$(1+\rho)$-Approximation for Selected-Internal Steiner Minimum Tree*

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Abstract. Selected-internal Steiner minimum tree problem is a generalization of original Steiner minimum tree problem. Given a weighted complete graph $G = (V, E)$ with weight function c, and two subsets $R^{'} \subsetneq R \subseteq V$ with $|R - R^{'}| \geq 2$, selected-internal Steiner minimum tree problem is to find a Steiner minimum tree T of G spanning R such that any leaf of T does not belong to $R^{'}$. In this paper, suppose c is metric, we obtain a $(1+\rho)$-approximation algorithm for this problem, where ρ is the best-known approximation ratio for the Steiner minimum tree problem.

Keywords: Selected-internal Steiner tree, Approximation algorithm

1 Introduction

Given a weighted complete graph $G = (V, E)$ with weight function c, and a subset R, Steiner minimum tree problem is to find a minimum subtree of G spanning R. Steiner tree can be applied in many fields such as VLSI routing [11], network routing [12], phylogeny [5, 10], et.al [2, 4, 8]. In the past years, some generalizations of Steiner minimum tree problem are arisen, such as Steiner minimum tree problem on some special metric space [6, 7], the full Steiner tree problem [3, 13] and the k-size Steiner tree problem [1].

In this paper, we study selected-internal Steiner minimum tree problem, also a generalization of Steiner minimum tree problem. Given a weighted complete graph $G = (V, E)$ with weight function c, and two subsets $R^{'} \subsetneq R \subseteq V$ with $|R - R^{'}| \geq 2$, selected-internal Steiner minimum tree problem is to find a Steiner minimum tree T of G spanning R such that any leaf of T does not belong to $R^{'}$. Since Steiner minimum tree problem is a special case ($R^{'} = \emptyset$) of this problem, the NP-completeness and MAX SNP-hardness of this problem can immediately follow from the hardness results of Steiner minimum tree problem [9]. Hsieh and

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Yang [9] gave a 2ρ-approximation algorithm for this problem when c is metric, where $\rho = 1 + \frac{\ln 3}{2} \approx 1.55$ is the best-known approximation ratio for Steiner minimum tree problem [14].

In this paper, we present a $(1+\rho)$-approximation algorithm for this problem when c is metric. Given a complete graph $G = (V, E)$ with weight function c on its edges and a set $R \subseteq V$, let T be a Steiner tree of G spanning R. We call an edge as an *in-edge* if both its endpoints belong to R, otherwise call it an *out-edge*. Denote E_{in}^T and E_{out}^T as the sets of the in-edges and out-edges of T, respectively. For two subsets V_1 and V_2 of $V(G)$, denote $dist(V_1, V_2) = \min_{v_1 \in V_1, v_2 \in V_2} c(v_1 v_2)$ as the distance between V_1 and V_2.

The rest of this paper is organized as follows. In section 2, we present a $(1+\rho)$-approximation algorithm for selected-internal Steiner minimum tree problem by 3 subsections. In subsection 2.1, we firstly give Algorithm 1 to divide R to a pairwise disjoint tree sequence $\mathcal{T} = \{T_1, \ldots T_m\}$. Then, we contract every T_i to v_{T_i} to obtain a new graph G_1 and set $R_1 = \{v_{T_i}\}_{i=1}^m$. Based on Algorithm 1, we present Algorithm 2 to construct a Steiner tree $T^{'}$ of G_1 spanning R_1 with approximation ratio ρ and a Steiner tree T of G spanning R such that there is a Steiner minimum tree T_{opt} of G spanning R, satisfying

$$E(T) = E(T^{'}) \bigcup E(\mathcal{T}),\ E(\mathcal{T}) \subseteq E_{in}^T \text{ and } c(E(T) \setminus E(\mathcal{T})) \leq \rho\, c(E(T_{opt}) \setminus E(\mathcal{T})). \quad (1)$$

In subsection 2.2, we modify $T^{'}$ to a new Steiner Tree $T^{''}$ of G_1 spanning R_1 with approximation ratio $(1+\rho)$. In $T^{''}$, every v_{T_i} has degree no less than 2 unless T_i has vertex belonging to $R - R^{'}$. In final subsection, we"blossom" very v_{T_i} of $T^{''}$ to T_i and modify it to a selected-internal steiner tree $T^{'''}$ of G spanning R with approximation ratio $(1+\rho)$. Finally, we conclude our results and discuss a special case when $R = V$, which is the selected-internal minimum spanning tree problem.

