Problem 6.1.4 Solution

We can solve this problem using Theorem 6.2 which says that

\[ \text{Var}[W] = \text{Var}[X] + \text{Var}[Y] + 2 \text{Cov}[X, Y] \quad (1) \]

The first two moments of \( X \) are

\[ E[X] = \int_0^1 \int_0^{1-x} 2x \, dy \, dx = \int_0^1 2x(1-x) \, dx = \frac{1}{3} \quad (2) \]
\[ E[X^2] = \int_0^1 \int_0^{1-x} 2x^2 \, dy \, dx = \int_0^1 2x^2(1-x) \, dx = \frac{1}{6} \quad (3) \]

Thus the variance of \( X \) is \( \text{Var}[X] = E[X^2] - (E[X])^2 = 1/18 \). By symmetry, it should be apparent that \( E[Y] = E[X] = 1/3 \) and \( \text{Var}[Y] = \text{Var}[X] = 1/18 \). To find the covariance, we first find the correlation

\[ E[XY] = \int_0^1 \int_0^{1-x} xy \, dy \, dx = \int_0^1 x(1-x)^2 \, dx = 1/12 \quad (5) \]

The covariance is

\[ \text{Cov}[X, Y] = E[XY] - E[X]E[Y] = 1/12 - (1/3)^2 = -1/36 \quad (6) \]

Finally, the variance of the sum \( W = X + Y \) is

\[ \text{Var}[W] = \text{Var}[X] + \text{Var}[Y] - 2 \text{Cov}[X, Y] = 2/18 - 2/36 = 1/18 \quad (7) \]

For this specific problem, it’s arguable whether it would be easier to find \( \text{Var}[W] \) by first deriving the CDF and PDF of \( W \). In particular, for \( 0 \leq w \leq 1 \),

\[ F_W(w) = P[X + Y \leq w] = \int_0^w \int_0^{w-x} 2 \, dy \, dx = \int_0^w 2(w-x) \, dx = w^2 \quad (8) \]

Hence, by taking the derivative of the CDF, the PDF of \( W \) is

\[ f_W(w) = \begin{cases} 2w & 0 \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (9) \]

From the PDF, the first and second moments of \( W \) are

\[ E[W] = \int_0^1 2w^2 \, dw = 2/3 \quad E[W^2] = \int_0^1 2w^3 \, dw = 1/2 \quad (10) \]

The variance of \( W \) is \( \text{Var}[W] = E[W^2] - (E[W])^2 = 1/18 \). Not surprisingly, we get the same answer both ways.
Problem 6.6.1 Solution

We know that the waiting time, \( W \) is uniformly distributed on \([0,10]\) and therefore has the following PDF.

\[
f_w(w) = \begin{cases} 
\frac{1}{10} & 0 \leq w \leq 10 \\
0 & \text{otherwise}
\end{cases}
\] (1)

We also know that the total time is 3 milliseconds plus the waiting time, that is \( X = W + 3 \).

(a) The expected value of \( X \) is \( E[X] = E[W + 3] = E[W] + 3 = 5 + 3 = 8 \).

(b) The variance of \( X \) is \( \text{Var}[X] = \text{Var}[W + 3] = \text{Var}[W] = \frac{25}{3} \).

(c) The expected value of \( A \) is \( E[A] = 12E[X] = 96 \).

(d) The standard deviation of \( A \) is \( \sigma_A = \sqrt{\text{Var}[A]} = \sqrt{12(\frac{25}{3})} = 10 \).

(e) \( P[A > 116] = 1 - \Phi(\frac{116-96}{10}) = 1 - \Phi(2) = 0.02275 \).

(f) \( P[A < 86] = \Phi(\frac{86-96}{10}) = \Phi(-1) = 1 - \Phi(1) = 0.1587 \)
Problem 6.6.2 Solution

Knowing that the probability that voice call occurs is 0.8 and the probability that a data call occurs is 0.2 we can define the random variable $D_i$ as the number of data calls in a single telephone call. It is obvious that for any $i$ there are only two possible values for $D_i$, namely 0 and 1. Furthermore for all $i$ the $D_i$'s are independent and identically distributed with the following PMF:

$$P_D (d) = \begin{cases} 
0.8 & d = 0 \\
0.2 & d = 1 \\
0 & \text{otherwise}
\end{cases} \quad (1)$$

From the above we can determine that

$$E[D] = 0.2 \quad \text{Var}[D] = 0.2 - 0.04 = 0.16 \quad (2)$$

With the previous descriptions, we can answer the following questions.