Working with Fields: A Pragmatic Perspective

by

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Foreword

This document follows closely the first few lectures I give in my electromagnetic courses here at UT Dallas: either our undergraduate Engineering Electromagnetics 4301 or our graduate course, Fields and Waves, 6316.

The approach to understanding and manipulating scalar fields and vector fields is presented here in a purposely unsophisticated manner that I hope will de-mystify the mathematics behind electromagnetic and thereby allow you to make decisions with confidence and to communicate your ideas with clarity.

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Scalar Fields and Vector Fields

In many of your electrical engineering experiences to date, you have looked at the temporal, or time varying, behavior of properties, or signals, of interest. For example, a voltage as a function of time may be written as $V(t)$, or a current as a function of time may be written as $I(t)$. These are examples of functions of one variable. In electromagnetics these (as well as many other quantities of interest) are also functions of space. Here we review how to handle functions of three dimensional space.

Once we define a convention consisting of a coordinate system and an origin, three numbers are needed to determine a point in space. For example, a longitude, a latitude, and an altitude will give the position of an airplane in flight. These three numbers may be three distances, two distances and an angle, one distance and two angles, or three angles. In this course we will mostly consider the simplest coordinate system – the Cartesian coordinate system – for keeping track of where a point in space is. A simple coordinate system is most appropriate for the study of complicated geometries which characterize many modern devices.

Thus, we may write for a voltage "field,"

$$V(t) \rightarrow V(x, y, z, t) \tag{1}$$

The fact that this quantity is now a function of three dimensional space makes this a *field*.[1] The fact that this quantity is a scalar as opposed to a vector, makes this a *scalar field*. To begin to visualize an example of a voltage field, think about constructing a large three dimensional circuit. In describing the operation of this system, it might prove convenient to keep track of the voltage at different (x, y, z) positions in space relative to some ground point (origin). I have seen a large room full of large suitcase-sized capacitors wired together for high voltage switching applications. In such a structure these capacitors were wired together for high voltage switching applications. In such a structure, these capacitors were indexed by an (x, y, z) code which suggested their location in the room. This indexing scheme was also used in modeling the changing voltage in space and time which occurred when a switch was closed.

Other examples of scalar fields include the temperature at different locations in the atmosphere. A hot air balloonist traveling from Plano, Texas to Rockwall, Texas, might care about temperatures at different altitudes (z), at different points, (x, y) along his route. While we are dangling up in the air, we can also think a bit about lightning. Caused by air breakdown (ionization) due to voltage differentials between clouds and the ground, the fact that lightning occurs in some places and not others suggests that voltage in the atmosphere also varies with (x, y, z) space.

So far, we have tried to motivate why we might want to keep track of a quantity as a function of three dimensional space. Since vector quantities are often of interest (force, velocity, momentum...), it is reasonable to want to consider *vector fields*,

[1] My "definition" here is rather mathematical and pragmatic, and inherently ignores the philosophical view that a field implies some sort of intermediary which replaces the doctrine of "action-at-a-distance."

$$\mathbf{J}(t) \rightarrow \mathbf{J}(x, y, z, t) \tag{2}$$

We have picked the symbol **J** because we are thinking about the electromagnetic quantity, current density. Current is a vector quantity because it is the flow of charged particles and both the magnitute (number) and the direction of the current is important. In Cartesian coordinates, the vector field decomposition becomes[2],

$$\mathbf{J}(x, y, z, t) = J_x(x, y, z, t)\mathbf{a}_x + J_y(x, y, z, t)\mathbf{a}_y + J_z(x, y, z, t)\mathbf{a}_z \tag{4}$$

The three components of **J**, J_x, J_y, J_z are themselves, scalar fields, and so we can say that to describe the information in a vector field we need to keep track of three scalar fields. $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are *unit vectors* (vectors of length one) and balance the vector nature of the left hand side of Eq. (4) to the vector of the right hand side of Eq. (4).

Taking derivatives of fields: The del operator, the gradient, the divergence and the curl.

You are all well familiar with manipulating functions of one variable, and hardly need to be reminded of the utility of their calculus. Derivatives, for example, lead to optimizations which in turn may lead to improved design. The laws of nature are often cast in differential equations, whose solution may yield a quantitative understanding of the physical nature of the system or device under study. Here we look at differentiating scalar and vector fields.

The rules for taking the time derivative of a scalar or a vector field are straight forward examples of *partial differentiation*:

$$\frac{d}{dt}\left[V(x, y, z, t)\right] = \frac{\partial V(x, y, z, t)}{\partial t} \tag{5}$$

$$\frac{d}{dt}\left[\mathbf{J}(x, y, z, t)\right] = \frac{\partial J_x(x, y, z, t)}{\partial t}\mathbf{a}_x + \frac{\partial J_y(x, y, z, t)}{\partial t}\mathbf{a}_y + \frac{\partial J_z(x, y, z, t)}{\partial t}\mathbf{a}_z \tag{6}$$

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[2] Your choice of coordinate system for the vector decomposition may be different from your choice of coordinates to describe the field's extent. For example, in writing,

$$\mathbf{J}(x, y, z, t) = J_r(x, y, z, t)\mathbf{a}_r + J_\theta(x, y, z, t)\mathbf{a}_\theta + J_z(x, y, z, t)\mathbf{a}_z \tag{3}$$

we have used cylindrical coordinates (r, θ, z) for the vector decomposition, but Cartesian coordinates (x, y, z) for the dependence of the scalar field.

Similarly, taking the derivative with respect to a particular direction in space is also straight forward:

$$\frac{d}{dx}\left[V(x,y,z,t)\right]=\frac{\partial V(x,y,z,t)}{\partial x} \tag{7}$$

$$\frac{d}{dy}\left[\mathbf{J}(x,y,z,t)\right]=\frac{\partial J_x(x,y,z,t)}{\partial y}\mathbf{a}_x+\frac{\partial J_y(x,y,z,t)}{\partial y}\mathbf{a}_y+\frac{\partial J_z(x,y,z,t)}{\partial y}\mathbf{a}_z \tag{8}$$

As a practical matter, when taking the partial derivative with respect to one variable, pretend – for the moment – that the other parameters are constants.