2 $(1+\rho)$-approximation algorithm

In this section, we will give a $(1+\rho)$-approximation algorithm for selected-internal Steiner minimum tree problem in the following three steps.

2.1 Construction of $T^{'}$ and T

In this subsection, we first give Algorithm 1 to divide R to a disjoint tree sequence $\mathcal{T} = \{T_1, \ldots T_m\}$. Then, we contract every T_i to v_{T_i} to obtain a new graph G_1 and set $R_1 = \{v_{T_i}\}_{i=1}^m$. Based on Algorithm 1, we present Algorithm 2 to construct a Steiner tree $T^{'}$ of G_1 spanning R_1 with approximation ratio ρ and a Steiner tree T of G spanning R satisfying (1).

Algorithm 1
Input: A weighted complete graph $G = (V, E)$ with weight function c and a set $R \subseteq V$.

Output: A sequence of pairwise disjoint subtrees $\mathcal{T} = \{T_1, T_2, \ldots\}$ such that $\bigcup V(T_i) = R$.
1. Set $S = R$, $i = 1$ and $\mathcal{T} = \emptyset$. /*S records the vertices in R not covered by $\mathcal{T}$ */
2. **If** $S = \emptyset$, output $\mathcal{T}$, stop;
Else Choosing a vertex v of S and creating a new tree T_i, set $V(T_i) := \{v\}$, $E(T_i) := \emptyset$ and $S := S - \{v\}$.
3. Search $V - V(T_i)$ with the vertex ordering $S, R-S-V(T_i), V-R$, finding the first vertex u such that $dist(u, T_i) = \min\limits_{w \in V(G-T_i)} dist(w, T_i)$. Let the corresponding edge be ut.
4. **If** $u \in S$, set $V(T_i) := V(T_i) \bigcup \{u\}$, $E(T_i) := E(T_i) \bigcup \{ut\}$ and $S := S - \{u\}$. Goto 3.
If $u \in R - S - V(T_i)$, then u is contained in some T_j with $1 \leq j < i$. Set $T_j := T_j \bigcup T_i + ut$ and delete T_i. Goto 2.
If $u \in V - R$ or $u =$NULL, set $\mathcal{T} := \mathcal{T} \bigcup T_i$ and $i := i + 1$. Goto 2.

At the step 4, if $u \in V - R$, then a new tree T_i will be included in $\mathcal{T}$ and holds the following property: there is a vertex $w \in V - R$ such that

$$dist(w, T_i) < dist(v, T_i) \text{ for any } v \in R - V(T_i). \tag{2}$$

We claim that when the second case of step 4 ($u \in R - S - V(T_i)$) is true, if the old tree T_j follows property (2), the modified tree $T_j^{'} := T_j \bigcup T_i \bigcup \{e^{'}\}$ also follows, where $e^{'}$ is a shortest edge between $V(T_i)$ and $V - V(T_i)$ and one of its endpoint is in T_j. For this purpose, since T_j has property (2), there is $w \in V - R$ such that $dist(w, T_j) < dist(v, T_j)$ for any other $v \in R - T_j$. Denote the corresponding edge as e. It implies that $c(e) < c(e^{'})$. For any edge $e^{''}$ between $V(T_i)$ and $R - V(T_j^{'})$, $c(e^{''}) \geq c(e^{'}) > c(e)$. Hence, the modified tree $T_j^{'}$ also has the property (2).

By recurrence, we obtain the following result.

