x_n$, t, Is there a subset whose sum is t

if $t > x_1 + x_2 + \ldots + x_n$ then $t \leftarrow x_1 + x_2 + \ldots + x_n$

while $\langle x_1 \ldots x_n, t \rangle$ is a No Instance of Subset sum do

$t--$

~~Until $t = 0 \longrightarrow$ No such subset~~

$X \leftarrow \{x_1 \ldots x_n\}$

for $i \leftarrow 1$ to n do

Remove x_i from X

If X is ~~still~~ a No Instance of Subset Sum

then Put x_i back into X.

Output X, t

Some NP Completeness proofs

Known NP-complete problems: Satisfiability, Clique, Subset Sum

New problem: Subgraph isomorphism

$\langle G_1, G_2 \rangle$: Is there a subgraph of G_2 that is isomorphic to G_1?

claim: Subgraph isomorphism (SI) is NP-complete.

Proof: (i) SI $\in$ NP. Given a guess: mapping from $G_1 \to G_2$, we can verify in poly time that this mapping is ~~given~~ is correct

$m: V_1 \to V_2$

[verify that m is 1 to 1 and also for each $(u,v) \in E_1$, $(m(u), m(v)) \in E_2$]

(ii) ~~All problems in NP~~ Known NP-complete problem can be reduced to S.I.

Clique $\leq_p$ S.I.

Consider an Input $\langle G, k \rangle$ to the Clique problem.

Construct an instance of SI:

$$G_1 = K_k = \text{~~clique~~ complete graph on } k \text{ nodes.}$$

(k-clique)

$$G_2 = G$$

S.I. instance: $\langle G_1, G_2 \rangle$.

($\Rightarrow$) If G has a clique of size k, then $\langle G_1, G_2 \rangle$ is a Yes instance of S.I. A k-clique in $G = G_2$ is isomorphic to G_1. $\Rightarrow$ Yes instance of S.I.

($\Leftarrow$) If G does not not have a clique of size k (No Instance) then $\langle G_1, G_2 \rangle$ is an No Instance of S.I.

Proof by contradiction: Suppose ~~not~~. $\langle G_1, G_2 \rangle$ is a Yes instance of S.I.

There is a subgraph S of G_2 isomorphic to G_1

Since G_1 is a k-clique, S is a k-clique of $G_2 = G$

– contradicts the claim that $\langle G, k \rangle$ is a no instance of clique

Partition is NP-complete

Partition: Given $X = \{x_1 \ldots x_n\}$, can X be partitioned into A and B ($A \cup B = X$, $A \cap B = \emptyset$) such that

$$\sum_{x_i \in A} x_i = \sum_{x_i \in B} x_i \; ?$$

(i) Partition $\in NP$. Guess A and B - Verify that $A \cup B = X$, $A \cap B = \emptyset$, $\sum_{x_i \in A} x_i = \sum_{x_i \in B} x_i$.

(ii) Subset sum $\leq_p$ Partition.

$$\left.\begin{array}{l} x_1 \ldots x_n, t \\ \text{Is there a subset of} \\ x_1 \ldots x_n \text{, whose} \\ \text{sum is exactly } t \end{array}\right\} \text{Input instance of S.S.}$$

Let $x = \sum_{i=1}^{n} x_i$

If $t \leq \frac{x}{2}$, $x_{n+1} = x - 2t$

Otherwise $x_{n+1} = 2t - x$

$x_1 \ldots x_n, x_{n+1}$ is a Yes Instance of Partition

$\Longleftrightarrow$ $x_1 \ldots x_n, t$ is a Yes Instance of S.S.