We would really like to take some sort of first derivative with respect to space, and have this derivative be independent with respect to the particular (x,y,z) coordinate system used. In constructing such a derivative it makes some sense to mix together equal parts of derivatives in each direction,

$$\frac{\partial V(x,y,z,t)}{\partial x},\frac{\partial V(x,y,z,t)}{\partial y},\frac{\partial V(x,y,z,t)}{\partial z} \tag{11}$$

The easiest way to combine these three terms is to add them together in some way. (Multiplying them together might occur to us but, among other objections, the dimensions of the final quantity would be out of line with a derivative.) If we add these three terms together in a scalar manner, we might lose track of which term came from which derivative. Therefore, it makes the most sense to add them up vectorially:

$$\frac{\partial V(x,y,z,t)}{\partial x}\mathbf{a}_\mathbf{x}+\frac{\partial V(x,y,z,t)}{\partial y}\mathbf{a}_\mathbf{y}+\frac{\partial V(x,y,z,t)}{\partial z}\mathbf{a}_\mathbf{z} \tag{12}$$

It is a comforting consequence of the deep significance of our mathematics that this quantity has a real physical meaning: *The direction of this vector field points in the direction of the largest slope. The magnitude of this vector field is the magnitude of this slope*. This vector field, therefore, is called the *gradient* of the scalar field.

$$gradV(x,y,z,t)=\frac{\partial V(x,y,z,t)}{\partial x}\mathbf{a}_\mathbf{x}+\frac{\partial V(x,y,z,t)}{\partial y}\mathbf{a}_\mathbf{y}+\frac{\partial V(x,y,z,t)}{\partial z}\mathbf{a}_\mathbf{z} \tag{13}$$

Consider for a second the elevation of a mountain in Colorado. This scalar field is a function of (x,y) (or longitude and latitude, or by USGS convention, “Northing and Easting”). If you are hiking at any point on this mountain and pour a little water out from your canteen, the water will flow in the direction[3] of the gradient of the elevation field evaluated at that point and its initial acceleration will be related to the magnitude of the gradient of the elevation field evaluated at that point.

One of your difficult tasks in electromagnetics is to try to visualize the various fields in space. It is sometimes convenient to think about the surfaces formed by a constant field values. To be specific, there will be surfaces upon which the voltage will be constant. To be confusing we'll call these "isopotentials". A surface of metal would be an isopotential, since, in steady state, we would expect the charges and thus the voltage to equalize out upon the surface. It is an interesting and important characteristic of the gradient that is always perpendicular to the surfaces of constant value. Returning to our mountain hike, we pull out of our contour map and find the location where we are spilling water from our canteen. The direction where the water is flowing is perpendicular to the contour which runs through our location. This gives us a convenient way to calculate the unit vector which is normal to a surface:

$$\hat{\mathbf{n}} = \frac{grad(V)}{|grad(V)|} \tag{14}$$

It is well worth our time to introduce the *del operator*:

$$\nabla = \frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z} \tag{15}$$

We will routinely use the del operator to signify some sort of first derivative with respect to space. For example, the gradient of a scalar field V may be written,

$$\begin{aligned} gradV(x,y,z,t) &= \nabla V(x,y,z,t) \\ &= \left[\frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z}\right]V(x,y,z,t) \\ &= \frac{\partial V(x,y,z,t)}{\partial x}\mathbf{a_x} + \frac{\partial V(x,y,z,t)}{\partial y}\mathbf{a_y} + \frac{\partial V(x,y,z,t)}{\partial z}\mathbf{a_z} \end{aligned} \tag{16}$$

We now turn to taking spatial derivatives of vector fields, and we will make considerable use of ∇. Notice that ∇ is a vector. Counting ∇ and a vector field $\mathbf{J}(x,y,z,t)$, we realize that we are working with two vectors. Two vectors may interact together by either the dot product or the cross product. Therefore, first spatial derivatives of vector fields come in two flavors: "del dotted with **J**",

[3] Exact opposite, actually.

$$\nabla \cdot \mathbf{J}(x, y, z, t) \tag{18}$$

And "del crossed with **J**",

$$\nabla \times \mathbf{J}(x, y, z, t) \tag{19}$$

Both of these have useful physical meanings, and are important in our study of electromagnetetics. Therefore let's learn the mechanics of evaluating them.

$\nabla \cdot \mathbf{J}(x, y, z, t)$ is called the *divergence*, or "div" for short. Evaluating the dot product shows us the mathematical meaning,

$$\mathbf{div}\ \mathbf{J}(x, y, z, t) = \nabla \cdot \mathbf{J}(x, y, z, t)$$

$$= \left[\frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z}\right] \cdot \left[J_x(x, y, z, t)\mathbf{a_x} + J_y(x, y, z, t)\mathbf{a_y} + J_z(x, y, z, t)\mathbf{a_z}\right]$$

$$= \frac{\partial J_x}{\partial x}(\mathbf{a_x} \cdot \mathbf{a_x}) + \frac{\partial J_y}{\partial x}(\mathbf{a_x} \cdot \mathbf{a_y}) + \frac{\partial J_z}{\partial x}(\mathbf{a_x} \cdot \mathbf{a_z})$$

$$+ \frac{\partial J_x}{\partial y}(\mathbf{a_y} \cdot \mathbf{a_x}) + \frac{\partial J_y}{\partial y}(\mathbf{a_y} \cdot \mathbf{a_y}) + \frac{\partial J_z}{\partial y}(\mathbf{a_y} \cdot \mathbf{a_z})$$

$$+ \frac{\partial J_x}{\partial z}(\mathbf{a_z} \cdot \mathbf{a_x}) + \frac{\partial J_y}{\partial z}(\mathbf{a_z} \cdot \mathbf{a_y}) + \frac{\partial J_z}{\partial z}(\mathbf{a_z} \cdot \mathbf{a_z})$$

$$= \frac{\partial J_x(x, y, z, t)}{\partial x} + \frac{\partial J_y(x, y, z, t)}{\partial y} + \frac{\partial J_z(x, y, z, t)}{\partial z}$$

Notice that $\nabla \cdot \mathbf{J}(x,y,z,t)$ returns a scalar field. Notice that this new scalar field includes the "diagonal" terms – the x derivative of the x component, the y derivative of the y component, and the z derivative of the z component.