Lemma 1 *Let $T_1, T_2, \ldots, T_m$ be the output tree sequence of Algorithm 1. If $R \neq V$, for any T_i, there is a vertex $w \in V - R$ such that $dist(w, T_i) < dist(v, T_i)$ for any $v \in R - V(T_i)$.*

Lemma 2 *For a weighted complete graph $G = (V, E)$ with weight function c and a set $R \subseteq V$, let $T_1, T_2, \ldots, T_m$ be the output of Algorithm 1. Then there exists a Steiner minimum tree T of G spanning R such that every T_i is a subtree of T.*

Proof. Firstly, order edges of $E^{'} = \bigcup\limits_{i=1}^{m} E(T_i)$ as $\{e_1, e_2, \ldots\}$, according to their order of appearance in Algorithm 1. Among all Steiner minimum trees of G spanning R, choose T satisfying the following conditions: (1) $|E(T) \bigcap E^{'}|$ as large as possible; (2) under the condition (1), choosing T such that the index of the first edge in $E^{'} \setminus E(T)$ as large as possible. We shall show that T contains $T_1, \ldots, T_m$ as subtrees.

Suppose to the contrary that the first edge in $E^{'} \setminus E(T)$ is $e_j = uv$. By Algorithm 1, there exists some stage at which there is a subtree $T^{'}$ (which is a subtree of some T_i) such that e_j is a shortest edge between $V(T^{'})$ and $V - V(T^{'})$. Since T is a tree, there is a unique path P between u and v on T . Let e be the unique edge on P between $V(T^{'})$ and $V - V(T^{'})$. Then $c(e_j) \leq c(e)$ and $\widetilde{T} = T + e_j - e$ is also a Steiner tree. Since T is a Steiner minimum tree, we have that $c(e_j) = c(e)$ and $c(T^{''}) = c(T)$. By the choice of T, e must be in $E^{'}$, otherwise $|E(\widetilde{T}) \bigcap E^{'}| > |E(T) \bigcap E^{'}|$. Let $e = e_t$. By the structure of $T^{'}$, we have $t > j$. But then, the index of the first edge in $E^{'} \setminus E(\widetilde{T})$ is larger than that of $E^{'} \setminus E(T)$, contradicting to condition (2). So, T contains all edges of $T_1, \ldots, T_m$. □

Now, we construct a new graph G_1 by contracting every T_i to a new vertex, denoted it by v_{T_i}. Let $R_1 = \{v_{T_1}, \ldots, v_{T_m}\}$. By Lemma 2, we obtain the following corollary.

Corollary 3 *For any Steiner minimum tree T_{R_1} of G_1 spanning R_1, if we "blossom" every v_{T_i} to T_i, then the new tree T_R is a Steiner minimum tree of G spanning R.*

Proof. Suppose to the contrary that the new tree T_R is not a Steiner minimum tree of G spanning R. By Lemma 2, there is a Steiner minimum tree $T^{'}_R$ spanning R such that every T_i is a subtree of $T^{'}_R$ and $c(T^{'}_R) < c(T_R)$. Contract every T_i to a new vertex in $T^{'}_R$. The resulting tree is a Steiner tree of G_1 spanning R_1 which has less weight than T_{R_1}, a contradiction. □

Based on Algorithm 1, we present the following algorithm which constructs two Steiner trees $T^{'}$ and T. By Corollary 3, there exists a Steiner minimum tree T_{opt} such that T satisfying (1).

Algorithm 2
Input: A weighted complete graph G with weight function c on $E(G)$ and a set $R \subseteq V$.
Output: Two Steiner trees $T^{'}$ and T.
1. Use Algorithm 1 to obtain $T_1, \ldots, T_m$.
2. Contract T_i to v_{T_i} $(i = 1, \ldots, m)$ to construct G_1. Set $R_1 = \{v_{T_i}\}_{i=1}^{m}$.
3. Use a $\rho-$approximation algorithm to obtain a Steiner tree $T^{'}$ of G_1 spanning R_1.
4. "Blossom" v_{T_i} to T_i $(i = 1, \ldots, m)$ to form a Steiner tree T of G spanning R.

2.2 Modification of $T^{'}$

In subsection 2.1, we obtain a Steiner Tree $T^{'}$ of G_1 spanning R_1 with approximation ratio ρ to the Steiner minimum Tree of G_1 spanning R_1. In this subsection we modify $T^{'}$ to a new Steiner tree $T^{''}$ with approximation ratio $(1 + \rho)$. In $T^{''}$, every v_{T_i} has degree no less than 2 unless T_i has vertex belonging to $R - R^{'}$. In the following study, we assume that the weight function c is *metric*.

