8.4: Uniform Convergence of Series; Power Series.

Defn: The series $s(x) = \Sigma_{k=1}^{\infty} u_k(x)$ **converges uniformly to the function** s if the functions $s_n(x) = \Sigma_{k=1}^{n} u_k(x)$ converges uniformly to the function s

Thm 8.18 (8.12). Suppose the $u_n : A \to Y$, are continuous functions where A, Y are metric space. If $s(x) = \Sigma_{k=1}^{\infty} u_k(x)$ converges uniformly, then s is continuous.

Thm 8.19 (8.13): If u_n are integrable functions on $[a, b]$ and if $s(x) = \Sigma_{k=1}^{\infty} u_k(x)$ converges uniformly, then $\int_a^b \Sigma_{k=1}^{\infty} u_k(x) = \Sigma_{k=1}^{\infty} \int_a^b u_k(x)$

Thm 8.20 (8.14): Suppose that

(1) $u_n : R \to R$ is differentiable *and that the derivatives of* u_n *are continuous*,

(2) $s_n(x) = \Sigma_{k=1}^{n} u_k(x)$ converges pointwise to $s(x) = \Sigma_{k=1}^{\infty} u_k(x)$ *for all* x, and

(3) $u_k'(x)$ converges uniformly.

Then $s'(x) = [\Sigma_{k=1}^{\infty} u_k(x)]' = \Sigma_{k=1}^{\infty} u_k'(x)$

Thm 8.21 (Weierstrass M-test): Suppose that

(1) $u_n : A \to R$, where A is a metric space.

(2) $|u_n(x)| \leq M_n$ for all n and for all $x \in A$.

(3) $\Sigma_{n=1}^{\infty} M_n$ converges.

Then $\Sigma_{k=1}^{\infty} u_k(x)$ and $\Sigma_{k=1}^{\infty} |u_k(x)|$ converge uniformly on A.

Theorem 8.22 (8.11):
Given the power series $\Sigma_{n=0}^{\infty} c_n (x-a)^n$,

let $\alpha = \overline{lim}_{n\to\infty} |c_n|^{\frac{1}{n}}$, and $R = \begin{cases} 0 & \alpha = +\infty \\ \frac{1}{\alpha} & \alpha \in (0, +\infty) \\ +\infty & \alpha = 0 \end{cases}$.

Then

(1) if $\alpha = +\infty$, then the series converges absolutely only for $x = a$ and diverges for all other values of x.

(2.) if $\alpha = 0$, the series converges absolutely for all x and the series converges uniformly on any closed interval $[c, d] \subset \mathcal{R}^1$.

(3) Else, the series converges absolutely for all $|x - a| < R$ and diverges for all $|x - a| > R$. The series converges uniformly on any closed interval $[c, d] \subset (a - R, a + R)$.

Thm 8.23 and 8.24ii: Suppose that the series $f(x) = \Sigma_{n=0}^{\infty} c_n (x-a)^n$, converges for $|x-a| < R$ with $R \in (0, \infty)$. Then

(i) f and f' are continuous on $|x-a| < R$,

(ii) $f'(x) = \Sigma_{n=0}^{\infty} n c_n (x-a)^{n-1}$ for $|x-a| < R$ and

(iii) $\int_a^x f(x)dx = \Sigma_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$ for $|x-a| < R$

Thm 8.24:
Suppose that the series $f(x) = \Sigma_{n=0}^{\infty} c_n (x-a)^n$, converges for $|x-a| < R$ with $R \in (0, \infty)$.

(i) Then f possesses derivatives of all orders. For each positive integer m, the derivative $f^{(m)}(x)$ is given for $|x-a| < R$ by the term-by-term differentiation of $f(x) = \Sigma_{n=0}^{\infty} c_n (x-a)^n$ m times.

(iii) $c_n = \frac{f^{(n)}(a)}{n!}$

Thm 8.25 (Taylor's theorem) Suppose $f : [c, d] \to R$,
(1) $f^{(n)}$ is continuous on $[c, d]$ for all $n = 0, 1, 2, ...$,
(2) $f^{(n+1)(x)}$ exists for all $x \in (c, d)$., and
(3) $a, b \in [c, d], a \neq b$.

$$\text{Define } P(x) = \Sigma_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j.$$

Then there exists a point ξ between a and b such that

$$f(b) = P(b) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}.$$