$\nabla \times \mathbf{J}(x,y,z,t)$ is called the *curl*. Evaluating the cross product shows us the mathematical meaning of the curl.

$$curl\mathbf{J}(x, y, z, t) = \nabla \times \mathbf{J}(x, y, z, t)$$

$$= \left[\frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z} \right] \times \left[J_x(x,y,z,t)\mathbf{a_x} + J_y(x,y,z,t)\mathbf{a_y} + J_z(x,y,z,t)\mathbf{a_z} \right]$$

$$= det \begin{vmatrix} \mathbf{a_x}\mathbf{a_y}\mathbf{a_z} \\ \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial z} \\ J_x J_y J_z \end{vmatrix} \tag{21}$$

$$= \left[\frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z} \right]\mathbf{a_x} + \left[\frac{\partial J_x}{\partial z} - \frac{\partial J_z}{\partial x} \right]\mathbf{a_y} + \left[\frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y} \right]\mathbf{a_z}$$

Perhaps the use of a 3x3 matrix for taking the cross product of two vectors is familiar to you. I find it convenient to remember where the minus signs go in the final expression.

Notice that $\nabla \times \mathbf{J}(x,y,z,t)$ returns a vector field, and that this new vector field contains the "off-diagonal" terms, which were not included in the divergence.

To begin to get an appreciation for when to use the divergence and when to use the curl, note two rules of thumb:

1. Use the divergence when you want the first derivative to match a scalar quantity. Use the curl when you want the first derivative to match a vector property. (Beware, though, that zero may be either a vector or a scalar).
2. Use the divergence when the parallel, diagonal terms are important to the problem. Use the curl when the orthogonal, off-diagonal terms are important.

A physically meaningful example of a curl's significance can be borrowed from fluid mechanics. Consider an arbitrary flow of water; the velocities of the water particles form a velocity field, $\mathbf{v}(x,y,z,t)$ which is a vector field. If I throw into this flow a small twig, or drinking straw, there is a good chance that this twig will begin to rotate. The axis of this rotation is in the direction of $\nabla \times \mathbf{v}$, and its speed of rotation will be proportional[4] to the magnitude of $\nabla \times \mathbf{v}$. It is a significant arrangement if the curl of a fluid's velocity field is zero; if $\nabla \times \mathbf{v}(x,y,z,t) = 0$, then the flow is called "irrotational."

Also in a fluid flow[5], the conservation of mass law may be written as, $\rho \nabla \cdot v = \frac{-\partial \rho}{\partial t}$ where $\mathbf{v}(x,y,z,t)$ is once again the velocity field and ρ is the density. That the rate of change of density of the

[4] To be rigorous, the "vorticity" vector $2w$ is given by $w = \frac{1}{2}\nabla \times \mathrm{v}$. The vorticity is twice the instantaneous rate of rigid body rotation in a flow field.

[5] which is incompressible

flow at a point can be written in terms of a divergence suggests that, physically, the divergence represents a flux. We will obtain a clearer picture of this when we consider one of the integral theorems below.

Second derivatives

Second derivatives of fields are particularly important in electromagnetics because the propagation of electromagnetic energy, and therefore information, is governed by a *wave equation*, which, as we will see, is a partial differential equation which includes a second derivative with respect to time and a second derivative with respect to space. So, given the gradient, the divergence and the curl, let us list our choices for second derivatives with respect to space.

If I start with a scalar field and take its gradient, I create a vector field. There are two ways to take the derivative with respect to this new vector field – the divergence and the curl. Thus there are exactly two ways to take a general second derivative of a scalar field with respect to space:

$$\nabla \cdot (\nabla V) \tag{24}$$

and

$$\nabla \times (\nabla V) \tag{25}$$

If I start with a vector field and take its divergence, I create a scalar field. There is one way to take the derivative of this new scalar field, the gradient. On the other hand, if I start with a vector field and take its curl, I create another vector field. There are two ways to take the derivative with respect to this new vector field – the divergence and the curl. Thus there are exactly three ways to take a general second derivative of a vector field with respect to space:

$$\nabla (\nabla \cdot \mathbf{J}) \tag{26}$$

$$\nabla \cdot (\nabla \times \mathbf{J}) \tag{27}$$

and

$$\nabla \times (\nabla \times \mathbf{J}) \tag{28}$$

Let us explore each of these five new fields to see which quantities might be useful. Consider the first quantity, the div(gradV):

$$\nabla \cdot (\nabla V) = \left[\frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z} \right] \cdot \left[\frac{\partial V}{\partial x}\mathbf{a_x} + \frac{\partial V}{\partial y}\mathbf{a_y} + \frac{\partial V}{\partial z}\mathbf{a_z} \right]$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \qquad (29)$$

$$= \nabla^2 V$$

which has an aesthetically nice intuitive form. Now consider the curl$(\text{grad}V)$:

$$\nabla \times (\nabla V) = det \begin{vmatrix} \mathbf{a_x} & \mathbf{a_y} & \mathbf{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y} \right]\mathbf{a_x} + \left[\frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z} \right]\mathbf{a_y} + \left[\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x} \right]\mathbf{a_z} \qquad (30)$$

$$= 0$$

Since the order of differentiation never matters. For all scalar field, then $\nabla \times (\nabla V) = 0$. While this is a useful vector identity, it does not yield any new field or quantity which we can put to work for us. Therefore the unambiguous and highly useful choice for a second spatial derivative of a scalar field is $\nabla \cdot (\nabla V) = \nabla^2 V$, which we will call the *Laplacian* of a scalar field.