If $m = 1$, obviously v_{T_1} is the Steiner minimum tree $T^{'}$ of G_1 spanning R_1. Set $T^{''} = T^{'}$. If $m = 2$, let e be the edge connecting v_{T_1} and v_{T_2}. Then $T^{'} = (\{v_{T_1}, v_{T_2}\}, \{e\})$ is a Steiner minimum tree. Since $|R - R^{'}| \geq 2$, one of T_1 and T_2, say T_1 should contain some vertex of $R - R^{'}$. By Lemma 1, there is a vertex $w \in V - R$ such that $c(wv_{T_2}) = dist(w, T_2) < dist(T_1, T_2) = c(e)$. Set $T^{''} = T^{'} + \{wv_{T_2}\}$. Then

$$c(T^{''}) = c(T^{'}) + c(w, v_{T_2}) < 2c(e) = 2c(T^{'}).$$

In the following, suppose $m \geq 3$. Firstly, we give some useful results.

Lemma 4 *Let G be a weighted complete graph on at least two vertices with metric weight function c and T a minimum spanning tree of G. Then, for any two vertices u and v, G has a hamilton cycle C such that edge $uv \in E(C)$ and $c(C) \leq 2c(T)$.*

Proof. We use induction on $|G|$. If $|G| = 2$, the digon through u and v is as desired. Suppose $|G| = k \geq 3$ and the assertion is true for all graphs with less than k vertices. If there is a leaf w different from u and v, let $e = ws$ be the unique edge incident with w and $\widetilde{T} = T - w$. Then $\widetilde{T}$ is a minimum spanning tree of $G[V(\widetilde{T})]$ and $\{u, v\} \subseteq V(\widetilde{T})$. By inductive hypothesis, there is a hamilton cycle $C^{'}$ of $G[V(\widetilde{T})]$ with $c(C^{'}) \leq 2c(\widetilde{T})$ and $uv \in E(C^{'})$. Let t be a neighbor of s on $C^{'}$ with $st \neq uv$. Set $C = C^{'} - st + sw + wt$. Then

$$\begin{aligned} c(C) &= c(C^{'}) - c(st) + c(sw) + c(wt) \\ &\leq 2c(\widetilde{T}) + 2c(ws) \\ &= 2c(T). \end{aligned}$$

If such a vertex w does not exists, then T is a hamiltion path from u and v, and $C = T + uv$ is as desired. □

As a consequence, we have

Corollary 5 *Let G be a weighted complete graph with metric weight function c and T a minimum spanning tree of G. For any two vertices u and v, there exists a hamilton path P from u to v such that $c(P) \leq 2c(T)$.*

In the following, we modify $T^{'}$ for the case $m \geq 3$. Let Q be the set of all leaves of $T^{'}$ in R_1. Then $Q \subseteq R_1$. If $|Q|$ is odd, we modify Q as follows. If $Q = R_1$, then there exists a vertex v_{T_i} such that T_i contains some vertex of $R - R^{'}$. In this case, set $Q := Q - v_{T_i}$. If $Q \neq R_1$, then there exists a vertex $v \in R_1 - Q$ and set $Q := Q \bigcup \{v\}$. Now, Q is a subset of R_1 and $|Q|$ is even. Find a minimum perfect matching M of Q in polynomial time. We claim that $c(M) \leq c(T_{opt})$, where T_{opt} is a Steiner minimum tree of G_1 spanning R_1. For this purpose, let C be a hamilton cycle as in Lemma 4. "Short cut" C to a cycle $C^{'}$ containing only vertices of Q. Since c is a metric weight function, we have $c(C^{'}) \leq c(C) \leq 2c(T_{opt})$. Then, one of the two perfect matchings of $C^{'}$, say M_1, satisfies $c(M_1) \leq c(T_{opt})$. The claim follows from $c(M) \leq c(M_1)$.

Now, add the edges of M one by one to modify $T^{'}$ such that every v_{T_i}, for which T_i contains only vertices in $R^{'}$, has degree at least 2.

