Math 2417 Student Notes #1H

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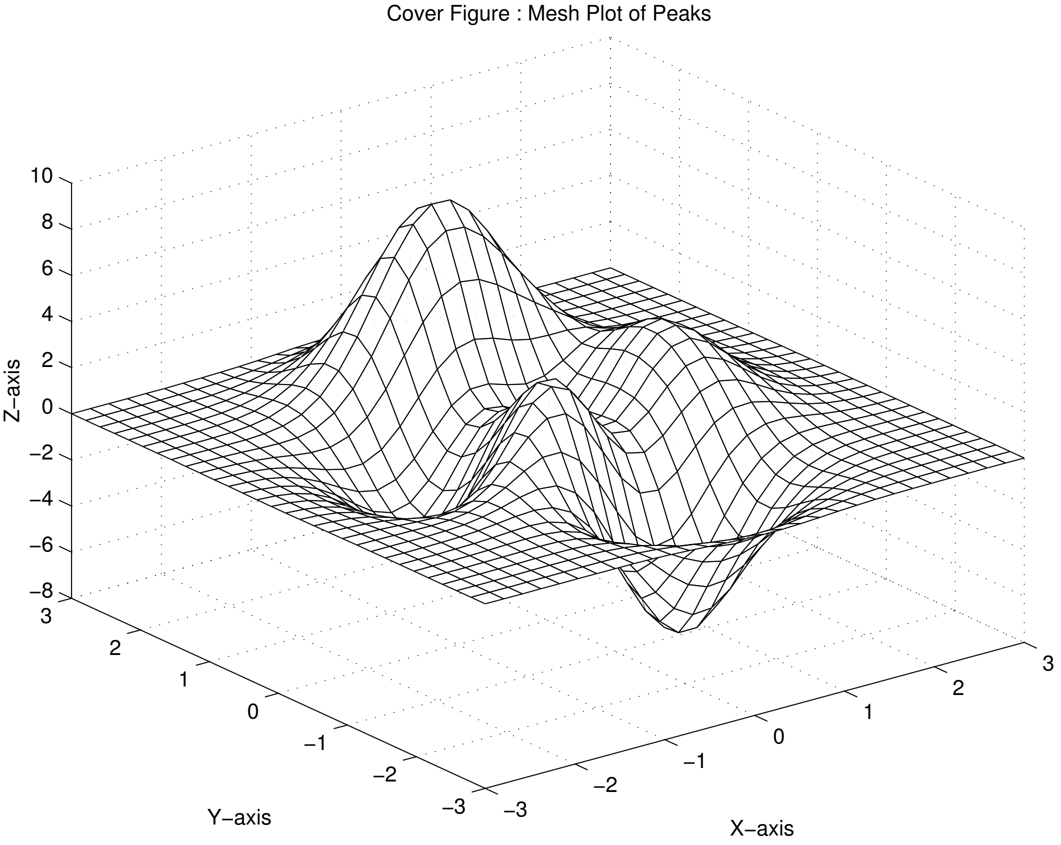

Figure 1: Cover Figure Mesh Plot of Peaks

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1 Introduction

As we pursue our course of study this semester, we will encounter many definitions and Theorems. It is the purpose of these notes to draw your attention to several of these definitions and Theorems and to indicate the type of question which might be associated with each one.

1. A list of important Theorems and definitions.
   - (a) The epsilon delta definition of the
     $$\lim_{x \to c} f(x) = L$$
     (Reference Page 52 of text)
   - (b) The Squeeze Theorem (page 65)
   - (c) Continuity at a point (page 70)
   - (d) Continuity over a closed interval (page 73)
   - (e) Intermediate Value Theorem (page 77)
   - (f) Definition of infinite limits (page 83)
     $$\lim_{x \to c} f(x) = \infty, \qquad \lim_{x \to c} f(x) = -\infty$$
   - (g) Definition of Vertical Asymptote (page 84/85)
   - (h) Definition of limits at infinity,(page 198)
     $$\lim_{x \to \infty} f(x) = L, \qquad \lim_{x \to -\infty} f(x) = L$$
   - (i) Definition of Horizontal Asymptote (page 199)
   - (j) Definition of the Derivative of a Function (Δ definition)(page 99)
   - (k) Derivatives of Sine and Cosine Functions (page 112)
   - (l) Rolles Theorem (page 172)
   - (m) The Mean Value Theorem (page 174)
   - (n) The Fundamental Theorem of Calculus (page 282)
   - (o) The Second Fundamental Theorem of Calculus (page 289)

2 Algebra

2.1 Some Algebra Errors

Warning. The following item was written immediately following a week end of MATH 2417/MATH 2419 correction by the author. Students must realize that it is impossible to obtain good grades in Calculus if they do not have a good working knowledge of basic Algebra. The following comments cover many, but not all, of the algebra mistakes made by Calculus students.

(a) $\dfrac{1}{a+b} \neq \dfrac{1}{a} + \dfrac{1}{b}$

(b) $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$

(c) $\sqrt{a^2+b^2} \neq a + b$

(d) $(a+b)^2 \neq a^2 + b^2$

(e) $\dfrac{a^2+b^2}{\sqrt{x}} \neq a^2 + b^2 - \sqrt{x}$

(f) $a^3a^4 \neq a^{12}$

(g) $\dfrac{a^8}{a^2} \neq a^4$

(h) $\dfrac{a}{b} + \dfrac{c}{d} \neq \dfrac{a+c}{b+d}$

(i) $\dfrac{ab+cd}{ae} \neq \dfrac{b+cd}{e}$

(j) If $x(x-2) = 6$ then $x \neq 6$ and $x - 2 \neq 6$

(k) $u^7 - u^5 \neq u^2$

(l) $\sqrt{36} \neq \pm 6$

(m) $\sin(A+B) \neq \sin A + \sin B$

(n) $\sin(3x) \neq 3\sin(x)$

(o) $\dfrac{\sin 5x}{\sin 3x} \neq \dfrac{5}{3}$

(p) $\sin^7 x - \sin^2 x \neq \sin^5 x$

(q) $\sin^{-1} x \neq \dfrac{1}{\sin x}$

(r) $\ln(A+B) \neq \ln A + \ln B$

(s) $\dfrac{\ln A}{\ln B} \neq \ln A - \ln B$

(t) $(\ln A)^4 \neq 4 \ln A$

2.2 Two Algebra Exercises

The following two exercises were originally in my precalculus notes.