That the div(curl**J**) equals zero may be shown in a manner similar to Eq (30).

$$\nabla \cdot (\nabla \times \mathbf{J}) = \left[\frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y} + \frac{\partial}{\partial z}\mathbf{a_z} \right] \cdot \left[\left(\frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z} \right)\mathbf{a_x} + \left(\frac{\partial J_x}{\partial z} - \frac{\partial J_z}{\partial x} \right)\mathbf{a_y} + \left(\frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y} \right)\mathbf{a_z} \right]$$

$$= \frac{\partial^2 J_z}{\partial x \partial y} - \frac{\partial^2 J_y}{\partial x \partial z} + \frac{\partial^2 J_x}{\partial y \partial z}$$

$$-\frac{\partial^2 J_z}{\partial y \partial x}+\frac{\partial^2 J_y}{\partial z \partial x}-\frac{\partial^2 J_x}{\partial z \partial y}$$

$$= 0 \tag{32}$$

It turns out that div(gradJ) is very messy and therefore of limited use, except in problems where a complicated material forces us to make use of it. Most of the time we try to dream up physical approximations which justify setting equal to zero. To be explicit:

$$\nabla(\nabla \cdot \mathbf{J})=\left(\frac{\partial}{\partial x}\mathbf{a_x}+\frac{\partial}{\partial y}\mathbf{a_y}+\frac{\partial}{\partial z}\mathbf{a_z}\right)\left(\frac{\partial J_x}{\partial x}+\frac{\partial J_y}{\partial y}+\frac{\partial J_z}{\partial z}\right)$$

$$=\left(\frac{\partial^2 J_x}{\partial x^2}+\frac{\partial^2 J_y}{\partial x \partial y}+\frac{\partial^2 J_z}{\partial x \partial z}\right)\mathbf{a_x} \tag{33}$$

$$+\left(\frac{\partial^2 J_x}{\partial y \partial x}+\frac{\partial^2 J_y}{\partial y^2}+\frac{\partial^2 J_z}{\partial y \partial z}\right)\mathbf{a_y}$$

$$+\left(\frac{\partial^2 J_x}{\partial z \partial x}+\frac{\partial^2 J_y}{\partial z \partial y}+\frac{\partial^2 J_z}{\partial z^2}\right)\mathbf{a_z}$$

What makes this expression complicated is the presence of J_y and J_z in the $\mathbf{a_x}$ term; the presence of J_x and J_z in the $\mathbf{a_y}$ term and the presence of J_x and J_y in the $\mathbf{a_z}$ term. This "coupling" will limit our ability to analytically solve differential equations derived with $\nabla(\nabla \cdot \mathbf{J})$. In this course we can usually find a reason to avoid it.

Finally, let us write out the curl(curl**J**):

$$\nabla \times (\nabla \times \mathbf{J}) = det\begin{vmatrix} \mathbf{a_x} & \mathbf{a_y} & \mathbf{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial J_z}{\partial y}-\frac{\partial J_y}{\partial z}\right) & \left(\frac{\partial J_x}{\partial z}-\frac{\partial J_z}{\partial x}\right) & \left(\frac{\partial J_y}{\partial x}-\frac{\partial J_x}{\partial y}\right) \end{vmatrix}$$

$$= \left[\frac{\partial^2 J_y}{\partial y \partial x} - \frac{\partial^2 J_x}{\partial y^2} - \frac{\partial^2 J_x}{\partial z^2} + \frac{\partial^2 J_z}{\partial z \partial x} \right] \mathbf{a_x} \tag{34}$$

$$+ \left[\frac{\partial^2 J_z}{\partial z \partial y} - \frac{\partial^2 J_y}{\partial x^2} - \frac{\partial^2 J_y}{\partial z^2} + \frac{\partial^2 J_x}{\partial x \partial y} \right] \mathbf{a_y}$$

$$+ \left[\frac{\partial^2 J_x}{\partial x \partial z} - \frac{\partial^2 J_z}{\partial x^2} - \frac{\partial^2 J_z}{\partial y^2} + \frac{\partial^2 J_y}{\partial y \partial z} \right] \mathbf{a_z}$$

What makes this expression complicated is the presence of J_y and J_z in the $\mathbf{a_x}$ term; the presence of J_x and J_z in the $\mathbf{a_y}$ term and the presence of J_x and J_y in the $\mathbf{a_z}$ term. These are the very same objectional terms as in Eq (33). We can combine Eqs (33) and (34) in a way which gets rid of these coupling terms, if we define the eminently useful Laplacian of a vector,

$$\nabla^2 \mathbf{J} = \nabla(\nabla \cdot \mathbf{J}) - \nabla \times \nabla \times \mathbf{J}$$

$$= \left(\frac{\partial^2 J_x}{\partial x^2} + \frac{\partial^2 J_x}{\partial y^2} + \frac{\partial^2 J_x}{\partial z^2} \right) \mathbf{a_x}$$

$$\left(\frac{\partial^2 J_y}{\partial x^2} + \frac{\partial^2 J_y}{\partial y^2} + \frac{\partial^2 J_y}{\partial z^2} \right) \mathbf{a_y} \tag{35}$$

$$\left(\frac{\partial^2 J_z}{\partial x^2} + \frac{\partial^2 J_z}{\partial y^2} + \frac{\partial^2 J_z}{\partial z^2} \right) \mathbf{a_z}$$

This vector quantity is often what we mean by a second spatial derivative of a vector field.

Two Integral Theorems

Next to taking derivatives, it may be said that doing integrals is the most exciting pastime. Here we will show, in Cartesian coordinates, the validity of two integral theorems of vector calculus, namely *Stokes theorem* and the *divergence theorem*. Their demonstration will also provide review examples of how to perform multiple integrals, line integrals, and area integrals.

Stokes Theorem

A real estate mogul, who buys New York City property by the square block, wants to know how many square blocks there are in a region defined by the street corners, 33rd street, 4th avenue; 82nd street, 3rd avenue; 101st street, 7th avenue; 54th street, 6th avenue. There is a surveyor's formula which relies on the coordinates of the vertices that allows the quick calculation of this area without a lot of hassle:

$$Area = \frac{1}{2}\left(\frac{x_1}{y_1} \times \frac{x_2}{y_2} \times \frac{x_3}{y_3} \times \frac{x_4}{y_4} \times \frac{x_1}{y_1}\right)$$

$$= \frac{1}{2}\left(\frac{33}{4} - \frac{82}{3} - \frac{101}{7} - \frac{54}{6} - \frac{33}{4}\right) \tag{37}$$

$$= \frac{1}{2}[(3)(33) + (7)(82) + (6)(101) + (4)(54) - (33)(6) - (7)(54) - (3)(101) - (4)(82)]$$

$$=288$$

The significance of this example is that it is natural to relate the area enclosed by a perimeter to the coordinates of the vertices of that perimeter.