Let e ba an edge in m. Since $T^{'}$ is a tree, $T^{'}+e$ has the unique cycle C. If C contains all vertices of $T^{'}$, then there exists a vertex $v \in R_1 \subseteq V(C)$ such that T_v contains some vertex of $R-R^{'}$. If some neighbor u of v on C with $u \in V-R$, let $T^{'} := T^{'}+e-uv$. Else, the both neighbors of v on C belong to R_1. Let u be one of them. By Lemma 1, there exists a vertex $w \in V(G_1)-R_1$ such that $c(wu) < c(uv)$. If w is not on C, let $T^{'} := T^{'}+e+uw-uv$. If $w \in V(C)$, let s and t be the neighbors of w on C and $T^{'} := T^{'}+e+uw-uv+st-sw-wt$. Then the new tree $T^{'}$ is a Steiner tree of G_1 on R_1 such that the degree of any vertex of R_1 is not less than 2 except v (but it does not matter since T_v contains some vertices in $R-R^{'}$), and the weight of new tree is less than the weight of the old tree plus $c(e)$ (by $c(uw) < c(uv)$ and the assumption that c is metric).

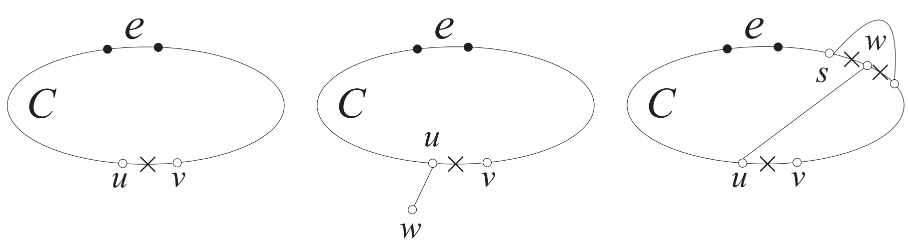

Figure 1. The three types of new tree

If C does not contain all vertices of $T^{'}$, there exists a vertex v of $V(C)$ such that $d_{T'}(v) \geq 3$. If some neighbor u of v on C has degree more than 2 or $u \in V(G_1)-R_1$, let $T^{'} := T^{'}+e-uv$. Otherwise, let u be a neighbor of v on C. Then $u \in R_1$ with degree 2 in $T^{'}$. By Lemma 1, there exists a vertex $w \in V(G_1)-R_1$ such that $c(wu) < c(uv)$. If $w \notin V(T^{'})$ or $w \in C$, then $T^{'}$ can be modified similarly to the above case. Hence, suppose $w \in V(T^{'}-C)$. In this case, the graph $H_1 = T^{'}+e+uw-uv$ has the unique cycle C_1. Clearly, $c(H_1) \leq c(T^{'})+c(e)$. If $d_{T'}(w) \geq 2$, let $s \neq u$ be the other neighbor of w on C_1, and t be another neighbor of w on $T^{'}$. Set $T^{'} := H_1 - sw - wt + st$. In all above case, the new tree $T^{'}$ is a Steiner tree satisfying the following conditions:

(1). all vertices of $V(C) \bigcap R_1$ have degree at least 2,

(2). the degree of any other vertex of R_1 does not alter, and

(3). the weight of the new tree is less than the weight of the old tree plus $c(e)$.

Next, we consider the final case $w \in V(T^{'}-C)$ with $d_{T'}(w) = 1$. Let s be the other neighbor of w on H_1. Set $H_2 := H_1 + us - w$. Then H_1 contains a Steiner tree of G_1 spanning R_1, and $c(H_2) \leq c(H_1) \leq c(T^{'})+c(e)$. Furthermore, H_2 has unique cycle C_2 and $|V(H_2)-R_1| < |V(H_1)-R_1|$. Repeat the above procedure. Since $|V(G_1)-R_1|$ is finite, there is a stage where we come across a case different from the final one, at which a modified tree $T^{'}$ satisfying conditions (1)-(3) is obtained.

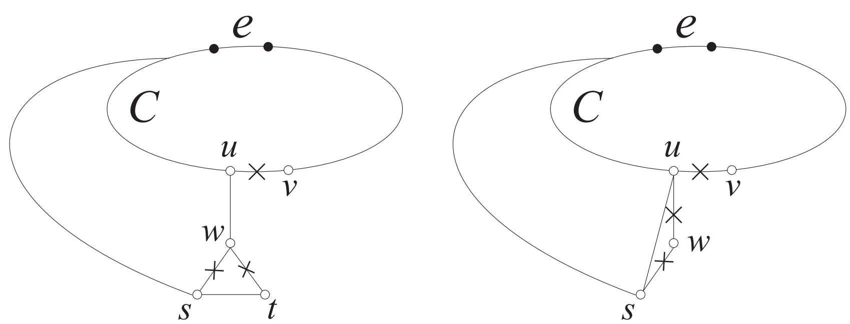

Figure 2. The two cases of $w \in V(T^{'} - C)$

After all edges of M having been added and all corresponding modifications having been made, let the final tree be $T^{''}$. Then $T^{''}$ is a Steiner tree of G_1 spanning R_1 such that v_{T_i} has degree 1 only when T_i contains vertex in $R - R^{'}$. Furthermore,