2. $f(x) = -x^3(9-x^2)^{-1/2} + 2x(9-x^2)^{1/2}$
   (a) For what x values is $f(x) = 0$?
   (b) For what x values is f(x) undefined?
   (c) For what x values is $f(x) < 0$?
   (d) For what x values is $f(x) > 0$?

3. $f(x) = 10(2x+1)^4(3x+1)^3 + 9(2x+1)^5(3x+1)^2$
   (a) Factor as completely as possible
   (b) For what x values is $f(x) = 0$?
   (c) For what x values is $f(x) < 0$?
   (d) For what x values is $f(x) > 0$?

2.3 Absolute Value Inequalities

4. Algerba Reminder In the definition of a limit, you will come across some absolute value inequalities. It is the purpose of this item to review this material. Consider the inequality $|x-3| < 5$ If we remove the absolute value signs , we have the following:

$$\begin{array}{ll} (x-3) < 5; & -(x-3) < 5 \\ x < 5+3; & -x+3 < 5 \\ x < 8; & -x < 2 \\ & x > -2 \end{array}$$

So we may write $-2 < x < 8$ This result may be obtained easily as follows. $|x-3| < 5$ implies $-5 < x-3 < 5$ or by adding 3 across the inequality $-2 < x < 8$ Thus ,the absolute value inequality statement $|x-3| < 5$ results in an interval on the number line centered at 3 and extending 5 units positive and negative from this number. Consider the statement $|x-3| > 0$ or equivalently $0 < |x-3|$ This leads to the two statements $x > 3$, $x < 3$. That is, any real number other than 3. If we combine the two absolute value inequality statements , we have $0 < |x-3| < 5$ which leads to the following intervals on the number line, namely $(-2,3) \cup (3,8)$. That is, an interval centered at 3 but not including 3 with an extension of 5 units above and below this central point. Consider changing from numbers to symbols. If $b > 0$,$|x-c| < b$ implies $-b < x-c < b$ or $c-b < x < c+b$. So if we write $0 < |x-c| < b$, it gives us an interval centered at c but not including c ,namely $(c-b,c) \cup (c,c+b)$. The distance from c on the number line is determined by "b".

5. Exercises

   (a) Solve for x. Write your answer in interval notation

   i. $|x-2| < 1$
   ii. $0 < |x-3| < 1$
   iii. $|x+5| < 1/2$
   iv. $|x-7| < 2$

   (b) Example If $|x-3| < 1$, then find the smallest k such that $|2-3x| < k$
   Solution $|x-3| < 1$ implies $-1 < x-3 < 1$ or by adding 3 to all entries in the inequality we have $2 < x < 4$. Once you have established a string of inequalities for x alone, then proceed to generate the required term by multiplying by appropriate quantities or by adding/subtracting the appropriate constants. In this case, multiply across by -3 , giving $-6 > -3x > -12$ where the direction of the inequality string has been reversed due to multiplication by a negative number. Rewriting , we have $-12 < -3x < -6$. Now add 2 to all terms, giving $-10 < 2-3x < -4$. Finally, we must put the inequality string in the form that will enable us to rewrite it as an absolute value inequality, namely $-10 < 2-3x < -4 < 10$ or $|2-3x| < 10$. Why is 10 the minimum value that satisfies the conditions of the problem? Clearly we could have written $-20 < -10 < 2-3x < -4 < 10 < 20$ leading to the absolute value inequality $|2-3x| < 20$. From this we conclude that any number greater than 10 can be used but that 10 is the smallest number that will satisfy the inequality.

   (c) If $|x-3| < 1$, then find the smallest k such that

   i. $|2x+3| < k$
   ii. $|3-5x| < k$
   iii. $|5x+7| < k$

(d) If $|x-5|<1$, then find the smallest k such that
   i. $|x-9| \ < k$
   ii. $|2-3x| \ < k$
   iii. $|4x-5| \ < k$

3 Limits: $\epsilon - \delta$ definition

6. Narative We include here a few comments on the definition of a limit. When we talk about a limit as x approaches c, it means that we are interested in what happens to the function as x gets close to c but we don't care what happens at $x=c$. In this definition, we must be able to be near "c", on either side of "c" but not necessarily at $x=c$. From item number 4,the statement $0<|x-c|<\delta$ does exactly that for this translates into $(c-\delta, c) \cup (c, c+\delta)$. This is an interval around c but not including c. The interval about c is determined by the positive quantity δ. When we say that the limit as x approaches c is L (a finite number) it means that as we get closer and closer to c, the value of f(x) gets closer to L . In some cases it may actually be equal to L near x = c. We measure the closeness to L by the positive quantity ϵ. This is similar to a tolerance when you go to the workshop and ask for a length of pipe 6 feet long. Usually they will ask how close to 6 feet do you want the pipe? Is 6ft ±3 inches close enough to 6 ft. for your needs or does it have to be ± .001 inch? How close do you want to be to the limiting value L. Once you settle on that question, you can calculate the interval around c in which $f(x)$ is within this tolerance. Note that in the definition, it states that for all challenges, that is for all ϵ, there is a response: there is an interval around c determined by the number δ such that $f(x)$ is within $\pm\ \epsilon$ of L whenever x is within $\pm\ \delta$ of c. When you calculate an answer to a $\epsilon - \delta$ problem, you will state either that δ is a positive constant for a linear function or that δ is dependent on ϵ for a non-linear function. Typical answers might be (a) $\delta = 2$; or (b) $\delta = \min(1, \epsilon/2)$

7. Question Give the $\epsilon - \delta$ definition of the statement

$$\lim_{x \to c} f(x) = L$$

   Answer Let f be a function defined on an open interval containing c

(except possibly at c itself) and let L be a real number. For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$.