This formula is a distant relative to Stokes theorem, which equates a line integral evaluated around a closed path, to an integral over the area enclosed by that path:

$$\oint \mathbf{J} \cdot \mathbf{dl} = \int\int (\nabla \times \mathbf{J}) \cdot \hat{\mathbf{n}} dA \tag{38}$$

Let's demonstrate this equality for a very small square of area $(\Delta x)(\Delta y)$ whose center is at some point (x, y) as shown in Figure 1. This square is sitting in some general vector field, $\mathbf{J}(x, y, z, t)$. Our path integral will begin at the point $\left(x - \frac{\Delta x}{2}, y - \frac{\Delta y}{2}\right)$ and travel in the x direction to the point $\left(x + \frac{\Delta x}{2}, y - \frac{\Delta y}{2}\right)$, the lower right hand corner of the square. From there, the path goes up to the point $\left(x + \frac{\Delta x}{2}, y + \frac{\Delta y}{2}\right)$, then back over to $\left(x - \frac{\Delta x}{2}, y + \frac{\Delta y}{2}\right)$, before returning to the starting point. The four line vectors for the four path segments are, $\mathbf{dl} = -\Delta y \mathbf{a_y}$, respectively. This counter-clockwise route, by the right-hand-rule, gives us a surface normal vector, $\hat{\mathbf{n}} = \mathbf{a_z}$.

The terms on the left hand side of Eq. (38) are, then, $\mathbf{J}\left(x, y-\frac{\Delta y}{2}\right)\cdot \Delta x\mathbf{a_x}$, $\mathbf{J}\left(x+\frac{\Delta y}{2}, y\right)\cdot \Delta y\mathbf{a_y}$, $\mathbf{J}\left(x, y+\frac{\Delta y}{2}\right)\cdot -\Delta x\mathbf{a_x}$, $\mathbf{J}\left(x-\frac{\Delta y}{2}, y\right)\cdot \Delta y\mathbf{a_y}$. Note that we have made use of the small nature of (Δx) and (Δy) to evaluate **J** along the midpoint of each line segment. Upon completing the dot product and arranging terms, the left hand side of Eq. (38) becomes,

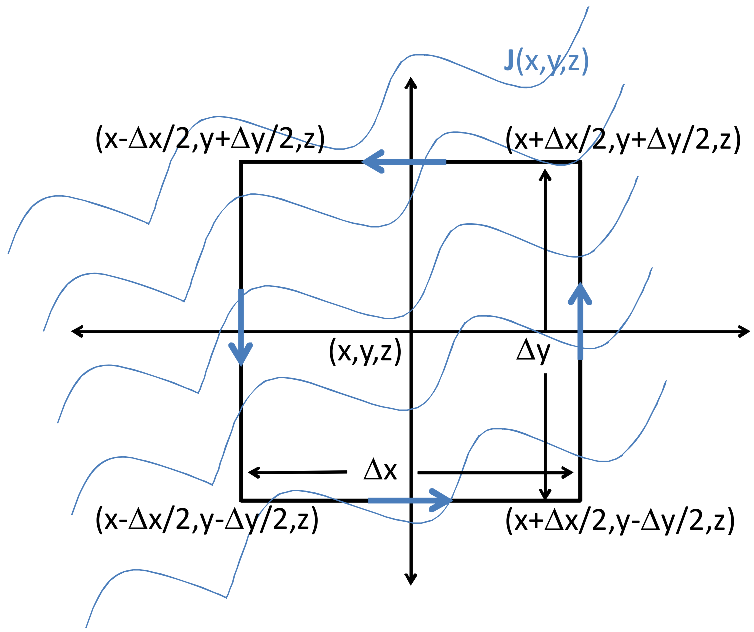


Figure 1. Differential loop in J(x,y,z) to show Stokes' Thereem.

$$\oint \mathbf{J}\cdot d\mathbf{l} = \left[J_x\left(x, y-\frac{\Delta y}{2}\right) - J_x\left(x, y+\frac{\Delta y}{2}\right)\right]\Delta x + \left[J_y\left(x-\frac{\Delta x}{2}, y\right) - J_y\left(x+\frac{\Delta x}{2}, y\right)\right]\Delta y \qquad (39)$$

Turning our attention to the right hand side of Eq. (38),

$$(\nabla\times\mathbf{J})\cdot\hat{\mathbf{n}} = (\nabla\times\mathbf{J})_z = \frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y} \qquad (40)$$

If $(\Delta x)(\Delta y)$ is sufficiently small, then the quantity, $\frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y}$ may be taken as a constant over the area and pulled through the integral signs. The right hand side of Eq. (38) then becomes

$$\iint (\nabla\times\mathbf{J})\cdot\hat{\mathbf{n}}dA = \left(\frac{\partial J_y}{\partial x} - \frac{\partial J_y}{\partial x}\right)\Delta x\Delta y \qquad (41)$$

Continuing our manipulations of Eq. (39),

$$\oint \mathbf{J}\cdot\mathbf{dl} = \left[J_x\left(x, y-\frac{\Delta y}{2}\right) - J_x\left(x, y+\frac{\Delta y}{2}\right)\right]\Delta x + \left[J_y\left(x-\frac{\Delta x}{2}, y\right) - J_y\left(x+\frac{\Delta x}{2}, y\right)\right]\Delta y$$

$$= \frac{\left[J_x\left(x, y-\frac{\Delta y}{2}\right) - J_x\left(x, y+\frac{\Delta y}{2}\right)\right]}{\Delta y}\Delta x\Delta y + \frac{\left[J_y\left(x-\frac{\Delta x}{2}, y\right) - J_y\left(x+\frac{\Delta x}{2}, y\right)\right]}{\Delta x}\Delta x\Delta y \tag{42}$$

If we take the limit as the lengths go to zero, we obtain

$$\lim_{\Delta y\to 0}\frac{\left[J_x\left(x, y-\frac{\Delta y}{2}\right) - J_x\left(x, y+\frac{\Delta y}{2}\right)\right]}{\Delta y}\Delta x\Delta y + \lim_{\Delta x\to 0}\frac{\left[J_y\left(x-\frac{\Delta x}{2}, y\right) - J_y\left(x+\frac{\Delta x}{2}, y\right)\right]}{\Delta x}\Delta x\Delta y$$

$$= \left(\frac{\partial J_y}{\partial x} - \frac{\partial J_x}{\partial y}\right)\Delta x\Delta y \tag{43}$$

Which is now in the form of Eq. (41).