$$c(T^{''}) \leq c(T^{'}) + c(M) \leq (1+\rho)c(T_{opt}).$$

Lemma 6 *We can obtain a Steiner tree $T^{''}$ of G_1 spanning R_1 with approximation ratio $(1+\rho)$ in polynomial time. In $T^{''}$, every v_{T_i} has degree no less than 2 unless T_i contains vertex in $R - R^{'}$.*

2.3 "Blossom" every v_{T_i} of $T^{''}$ and modification

In this subsection, we "blossom" every v_{T_i} to T_i and modify it to a new steiner tree $T^{'''}$ of G spanning R such that every vertex of $R^{'}$ is not a leaf.

If $d_{T''}(v_{T_i}) \geq 2$ and there are at least two vertices u and v of T_i connected to vertices outside of T_i. By Corollary 5, there is a path P from u and v containing all vertices of T_i with $c(P) \leq 2c(T_i)$. Replace T_i by P. If $d_{T''}(v_{T_i}) \geq 2$ and there is only one vertex u of T_i connected to vertices outside of T_i. Choose two neighbors of u in $V - V(T_i)$, saying v and w. By Lemma 4, there is a cycle C containing all vertices of T_i with $c(C) \leq 2c(T_i)$. Let t be a neighbor of u on C. Replace T_i by $C - ut$ and uw by wt. If $d_{T''}(v_{T_i}) = 1$, then T_i has a vertex $u \in R - R^{'}$. Let v be the unique vertex connected to some vertex outside of T_i. If $u \neq v$, replace T_i by P, where P is a path from v to u containing all vertices of T_i with $c(P) \leq 2c(T_i)$. If $u = v$, let w be the unique neighbor of u in $V - V(T_i)$ and t another vertex of T_i. By Lemma 4, there is a cycle C containing all vertices of T_i with $c(C) \leq 2c(T_i)$ and $ut \in C$. Replace T_i by $C - vt$ and uw by wt.

After modifying all T_i as above, the resulting graph $T^{'''}$ is a Steiner tree of G spanning R such that every vertex of $R^{'}$ is not a leaf. By Corollary 3, let T_{R_1}

be a Steiner minimum tree of G_1 spanning R_1 and T_R be a Steiner minimum tree of G spanning R with $\bigcup_{i=1}^{m} E(T_i) \subseteq E(T_R)$. Then

$$\begin{aligned} c(T^{'''}) &\leq (1+\rho)c(T_{R_1}) + 2c(T_1) + \ldots + 2c(T_m) \\ &\leq (1+\rho)(c(T_{R_1}) + c(T_1) + \ldots + c(T_m)) \\ &\leq (1+\rho)c(T_R). \end{aligned}$$

Hence, we obtain the following theorem.

Theorem 7 *Given a weighted complete graph $G = (V, E)$ with metric weight function c on its edges. Let $R^{'} \subset R \subseteq V$ with $|R - R^{'}| \geq 2$. We can obtain a selected-internal Steiner tree with approximation ratio $(1+\rho)$ in polynomial time.*

3 Conclusion and discussion

In this paper, we present a $(1+\rho)$-approximation algorithm for selected-internal Steiner minimum tree problem. It would be interesting and challenging to design a better approximation algorithm or design a approximation algorithm for general wight function c.

Then, we discuss a special case when $R = V$, i.e., the selected-internal minimum spanning tree problem. This problem is NP-complete since when $|R - R^{'}| = 2$, this problem is equivalent to the minimum hamilton path problem. Similarly, by Corollary 5, we can obtain a 2-approximation algorithm for selected-internal minimum spanning tree problem.

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