8. Question Using the above definition, find a suitable delta which proves

$$\lim_{x \longrightarrow 3}(5x - 2) = 13$$

Answer **General Remarks** In this, and in similar problems, one successful method is to setup the inequality

$$|f(x) - L| \leq K|x - c| \tag{1}$$

where K is a positive constant. Then take $\delta = \epsilon/K$.
Proof The proof that this works is very simple. Start with

$$0 < |x - c| < \delta$$

or

$$0 < |x - c| < (\epsilon/K)$$

since $\delta = \epsilon/K$. Multiply by K (no difficulties since K is positive). Hence

$$0 < K|x - c| < \epsilon$$

From equation 1, you have just established that

$$|f(x) - L| \leq K|x - c|$$

Hence we can write

$$0 < |f(x) - L| \leq K|x - c| < \epsilon$$

or simply

$$|f(x) - L| < \epsilon$$

as required. We want to setup the inequality

$$|f(x) - L| \leq K|x - c|$$

or in our particular case

$$|5x - 2 - 13| \leq K|x - 3|$$

or

$$|5x - 15| \leq K|x - 3|$$

Start with the left hand side namely

$$|f(x) - L| = |5x - 15|$$

$$|f(x) - L| = 5|x - 3|$$

take K = 5 and hence $\delta = (\epsilon/5)$ This is the simplest form of this type of question. More complicated versions follow.

Remember in this work ϵ is the challenge and δ is the response. So if the challenge is .1, that is you want the function to be within .1 of the limit 13, then the x values must be within .1/5 or .02 of 3. If a given value of δ works, then any smaller value will also work. For example in the above calculation, any positive number smaller than .02 will also work.

9. Question Using the above definition, find a suitable delta which proves

$$\lim_{x \longrightarrow 5} (x^2 - 2) = 23$$

Answer As before we try to set up the inequality

$$|f(x) - L| \leq K|x - c|$$

or in our case

$$|x^2 - 2 - 23| \leq K|x - 5|$$

That is

$$|x^2 - 25| \leq K|x - 5|$$

As before, we start with the left hand side

$$|x^2 - 25| = |x + 5||x - 5|$$

Clearly part of the required expression is present, namely $|x - 5|$. However there is no constant "K" in front of this term. We want $|x + 5| \leq K$. We begin by placing a restriction on the allowable values x may assume. To do this we assign a value of 1 to δ; that is, we force $|x - 5| < 1$. Expanding the absolute value inequality, we have $-1 < x - 5 < 1$ or $4 < x < 6$. From this statement we deduce information about the term $|x+5|$, that is, $4 < x < 6$ becomes $9 < x+5 < 11$ or $-11 < 9 < x+5 < 11$ which leads to the statement $|x+5| < 11$ which is true iff $\delta \leq 1$. Note that with $\delta = 1$, x is restricted to between 4 and 6 in this problem. With a smaller value of δ, x is even more restricted and the statement $|x+5| < 11$ is still true. Returning to our problem, and once again starting with the left hand side of our aim

$$|f(x) - L| = |x^2 - 25|$$

or

$$|f(x) - L| = |x+5||x-5|$$

or since $|x+5| \leq 11$ for $\delta \leq 1$,

$$|f(x) - L| \leq 11|x-5|$$

Take K = 11 and hence $\delta = \epsilon/11$ However this is only true if $\delta \leq 1$ so we take δ as the minimum of 1 and $\epsilon/11$ written as $\delta = \min(1, \epsilon/11)$

10. Example Using the above definition, find a suitable delta which proves

$$\lim_{x \to -3} (2x^2 + 3x) = 9$$

Answer As before we try to set up the inequality

$$|f(x) - L| \leq K|x - c|$$

or in our case

$$|2x^2 + 3x - 9| \leq K|x+3|$$

where K is a positive constant. We start with the left hand side

$$\begin{aligned} |2x^2 + 3x - 9| &= |(2x-3)(x+3)| \\ &= |(2x-3)||(x+3)| \end{aligned}$$

Clearly part of the required expression is present, namely $|x+3|$. However there is no constant "K" in front of this term. We want $|2x-3| \leq K$. We begin by placing a restriction on the allowable values x may assume. To do this we assign a value of 1 to δ; that is, we force $|x+3| < 1$. Expanding the absolute value inequality, we have $-1 < x+3 < 1$ or $-4 < x < -2$. From this statement we deduce information about the term $|2x-3|$, that is, $-4 < x < -2$

becomes $-8 < 2x < -4$; $-11 < 2x - 3 < -7$ or $-11 < 2x - 3 < -7 < 11$. Hence $-11 < 2x - 3 < 11$ which leads to the statement $|2x - 3| < 11$. This is true iff $\delta \leq 1$. Note that with $\delta = 1$, x is restricted to between -4 and -2 in this problem. With a smaller value of δ, x is even more restricted and the statement $|2x - 3| < 11$ is still true. Returning to our problem, and once again starting with the left hand side of our aim

$$\begin{aligned} |f(x) - L| &= |2x^2 + 3x - 9| \\ |f(x) - L| &= |2x - 3||x + 3| \\ |f(x) - L| &\leq 11|x + 3| \text{ since } |2x - 3| \leq 11 \text{ for } \delta \leq 1 \end{aligned}$$

Take K = 11 and hence $\delta = \epsilon/11$ However this is only true if $\delta \leq 1$ so we take δ as the minimum of 1 and $\epsilon/11$ written as $\delta = \min(1, \epsilon/11)$

Also note that if we begin with the statement $-11 < 2x - 3 < 11$,we could modify it to become $-20 < -11 < 2x - 3 < 11 < 20$ which would lead to the statement $|2x - 3| < 20$ which in turn leads to a K of 20 giving an answer of $\delta = \min(1, \epsilon/20)$. This work demonstrates that if a given value of δ works, then any smaller value of δ will also work.

This is an answer to the problem. It is not the only answer. In this case the choice of $\delta = 1$ was quite arbitrary. An equally valid choice could be $\delta = 2$ or in fact $\delta =$ any positive number. When this happens, the most common choice is 1. However this choice won't always work. Consider the next problem.