Stokes theorem in Eq. (31) holds macroscopically – for a large closed curve and any (!) surface (flat, concave, convex, etc.) bounded by that perimeter curve. Our demonstration is microscopic and, even if we restrict our discussion to Cartesian coordinates, is only a building block of the final proof. To this end, consider two adjacent squares, as shown in Figure 2, each of which is similar to the one which starred in our above analysis. If we write down all the line integrals, as we did in Eq. (42), we notice that the shared side is written twice, once for **dl** positive, the other for **dl** negative. The contribution of this shared side thus vanishes due to the vector nature of our theorem. Only the six exterior line segments match up to the (now doubled) area. By extension we can keep adding on small squares until we have mapped out a large, not necessarily flat area. All shared sides will cancel, and the line integral about the perimeter will yield area information about the enclosed surface.

Example:Faraday's law in differential form

Faraday's law of electromagnetics may be written "in integral form" as:

$$\oint \mathbf{E}\cdot\mathbf{dl} = -\frac{\partial}{\partial t}\iint \mathbf{B}\cdot\hat{\mathbf{n}}\,dA \tag{44}$$

The closed line integral on the left hand side of the equation defines the limits of integration of the right hand side. Applying Stokes theorem to the left hand side we can convert the line integral to an area integral.

$$\oint \mathbf{E}\cdot \mathbf{dl} = \iint (\nabla \times \mathbf{E})\cdot \hat{\mathbf{n}} dA = -\frac{\partial}{\partial t}\iint \mathbf{B}\cdot \hat{\mathbf{n}} dA \tag{45}$$

The area in both surface integrals must be the same, so we may combine them under the same limits of integration,

$$\iint \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial \mathbf{t}} \right)\cdot \hat{\mathbf{n}} dA = 0 \tag{46}$$

Which will equal zero if

$$\nabla \times \mathbf{E} = \frac{\partial \mathbf{B}}{\partial t} \tag{47}$$

This is Faraday's law of electromagnetics "in differential form".

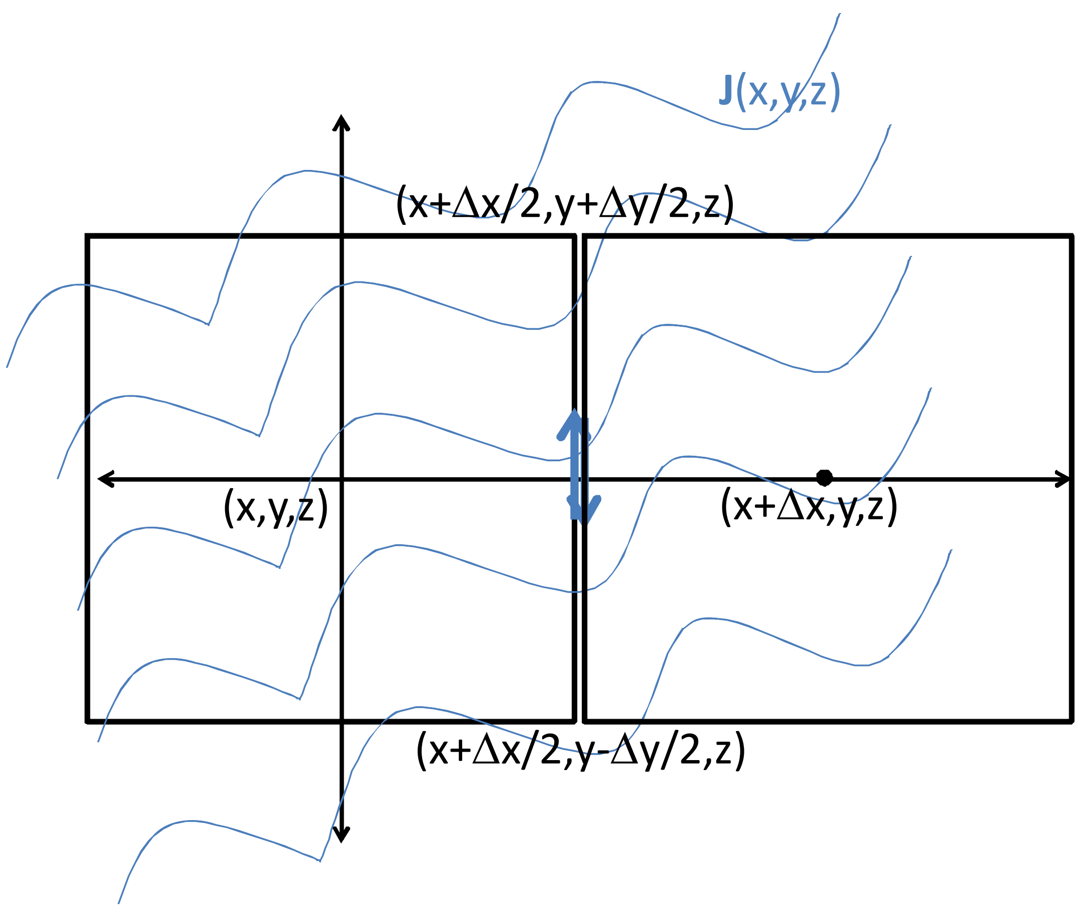


Figure 2. Two adjacent differential loops in J(x,y,z) to show Stokes' Theorem.

Divergence Theorem

Our second integral theorem is the Divergence theorem. For an arbitrary vector field,

$$\int \nabla \cdot \mathbf{J} dV = \oint \mathbf{J} \cdot \hat{\mathbf{n}} dA \tag{48}$$

The surface of the volume on the left hand side of Eq. (48) gives the area limits of integration on the right hand side of Eq. (48). We will demonstrate this theorem in Cartesian coordinates in a manner analogous to our demonstration of Stokes theorem above. Our building block is a small cube with sides $\Delta x, \Delta y,$ and Δz. The center of this cube is at the point (x, y, z), and it is aligned nicely with the x-y-z axis. This situation is shown in Figure 3.

