SOLUTION FOR HOMEWORK 3, STAT 4351

Welcome to your third homework which rounds out topics in Chapter 2. Now you know basics of Probability, Kolmogorov’s axioms, basics of algebra of sets, conditional probability, independence/dependence of events, notion of partition and prior probabilities, the rule (law, theorem) of total probability and Bayes’ theorem (rule).

Now let us look at your problems.

1. Problem 2.20. Given: \( P(ABC) \neq 0 \) and \( P(C|AB) = P(C|B) \). In words, conditionally on the event \( B \), events \( C \) and \( A \) are independent. Please recall that the conditional probability \( P(E|F) \) is a traditional probability whenever \( F \) is given; thus what I said earlier should be clear to you. [Also, recall that I may use shorthand \( AB = A \cap B \) whenever no confusion can occur.]

Then we should establish that \( P(A|BC) = P(A|B) \), which in words means that \( A \) and \( C \) are conditionally independent given \( B \).

If you compare what is given with what you are asked to prove then the assertion looks obvious; nonetheless we need to prove it using our algebra. Let us do this:

\[
P(A|BC) = \frac{P(ABC)}{P(BC)}. \]

Now I look at what is given and continue correspondingly

\[
P(A|BC) = \frac{P(ABC)}{P(BC)} = \frac{P(AB)P(C|AB)}{P(BC)} = \frac{P(B)P(A|B)P(C|B)}{P(B)P(C|B)} = P(A|B). \]

What was wished to show.

2. Problem 2.23. Well, if \( A \) and \( B \) are dependent then, obviously, \( A \) and \( B^c \) should be dependent as well because otherwise we can use the assertion proved in class (see also Theorem 2.11): if two events \( E \) and \( F \) are independent then \( E \) and \( F^c \) are also independent. Nonetheless, let us prove the assertion directly.

Given: \( P(AB) \neq P(A)P(B) \). Check that \( P(AB^c) \neq P(A)P(B^c) \).

Indeed, using our familiar identity (the law of total probability in its simplest form with the partition of sample space into \( B \) and \( B^c \))

\[
P(A) = P(AB) + P(AB^c), \tag{1} \]

we get for the probability in the question:

\[
P(AB^c) = P(A) - P(AB), \]

and then using the given dependence between \( A \) and \( B \) we finish with

\[
P(AB^c) \neq P(A) - P(A)P(B) = P(A)[1 - P(B)] = P(A)P(B^c). \]
3. Problem 2.28. Given: \( P(A|B) < P(A) \). Check that \( P(B|A) < P(B) \). Well, we need to begin with the probability in question and then, somewhere in the midst of rewriting it, use what is given. Let us begin:

\[
P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(B)P(A|B)}{P(A)}.
\]

Well, this is what I have been looking after: I got the given probability and I can use the given inequality in the right-hand side of the last equality. By doing this I get

\[
P(B|A) < \frac{P(B)P(A)}{P(A)} = P(B).
\]

What I was supposed to verify.

4. Problem 2.76. In what follows \( G \) - college and \( T \) - at least 3 years of experience. By using the table we get:

(a) \( P(G) = (18 + 36)/90. \)
(b) \( P(T^c) = (36 + 27)/90. \)
(c) \( P(GT) = 18/90. \)
(d) \( P(G^c \cap T^c) = 27/90. \)
(e) \( P(T|G) = 18/(18 + 36). \)
(f) \( P(G^c|T^c) = 27/(36 + 27). \)

5. Problem 2.93. Here \( A \) denote the event that an even number comes up after the first trial, \( B \) - after the second one, and \( C \) - both trials result in the same number. Then we have:

\[
P(A) = 1/2, \ P(B) = 1/2, \ P(C) = 1/6, \ P(AB) = 1/4, \ P(AC) = 3/36, \ P(ABC) = 1/12.
\]

\( (2) \)

(a). Let us check pairwise independence of the events. By a direct calculation based on the quantities in (2), we verify that

\[
P(AB) = P(A)P(B); \ P(AC) = P(A)P(C), \ P(BC) = P(B)P(C).
\]

This implies that the events \( (A, B, C) \) are pairwise independent.

But is this enough to conclude that these three events are mutually independent? We know that to make this conclusion we need to check part (b) of the problem at hand.

(b) A direct calculation, based on using (2), shows that

\[
P(ABC) \neq P(A)P(B)P(C).
\]

This is a nice example which shows that three events can be pairwise independent and mutually dependent. There are plenty of similar examples, so remember that you need to check all relationships for declaring Independence of more than 2 events.
6. Problem 2.100. Let us denote by $p_i$ the event that a customer pays promptly after $i$th month. Then, using this notation, we can write:

It is given that $P(p_{i+1}|p_i) = 0.9$ and $P(p_{i+1}|p_i^c) = 0.4$. Now we can consider the questions.

(a) Write using Theorem 2.10:

$$P(p_2p_3p_4|p_1) = P(p_2|p_1)P(p_3|p_2p_1)P(p_4|p_3p_2p_1).$$

As it is written in the problem (but I admit that it could be written more explicitly)

$$P(p_3|p_2p_1) = P(p_3|p_2) = 0.9, \quad P(p_4|p_3p_2p_1) = P(p_4|p_3) = 0.9.$$

Using this information we conclude that

$$P(p_2p_3p_4|p_1) = (0.9)^3.$$

[This is an interesting example of time series and Markov chain — remind me in class and I’ll tell you more about such observations.]