11. Question Using the above definition,find a suitable δ which proves

$$\lim_{x \longrightarrow 1} \left(\frac{1}{x}\right) = 1$$

Answer As before we try to set up the inequality

$$|f(x) - L| \leq K|x - c|$$

or in our case

$$\left|\frac{1}{x} - 1\right| \leq K|x - 1|$$

That is

$$\left|\frac{1 - x}{x}\right| \leq K|x - 1|$$

As before, we start with the left hand side

$$\begin{aligned}\left|\frac{1-x}{x}\right| &= \frac{|1-x|}{|x|}\\ &= \frac{|x-1|}{|x|}\end{aligned}$$

We need to place a bound on $1/|x|$, that is, we want to be able to write $1/|x| \leq$ a constant, K. If we take $\delta = 1$ then $|x-1| < \delta$ becomes $|x-1| < 1$ or $-1 < x-1 < 1$ which leads to $0 < x < 2$. Clearly if we try to get a reciprocal from this expression,we encounter very serious trouble. We might try to write $\infty > 1/x > 1/2$ which is telling us that 1/x is not bounded above. Try $\delta = 1/2$ Eventually this will lead to the statement $1/|x| < 2$ and hence we will take $\delta = \min(1/2, \epsilon/2)$ We leave you to fill in the details. If you have trouble ask your T.A in the problem session.

12. Question In some other course, you might be asked to find a suitable δ to prove

$$\lim_{x \longrightarrow a} \frac{1}{x} = \frac{1}{a}$$

What value of δ would you choose in order to put a bound on $1/|x|$. Again ask your T.A. for help if you cannot see the solution.

13. <u>Exercises</u> Find a suitable δ which proves the following limits.

(a) $\lim_{x \longrightarrow -3} 2 - 5x = 17$

(b) $\lim_{x \longrightarrow 7} 5 - 2x = -9$

(c) $\lim_{x \longrightarrow 4} 6x + 2 = 26$

(d) $\lim_{x \longrightarrow 6} 2x^2 - 5x + 3 = 45$

(e) $\lim_{x \longrightarrow 3} \frac{1}{x} = \frac{1}{3}$

(f) $\lim_{x \longrightarrow -2} x^2 + 6x = -8$

(g) $\lim_{x \longrightarrow 2} 5 - x^2 + 6x = 13$

(h) $\lim_{x \longrightarrow -1} 5x - x^2 = -6$

(i) $\lim_{x \to -2} 2x - x^2 + 5 = -3$

(j) $\lim_{x \to -3} 2x^2 + 5x - 1 = 2$

4 Limit Rules; $\frac{0}{0}$ Problem

You are encouraged to read carefully and understand the limit theorems listed in §1.3 of your textbook. An abbreviated summary is presented in these notes. These rules may be proved using the $\epsilon - \delta$ definition of a limit.
Let b,c, be any real number, let n be a positive integer and $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$

$$\lim_{x \to c} b = b \tag{2}$$

$$\lim_{x \to c} x = c \tag{3}$$

$$\lim_{x \to c} x^n = c^n \tag{4}$$

Scalar Multiple

$$\lim_{x \to c} [bf(x)] = b \lim_{x \to c} f(x) \tag{5}$$

$$\text{or} \lim_{x \to c} [bf(x)] = bL \tag{6}$$

Sum/Difference

$$\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) \tag{7}$$

$$\text{or } \lim_{x \to c} [f(x) \pm g(x)] = L \pm K \tag{8}$$

Product

$$\lim_{x \to c} [f(x)g(x)] = \lim_{x \to c} f(x) \lim_{x \to c} g(x) \tag{9}$$

$$\text{or } \lim_{x \to c} [f(x)g(x)] = LK \tag{10}$$

Quotient

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} \tag{11}$$

$$\text{or } \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K} \text{ Provided } K \neq 0 \tag{12}$$

Power

$$\lim_{x \to c} [f(x)]^n = [\lim_{x \to c} f(x)]^n \tag{13}$$

$$\text{or} \lim_{x \to c} [f(x)]^n = L^n \tag{14}$$

Radical, $n > 0$, and odd

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c} \tag{15}$$

or $n > 0$, and even, $c > 0$

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c} \tag{16}$$

It follows from the application of these rules that if P(x) is a polynomial, then $\lim_{x \to c} P(x) = P(c)$. It also follows that if $R(x) = \frac{P(x)}{Q(x)}$ where P(x) and

Q(x) are polynomials in x, then

$$\lim_{x\to c} R(x) = \frac{\lim\limits_{x\to c} P(x)}{\lim\limits_{x\to c} Q(x)} = \frac{P(c)}{Q(c)} \text{ provided } Q(c) \neq 0$$

It is important to be able to find trigonometric limits. Let c be any real number in the domain of the stated trigonometric function. Then the following limit theorems apply

$$\begin{array}{rclrcl}
\lim\limits_{x\to c} \sin x &=& \sin c & \lim\limits_{x\to c} \cos x &=& \cos c \\
\lim\limits_{x\to c} \tan x &=& \tan c & \lim\limits_{x\to c} \cot x &=& \cot c \\
\lim\limits_{x\to c} \sec x &=& \sec c & \lim\limits_{x\to c} \csc x &=& \csc c
\end{array}$$

The domain of the sine and cosine functions is all real numbers. However, for the other functions, there are undefined points, For example, there is a difficulty at $\pi/2$ for $\tan x$. See section 6 on page 26 for a discussion of the description of the functional behaviour at these undefined points,

14. Example Evaluate $\lim\limits_{x\to 2}(x^3 - 2x^2 + 7)$

    Answer $\quad \lim\limits_{x\to 2}(x^3 - 2x^2 + 7) = 8 - 8 + 7 = 7$

15. Example Evaluate $\lim\limits_{x\to\pi/2} \sin x - \cos 2x$

    Answer $\quad \lim\limits_{x\to\pi/2} \sin x - \cos 2x = 1 - (-1) = 2$

16. Example Evaluate $\lim\limits_{x\to 2} \dfrac{x+2}{x^2+3x+5}$

    Answer $\quad \lim\limits_{x\to 2} \dfrac{x+2}{x^2+3x+5} = \dfrac{\lim\limits_{x\to 2} x+2}{\lim\limits_{x\to 2} x^2+3x+5} = \dfrac{4}{15}$

17. Example Evaluate

$$\lim_{x\to -2} \frac{x+2}{x^2-4}$$

Answer

$$\begin{aligned} &\lim_{x\to-2}\frac{x+2}{x^2-4} \\ =&\lim_{x\to-2}\frac{x+2}{(x+2)(x-2)} \\ =&\lim_{x\to-2}\frac{1}{x-2}=-\frac{1}{4} \end{aligned}$$