For this cube, the left hand side of the theorem may be written,

$$\int \nabla \cdot \mathbf{J} dV = \iiint \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right) dxdydz \tag{49}$$

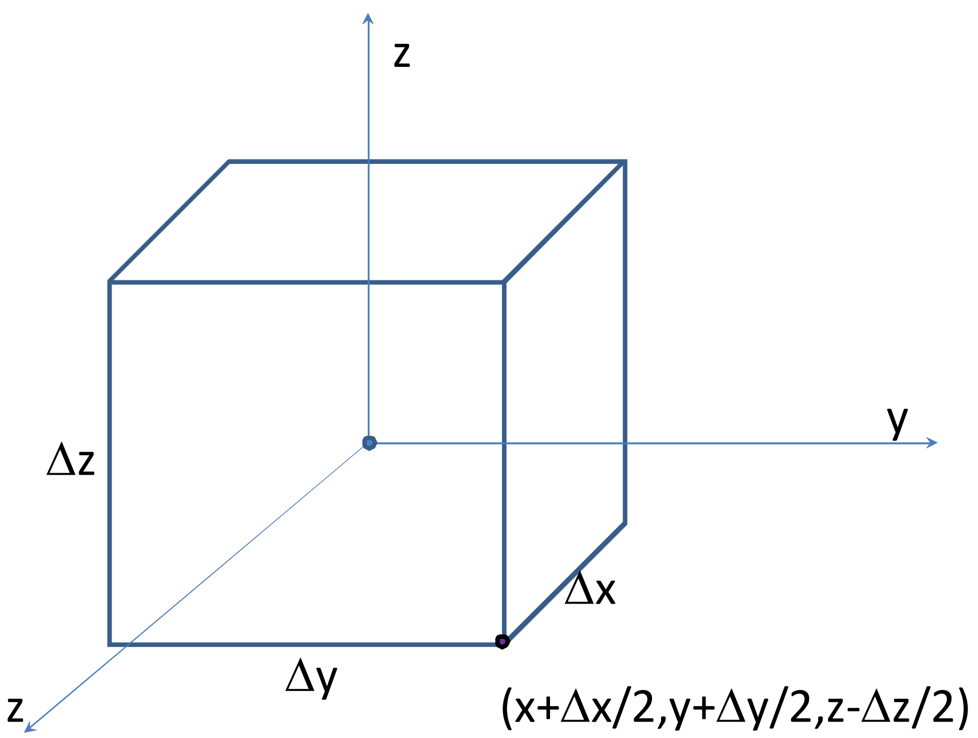


Figure 2. A small cube in J(x,y,z) to show the Divergence Theorem.

$$= \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

We have assumed that $\Delta x \Delta y \Delta z$ is small enough so that $\nabla \cdot \mathbf{J}$ does not significantly change over the differential volume.

There are six terms on the right hand side of Eq. (48), one for each face of the cube. Evaluating **J** in the middle of each face, and grouping by the normal unit vector of each face, we have,

$$\begin{aligned}\oint \mathbf{J} \cdot \hat{\mathbf{n}} dA = & \left[-J_x\left(x - \frac{\Delta x}{2}, y, z \right) + J_x\left(x + \frac{\Delta x}{2}, y, z \right) \right] \Delta y \Delta z \\ & + \left[-J_y\left(x, y - \frac{\Delta y}{2}, z \right) + J_y\left(x, y + \frac{\Delta y}{2}, z \right) \right] \Delta x \Delta z \\ & + \left[-J_z\left(x, y, z - \frac{\Delta z}{2} \right) + J_z\left(x, y, z + \frac{\Delta z}{2} \right) \right] \Delta x \Delta y \end{aligned} \tag{50}$$

Multiplying each term appropriately by one, and taking the limits, gives

$$\begin{aligned}\oint \mathbf{J} \cdot \hat{\mathbf{n}} dA = & \lim_{\Delta x \to 0} \frac{\left[-J_x\left(x - \frac{\Delta x}{2}, y, z \right) + J_x\left(x + \frac{\Delta x}{2}, y, z \right) \right]}{\Delta x} \Delta x \Delta y \Delta z \\ & + \lim_{\Delta y \to 0} \frac{\left[-J_y\left(x, y - \frac{\Delta y}{2}, z \right) + J_y\left(x, y + \frac{\Delta y}{2}, z \right) \right]}{\Delta y} \Delta x \Delta y \Delta z \\ & + \lim_{\Delta z \to 0} \frac{\left[-J_z\left(x, y, z - \frac{\Delta z}{2} \right) + J_z\left(x, y, z + \frac{\Delta z}{2} \right) \right]}{\Delta z} \Delta x \Delta y \Delta z \\ = & \left(\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right) \Delta x \Delta y \Delta z \end{aligned} \tag{51}$$

which is the same as Eq. (49).

The Divergence theorem in Eq. (48) holds macroscopically – for a large closed surface and its enclosed volume. Our demonstration is microscopic and, even if we restrict our discussion to Cartesian coordinates, is only a building block of the final proof. Our extension of our demonstration to the macroscopic world is analogous to the argument presented for Stokes theorem. Consider two adjacent cubes sharing one side as shown in Figure 4. If we write down all the surface integrals, as we did in Eq.