(b) Here we need to calculate $P(p_2^cp_3^cp_4|p_1^c)$. Again using the same technique we get:

$$P(p_2^cp_3^cp_4|p_1^c) = P(p_2^c|p_1^c)P(p_3^c|p_2^c)p_4|p_3^c = (1 - 0.4) \times (1 - 0.4) \times (0.4).$$

As you see, I skip a final calculation, but in general YOU should finish by a number; and do not forget that regardless of what you are doing the answer should be within $[0,1]$.

7. Problem 2.104. Here we have a classical case of partition: all labor-management disputes are either over wages (W), or over working conditions (C), or over fringe benefits (F). As a result, on the tree diagram you have 3 branches from the root node. Then, from each of these 3 branches you have 2 new branches that imply a Strike (S) or No Strike ($S^c$). Note that the probabilities for partitioning events W, C and F are given (.6,.15,.25), and they typically referred to as prior probabilities. Also, it is given:

$$P(S|W) = .55; \quad P(S|C) = .3, \quad P(S|F) = .6.$$ 

Here I used a pure algebra, but for such a simple example a tree diagram with the probabilities written above branches may be helpful.

The question is on using the law (rule, theorem) of total probability (recall Theorem 2.12). Write (note that here $S$ denotes the event strike and not the sample space; here I would prefer to use $\Omega$ for the sample space because $S$ is a very convenient notation for the strike)

$$P(S^c) = P(S^cW) + P(S^cC) + P(S^cF)$$

$$= P(W)P(S^c|W) + P(C)P(S^c|C) + P(F)P(S^c|F) = (.6)(.45) + (.15)(.7) + (.25)(.4).$$

Alternatively, you may begin with $P(S^c) = 1 - P(S)$ and then calculate $P(S)$. It is up to you.

8. Problem 2.107. This is a VERY typical problem in biostatistics and clinical trials. The partition here on persons with diabetes (D) and no diabetes ($D^c$). This is your root
node with two branches and $P(D) = .08$. Then you have secondary branches corresponding to the clinical diagnoses. Please note that there are two types of mistakes: one for $D$ persons and one for $D^c$ persons; both cases are bad and must be taken into account. Denote by $L$ the event “labeled”, that is, the diagnosis tells us that a patient has diabetes (but note that this patient may not have diabetes at all). It is given that

$$P(L|D) = .95; \quad P(L|D^c) = .02.$$ 

Again, drawing a tree diagram may be useful.

(a). The question is on using the law of total probability:

$$P(L) = P(D)P(L|D) + P(D^c)P(L|D^c) = (.08)(.95) + (.92)(.02).$$

(b) Here we use Bayes’ theorem (or the conditional probability definition plus the part (a)):

$$P(D|L) = \frac{P(DL)}{P(L)} = \frac{P(D)P(L|D)}{P(L)} = \frac{(.08)(.95)}{(.08)(.95) + (.92)(.02)}.$$  

9. Problem 2.109. This is again a classical situation with partitioning into 3 mutually exclusive events where either clerk $U$ or $V$ or $W$ In your tree-diagram this gives you 3 branches from the root node. Then a mistake can be made by any clerk (let us denote the event “Mistake” as $M$), or no mistake as $M^c$. It is given that $P(U) = .3$, $P(V) = .4$ and $P(W) = .3$. Also, $P(M|U) = .01$, $P(M|V) = .05$, $P(M|W) = .03$. Now we can solve the formulated problems.

(a) Plainly by the total probability law

$$P(M) = P(U)P(M|U) + P(V)P(M|V) + P(W)P(M|W) = (.3)(.01) + (.4)(.05) + (.3)(.03).$$

(b) Here we use Bayes’ Theorem and the result (a):

$$P(U|M) = \frac{P(U)P(M|U)}{P(M)} = \frac{(.3)(.01)}{P(M)}.$$  

(c) Similarly to (b)

$$P(V|M) = \frac{P(V)P(M|V)}{P(M)} = \frac{(.4)(.05)}{P(M)}.$$  

10. Problem 2.11. Here the partitioning is into the number of possible forgeries that ranges from 0 to 5 and will be denoted by these numbers. Note that prior probabilities are assigned to these 6 events, and they are .76, .09, .02, .01, .02, .1 respectively. One of the paintings is authenticated, and by $F$ we denote the event that this painting is Forged.

Now we are in a position to solve the problem. Note that given $F$, the event of interest is that all 5 paintings are forgeries. Write using Bayes’ theorem:

$$P(5|F) = \frac{P(5 \cap F)}{P(F)} = \frac{P(5)}{P(F)} = \frac{.1}{\sum_{i=0}^{5} P(i)P(F|i)}.$$  

(3)
Now note that $P(F|i) = i/5$, and we get

$$P(F) = (.76)(0) + (.09)(1/5) + (.02)(2/5) + (.01)(3/5) + (.02)(4/5) + (.1)(5/5).$$

Plug into (3) and get the answer.

I hope that I made no mistakes. Agree?