In example 16, we were able to use the limit rules discussed in §1.3 of your text book. However, in exercise 17,we could not apply those limit rules. Exercise 17 is an example of a $\frac{0}{0}$ indeterminate form. Another type of indeterminate form $\frac{\infty}{\infty}$ is discussed in §3.5 of your text book. Indeterminate forms are discussed later in this course, §8.7 of your text book. The answer to an indeterminate form depends on the functions involved. In general, the approach to solving an indeterminate form problem depends not only on the type involved but also on the functions in the exercise. For the present, we will focus our attention on a $\frac{0}{0}$type.

$$\text{Consider } \lim_{x\to c}\frac{f(x)}{g(x)} \qquad \frac{0}{0}$$

(a) If f(x) and g(x) are polynomials, then $(x-c)$ is a factor. This is guaranteed by a theorem from your algebra days viz. , that if P(a) = 0 then $(x-a)$ is a factor of the polynomial P(x). The cancellation of the common factor of $(x-c)$ will often lead to the solution of the problem because it most likely will remove the $\frac{0}{0}$ difficulty. This approach is justified by Theorem 1.7 on page 62 of your text book. Exercise 17 is an illustration of this method.

(b) If f(x) or g(x) contain radicals, rationalization of either the numerator or denominator is probably the best approach. Example 18 illustrates this type of problem. The rationalization process leads to a common factor which may be cancelled thus reducing the problem to one similar to that considered above.

(c) If f(x) and g(x) are trigonometric functions, then several approaches are available

  i. Use trigonometric identities

    ii. Use known limit $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ which is proved in §1.3 of your text book
    iii. Use known limit $\lim\limits_{x\to o} \dfrac{1-\cos x}{x} = 0$ which is proved in §1.3 of your text book

18. Example Evaluate

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}$$

Answer

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{(x-3)} \qquad\qquad \frac{0}{0}$$

$$= \lim_{x\to 3} \frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)}{(x-3)(\sqrt{x+1}+2}$$

$$= \lim_{x\to 3} \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3))\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2}$$

$$= \frac{1}{4}$$

19.

20. Example Evaluate $\lim\limits_{x\to 0} \dfrac{\sin 7x}{x}$

Introduce a new variable ”t” such that $t = 7x$. $t \to 0$ as $x \to 0$. Rewriting the problem in terms of the variable ”t”, we have
$\lim\limits_{x\to 0} \dfrac{\sin 7x}{x} = \lim\limits_{t\to 0} \dfrac{\sin t}{t/7} = 7 \lim\limits_{t\to 0} \dfrac{\sin t}{t} = 7$

21. Example Evaluate

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 3x}$$

Answer This is an example of a trigonometric $\frac{0}{0}$ problem. Clearly we must use our known limit involving the sine function namely $\lim_{x\to 0}\frac{\sin x}{x} = 1$ This answer is valid provided the argument of the sine function is the same as the denominator. For example

$$\lim_{x\to 0}\frac{\sin 5x}{5x} = 1$$

If this is not the case, then multiplying numerator and denominator by an appropriate factor is necessary.

$$\lim_{x\to 0}\frac{\sin 7x}{x} = 7\lim_{x\to 0}\frac{\sin 7x}{7x} = 7$$

In exercise 21 we need to multiply numerator and denominator by both 5x and 3x.

$$\lim_{x\to 0}\frac{\sin 5x}{\sin 3x}$$

$$= \lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)\left(\frac{5x}{3x}\right)\left(\frac{3x}{\sin 3x}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)\left(\frac{5x}{3x}\right)\frac{1}{\left(\frac{\sin 3x}{3x}\right)}$$

$$= \lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)\lim_{x\to 0}\left(\frac{5x}{3x}\right)\lim_{x\to 0}\frac{1}{\left(\frac{\sin 3x}{3x}\right)}$$

$$= \frac{5}{3}$$

22. Example Evaluate

$$\lim_{x\to \pi/4}\frac{1-\tan x}{\sin x - \cos x}$$

Answer

$$\lim_{x\to \pi/4}\frac{1-\frac{\sin x}{\cos x}}{\sin x - \cos x}$$

$$\lim_{x\to \pi/4}\frac{\cos x - \sin x}{\cos x(\sin x - \cos x)} = \lim_{x\to \pi/4}\frac{-1}{\cos x} = -\sqrt{2}$$

23. Exercises Evaluate the following limits.

(a) $\lim_{x\to 0} \dfrac{1-\cos 3x}{x}$

(b) $\lim_{x\to 0} \dfrac{\sqrt{x+5}-\sqrt{5}}{x}$

(c) $\lim_{x\to -2} \dfrac{x^2+3x+5}{x^2-7x+12}$

(d) $\lim_{x\to \pi/4} \dfrac{\mathrm{cosec}^2\mathrm{x}}{\sin x+\tan x}$

(e) $\lim_{x\to 0} \dfrac{\sin 4x}{\tan 2x}$

(f) $\lim_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x^2}$

(g) $\lim_{x\to 2} \dfrac{\dfrac{1}{x+3}-\dfrac{1}{5}}{x-2}$

(h) $\lim_{x\to 0} \dfrac{\dfrac{1}{x+7}-\dfrac{1}{7}}{x}$

(i) $\lim_{x\to 1} \dfrac{\sin(x^2-1)}{x-1}$

5 Squeeze Theorem,Continuity, Intermediate Value Theorem

24. Question State the Squeeze Theorem for a limit at a point. Illustrate your answer with a simple sketch.(Figure 2)
Answer If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\longrightarrow c} h(x) = L = \lim_{x\longrightarrow c} g(x)$$

then $\lim_{x\longrightarrow c} f(x)$ exists and is equal to L.