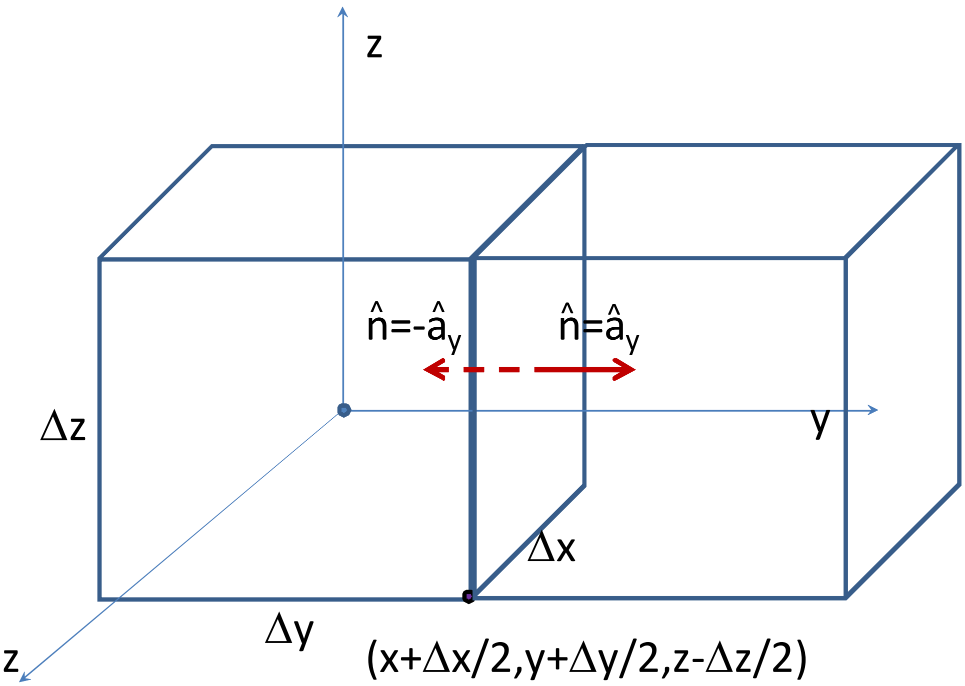


Figure 2. Two adjacent small cubes in J(x,y,z) to show the Divergence Theorem.

(51), we notice that the shared surface appears twice, once for $\hat{\mathbf{n}}$ positive, the other for negative $\hat{\mathbf{n}}$. The contribution of this shared surface thus vanishes due to the vector nature of our theorem. Only the ten exterior surfaces match up to the (now doubled) volume. By extension we can keep adding on small cubes until we have mapped out a large volume. All shared surfaces will cancel, and the surface integral about the exterior will yield volumetric information about the interior.

Example: Gauss' law in differential form

Gauss' law of magnetism may be written "in integral form" as:

$$\oint \mathbf{B}\cdot\hat{\mathbf{n}}dA = 0 \tag{52}$$

Applying the divergence theorem to the left hand side we can convert the area integral to an integral over the volume enclosed by the same area.

$$\int \nabla\cdot\mathbf{B}dV = 0 \tag{53}$$

Evaluating this integral in a very small region around a point gives,

$$\nabla \cdot \mathbf{B} = 0 \tag{54}$$

at that point. This is Gauss' law for magnetism "in differential form."

From our discussion of the divergence theorem, it follows that the divergence of a vector field may be written as

$$\nabla \cdot \mathbf{J} = \lim_{\Delta V \to 0} \oint \mathbf{J} \cdot \hat{\mathbf{n}} dA \tag{55}$$

In fact this is often the way mathematicians define the divergence. Physically we can say that $\nabla \cdot \mathbf{J}$ is the "efflux" (outflow) of $\mathbf{J}$ per unit volume at a point in space.

General vector identities

Equations (30), (32), and (35) are examples of "vector identities," truths which will be used in derivations throughout the electromagnetics course. You should become familiar with these expressions, but don't spend hours memorizing them. They are listed here, as they are in your text, for easy reference. It may be helpful to demonstrate the identities in Cartesian coordinates. Here, f and g are scalar fields; **A** and **B** are vector fields.

$$\nabla(f+g) = \nabla f + \nabla g \tag{56}$$

$$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B} \tag{57}$$

$$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B} \tag{58}$$

$$\nabla(fg) = f\nabla g + g\nabla f \tag{59}$$

$$\nabla\cdot(f\mathbf{A}) = \mathbf{A}\cdot\nabla f + f\nabla\cdot\mathbf{A} \tag{60}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\nabla\times\mathbf{A} - \mathbf{A}\cdot\nabla\times\mathbf{B} \tag{61}$$

$$\nabla\times(f\mathbf{A}) = (\nabla f)\times\mathbf{A} + f(\nabla\times\mathbf{A}) \tag{62}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\nabla\cdot\mathbf{B} - \mathbf{B}\nabla\cdot\mathbf{A} + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \tag{63}$$

$$\nabla\cdot\nabla f = \nabla^2 f \tag{64}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0 \tag{65}$$

$$\nabla\times\nabla f = 0 \tag{66}$$

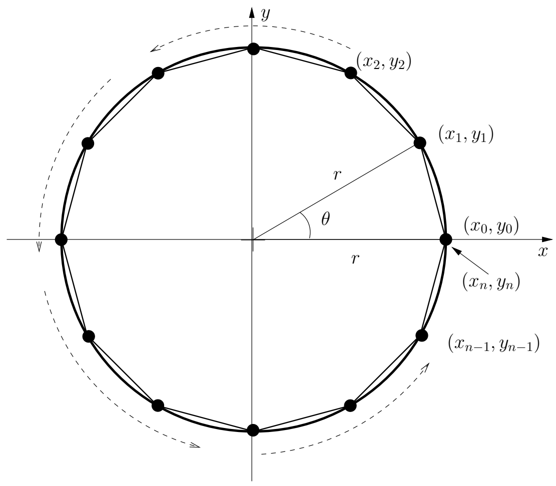
$$\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} \tag{67}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}(\nabla\times\mathbf{A}) \tag{68}$$

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} \tag{69}$$

$$\mathbf{A} \times \mathbf{B} \times \mathbf{C} = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) = \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{70}$$

Example: Area of Circle Using Surveyor's Formula.



The enclosed area of a regular n-sided polygon is given by

$$
\begin{aligned}
A &= \frac{1}{2}\left\{\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-1} & x_n \\ y_{n-1} & y_n \end{vmatrix}\right\} \\
&= \frac{1}{2}\left\{\begin{vmatrix} r\cos 0 & r\sin\frac{2\pi}{n} \\ r\sin 0 & r\sin\frac{2\pi}{n} \end{vmatrix} + \begin{vmatrix} r\cos\frac{2\pi}{n} & r\cos\frac{4\pi}{n} \\ r\sin\frac{2\pi}{n} & r\sin\frac{4\pi}{n} \end{vmatrix} + \right. \\
&\qquad \left. \cdots + \begin{vmatrix} r\cos\frac{2\pi(n-1)}{n} & r\cos 0 \\ r\sin\frac{2\pi(n-1)}{n} & r\sin 0 \end{vmatrix}\right\} \\
&= \frac{r^2}{2}\left[\sin\frac{2\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{2\pi}{n}\right] = \pi r^2\left(\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right)
\end{aligned}
$$

$$
\text{as } (n \to \infty) \qquad \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}} \to \frac{\sin 0}{0} \to 1
$$

$$
\boxed{\therefore A = \pi r^2 \qquad (n \to \infty)}
$$