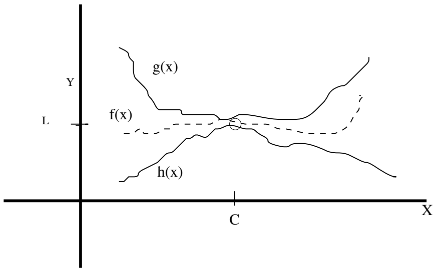

Figure 2: Illustrating the Squeeze Theorem for a limit at a point

25. Question Use a Squeeze Theorem to find $\lim_{x \longrightarrow 0} x^2 \sin\left(\frac{1}{x}\right)$

    Answer

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \text{ for } (-\infty, 0) \cup (0, \infty)$$

$$\text{That is } -x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Since $\lim_{x \longrightarrow 0} -x^2 = 0$ and $\lim_{x \longrightarrow 0} x^2 = 0$, it follows from the Squeeze Theorem that $\lim_{x \longrightarrow 0} x^2 \sin(\frac{1}{x}) = 0$

   (a) Exercise If possible use Matlab to draw a graph of $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ for $-0.4 \leq x < 0 \cup 0 < x \leq 0.4$
   Also show graphs of $h(x) = -x^2$ and $g(x) = x^2$ on the same set of axes.

   (b) Exercise For the same f(x) defined above, find the x intercepts.

   (c) Exercise For the same f(x) defined above, if possible use Matlab to draw a graph for $0.1 \leq x \leq 10$. On the same set of axes, draw a graph of $y = x$

   (d) Exercise Generate a table of values for f(x) over the interval $0.1 \leq x \leq 5$ What conclusion might you reach about the value of $x^2 \sin(\frac{1}{x})$ as x increases without bound.

26. Question Define the term " Continuity at a Point x equal c "
    Answer A function is continuous at a point x equal c if the following three conditions are met.

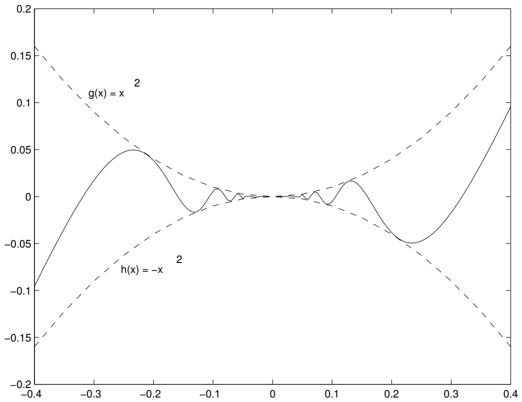

Figure 3: Graph of $f(x) = x^2 \sin(1/x)$

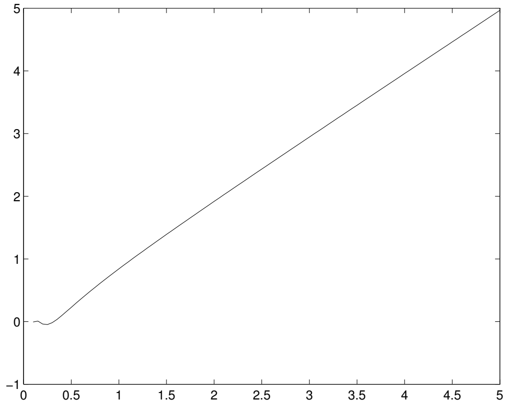

Figure 4: Graph of $f(x) = x^2 \sin(1/x), .1 \leq x \leq 5$

(a) f(c) is defined

(b) $\lim_{x \longrightarrow c} f(x)$ exists

(c) $\lim_{x \longrightarrow c} f(x) =$ f(c)

Note A function is continuous on an open interval (a,b) if it is continuous at each point in the interval.

27. Question Give the $\epsilon - \delta$ definition of the statement

$$\lim_{x \longrightarrow c^-} f(x) = L$$

This statement is referred to as a <u>left hand limit</u>

<u>Answer</u> Let f be defined in some open interval to the left of c on the number line. For <u>all</u> $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $c - \delta < x < c$

28. <u>Question</u> Give the $\epsilon - \delta$ definition of the statement

$$\lim_{x \longrightarrow c^+} f(x) = L$$

This statement is referred to as a <u>right hand limit.</u>

<u>Answer</u> Let f be defined on some open interval to the right of c on the number line. For <u>all</u> $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $c < x < c + \delta$

<u>Definition</u> Let f be a function and let c and L be real numbers.

$$\lim_{x \longrightarrow c} f(x) = L \text{ iff } \lim_{x \longrightarrow c^-} f(x) = L \text{ and } \lim_{x \longrightarrow c^+} f(x) = L$$

29. <u>Question</u> Give a definition of Continuity on a Closed Interval

<u>Answer</u> A function is continuous on a closed interval [a,b] if it is continuous on the open interval (a,b) and

$$\lim_{x \longrightarrow a^+} f(x) = f(a) \text{ and } \lim_{x \longrightarrow b^-} f(x) = f(b)$$

30. (a) <u>Exercises</u> Is f(x) continuous at x = -1 and is f(x) continuous at x = 3? Carefully explain your answers.

$$f(x) = \begin{cases} 2 & \text{if } x \leq -1; \\ -x + 1 & \text{if } -1 < x < 3; \\ -2 & \text{if } x \geq 3; \end{cases}$$

(b) Evaluate the limit $\lim\limits_{x \to 7^-} \dfrac{|8x - 56|}{7 - x}$

(c) Evaluate the limit $\lim\limits_{x \to 5^+} \dfrac{|3x - 15|}{5 - x}$

(d) Find the x values (if any) for which f is not continuous.

$$f(x) = \begin{cases} 3x + 2, & x < -1 \\ 2x^2 - 3x + 6, & x > -1 \\ 3, & x = -1 \end{cases}$$

If there is a discontinuity, decide whether it is an essential or a removable discontinuity.

31. Question State the Intermediate Value Theorem
Answer If f is continuous on the closed interval [a,b] and k is any number between f(a) and f(b) then there is at least one number c in [a,b] such that f(c) = k
NOTE The next Theorem follows from the Intermediate Value Theorem

32. Theorem If f is continuous on the closed interval [a,b] and if f(a) and f(b) have opposite signs that is if $f(a) > 0$ and $f(b) < 0$ or if $f(a) < 0$ and $f(b) > 0$ then there is at least one number c in (a,b) such that
f(c) = 0 , that is there is a solution to the equation f(x) = 0 in the interval (a,b)

    (a) Exercise The graph of $f(x) = 2x - \tan x$ is shown below (Figure 5).Clearly f(1) is positive and f(1.3) is negative. From the above theorem it follows that somewhere between x = 1 and x = 1.3 there is a solution to the equation $2x - \tan x = 0$. Find an approximation correct to 4 decimal places. Hint: Start by finding $f(1.15)$. This will determine whether the solution is between 1 and 1.15 or between 1.15 and 1.3.
    Answer 1.1656

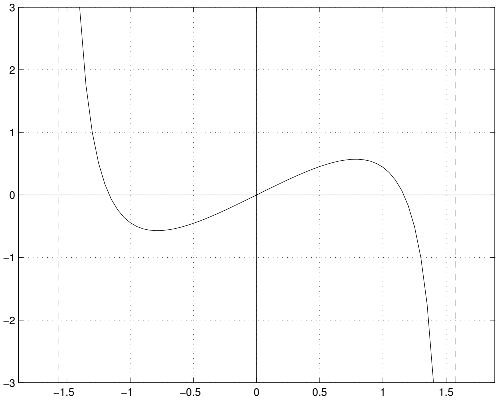

Figure 5: Graph of $f(x) = 2x - \tan x$

6 Infinite Limits,Vertical Asymptotes

33. Question Give the definition of the statement

$$\lim_{x \longrightarrow c} f(x) = \infty$$

Answer Let f be defined in some open interval containing the number "c" except that f need not be defined at c . We write

$$\lim_{x \longrightarrow c} f(x) = \infty$$

if for all $N > 0$, there exists a $\delta > 0$ such that $f(x) > N$ whenever $0 < |x-c| < \delta$

Comment: This definition implies that f increases without bound as x approaches c from either side of c on the number line.

34. Question Using the above definition, find a suitable N which proves

$$\lim_{x \longrightarrow 1} \frac{1}{(x-1)^2} = \infty$$

Answer We require $f(x) > N$ whenever $0 < |x-1| < \delta$

or $\dfrac{1}{(x-1)^2} > N$ whenever $0 < |x-1| < \delta$

That is $(x-1)^2 < \dfrac{1}{N}$ or $\sqrt{(x-1)^2} < \dfrac{1}{\sqrt{N}}$ Hence $|x-1| < \dfrac{1}{\sqrt{N}}$.

Take $\delta = \dfrac{1}{\sqrt{N}}$. Try to check your answer. Remember that N is the challenge and δ is the response. Take N = 50 , $\delta = 0.14142$.

Hence $.85858 < x < 1.14142$ f(1.1) = 100, f(1.13) = 59 and f(1.14) = 51.02 all of which are greater than 50 as required.

35. Question Give the definition of the statement

$$\lim_{x \to c^-} f(x) = \infty$$

Answer Let f(x) be defined in some open interval to the left of c on the number line. We write

$$\lim_{x \to c^-} f(x) = \infty$$

if for all $N > 0$ there exists a $\delta > 0$ such that $f(x) > N$ whenever $c-\delta < x < c$. Comment. This definition indicates that f increases without bound as x approaches c from the left of c (that is below c) on the number line. Obviously there are other definitions along these lines. Try the following exercises and check your answers with your T.A

(a) Exercise Give the definition of the statement

$$\lim_{x \to c^+} f(x) = -\infty$$

(b) Exercise Give the definition of the statement

$$\lim_{x \to c^-} f(x) = -\infty$$

36. Question Give the definition of the statement

$$\lim_{x \longrightarrow c} f(x) = -\infty$$

Answer Let f be defined in some open interval containing the number "c" except that f need not be defined at c . We write

$$\lim_{x \longrightarrow c} f(x) = -\infty$$

if for all $N > 0$, there exists a $\delta > 0$ such that $f(x) < -N$ whenever $0 < |x-c| < \delta$
Comment: This definition implies that f decreases without bound as x approaches c from either side of c on the number line.

37. Question Give the definition of the statement Vertical Asymptote
Answer If any one of the conditions listed below are met, then the line x = c is a vertical asymptote for the graph of y = f(x)

$$\lim_{x \longrightarrow c} f(x) = \infty \qquad \lim_{x \longrightarrow c^+} f(x) = \infty \qquad \lim_{x \longrightarrow c^-} f(x) = \infty$$

$$\lim_{x \longrightarrow c} f(x) = -\infty \quad \lim_{x \longrightarrow c^+} f(x) = -\infty \quad \lim_{x \longrightarrow c^-} f(x) = -\infty$$

Note The graph of f cannot cross a vertical asymptote, that is you cannot plot a point (c, ∞) . An examination of Figure 6 shows that $\lim_{x \to \pi/2} \tan x$ does not

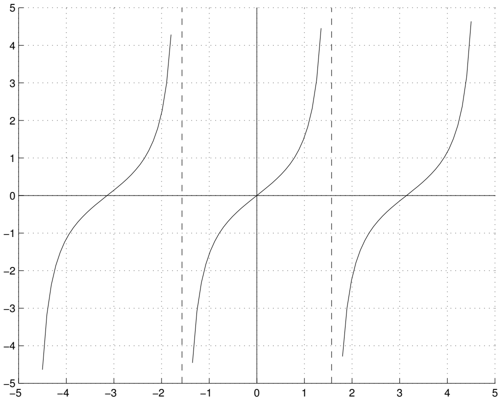

Figure 6: Graph of $f(x) = \tan x$,dashed lines $x = \pm\pi/2$

exist. Neither does $\lim\limits_{x\to\pi/2^-}$ nor $\lim\limits_{x\to\pi/2^+}$ exist. However, in the latter two cases , we can write down a complete description of their non existence, namely

$$\boxed{\lim_{x\to\pi/2^-} = \infty \text{ and } \lim_{x\to\pi/2^+} = -\infty}$$

The line $x = \pi/2$ is a vertical asymptote for the graph $f(x) = \tan x$

38. Exercise Examine the following functions for vertical asymptotes.

    (a) $f(x) = \dfrac{x^2+1}{x^2-1}$

    (b) $f(x) = \tan x; \quad -2\pi \le x \le 2\pi$

    (c) f(x) = sec x ; $-2\pi \le x \le 2\pi$

    (d) $f(x) = \dfrac{x^2+8x+15}{x^2+3x-10}$

39. Exercise:Discuss the continuity of the following functions. Indicate whether a discontinuity is an essential discontinuity or a removable discontinuity. Describe the type of discontinuity, for example, oscillatory discontinuity, jump discontinuity,hole, or an infinite discontinuity.

    (a) $f(x) = \sin\left(\dfrac{1}{x}\right)$

    (b) $f(x) = \dfrac{\sin x}{x}$

(c) $f(x) = \dfrac{x^2-1}{x-1}$

(d) $f(x) = \dfrac{x}{(x-1)^2}$

(e) $f(x) = \begin{cases} 2x+5 & \text{when } x \geq 2 \\ 3x-1 & \text{when } x < 2 \end{cases}$

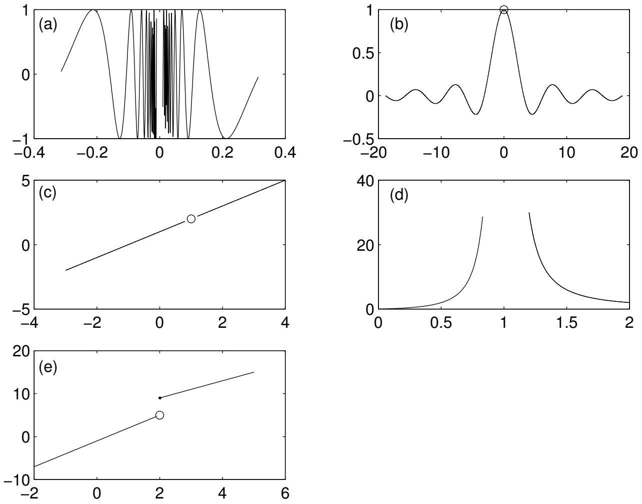

Figure 7: Graphs of various types of discontinuities.(a) $f(x) = \sin(1/x)$ (b) $f(x) = \dfrac{\sin x}{x}$ (c)$f(x) = \dfrac{x^2-1}{x-1}$ (d) $f(x) = \dfrac{x}{(x-1)^2}$ (e) $f(x) = 2x+\ 5$ when $x \geq 2$ and $f(x) = 3x-1$ when $x < 2$

7 Limits at infinity,Horizontal Asymptotes

40. Question Give the definition of the statement

$$\lim_{x \longrightarrow \infty} f(x) = L$$

Answer Let f be defined in some open interval (c, ∞). We write

$$\lim_{x \longrightarrow \infty} f(x) = L$$

if for all $\epsilon > 0$, there exists a $N > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > N > c$

41. <u>Question</u> Give the definition of the statement

$$\lim_{x \longrightarrow -\infty} f(x) = L$$

<u>Answer</u> Let f be defined in some open interval $(-\infty, c)$. We write

$$\lim_{x \longrightarrow -\infty} f(x) = L$$

if for <u>all</u> $\epsilon > 0$, there exists a $N > 0$ such that $|f(x) - L| < \epsilon$ whenever $x < -N < c$

42. <u>Question</u> Use one of the above definitions to find a suitable N which proves

$$\lim_{x \longrightarrow \infty} \frac{2x}{x-3} = 2$$

We want $|f(x) - L| < \epsilon$ whenever $x > N$ or in our particular case,

$$\left| \frac{2x}{x-3} - 2 \right| < \epsilon \text{ whenever } x > N$$

$$\text{That is } \left| \frac{2x - 2x + 6}{x-3} \right| < 3 \text{ whenever } x > N$$

$$\left| \frac{6}{x-3} \right| < \epsilon \text{ whenever } x > N.$$

Take $N_1 = 3$. For $x > N_1, x - 3 > 0$ and hence

$$\frac{6}{x-3} < \epsilon \text{ for } x > N_1$$

$$\frac{x-3}{6} > \frac{1}{\epsilon} \text{ for } x > N_1$$

$$x - 3 > \frac{6}{\epsilon} \text{ or } x > 3 + \frac{6}{\epsilon}$$

Take $N_2 = 3 + \frac{6}{\epsilon}$ and $N = \max(N_1, N_2) = N_2$ that is $N = 3 + \frac{6}{\epsilon}$

Test your result. If $\epsilon = 0.1, N = 63$: take $x = 64$ and $\frac{2x}{x-3}$ equals 2.098 which is within the stated tolerance of 2 ± 0.1

(a) Exercise Use one of the above definitions to find a suitable N which proves

$$\lim_{x \longrightarrow -\infty} \frac{2x}{x-3} = 2$$

Note : In solving a problem of the type

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)}$$

where P(x) and Q(x) are polynomials or polynomials under a radical sign, try to get reciprocals such as

$$\frac{1}{x}, \frac{1}{x^2},$$

etc. since the

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \text{ if n is a positive integer}$$

The same comment applies to

$$\lim_{x\to-\infty} \frac{1}{x^n}$$

43. Example Evaluate

$$\lim_{x\to\infty} \frac{x^5 - 2x^3 + 39}{2x^5 + 3x^2 - 21}$$

$$= \lim_{x\to\infty} \frac{x^5(1 - \frac{2}{x^2} + \frac{39}{x^5})}{x^5(2 + \frac{3}{x^3} - \frac{21}{x^5})}$$

$$= \lim_{x\to\infty} \frac{(1 - \frac{2}{x^2} + \frac{39}{x^5})}{(2 + \frac{3}{x^3} - \frac{21}{x^5})}$$

$$= \frac{1}{2}$$

44. Example Evaluate

$$\lim_{x\to\infty} \frac{x^5 + 2x^3 + 39}{3x^7 + x^2 + 35}$$

$$= \lim_{x\to\infty} \frac{x^5(1 + \frac{2}{x^2} + \frac{39}{x^5})}{x^5(3x^2 + \frac{1}{x^3} + \frac{35}{x^5})}$$

$$= \lim_{x\to\infty} \frac{(1 + \frac{2}{x^2} + \frac{39}{x^5})}{(3x^2 + \frac{1}{x^3} + \frac{35}{x^5})}$$

$$= 0$$

45. <u>Example</u> Evaluate

$$\lim_{x\to\infty} \frac{2x^5 + x^2 - 31}{3x^3 + x + 25}$$

$$= \lim_{x\to\infty} \frac{x^3(2x^2 + \frac{1}{x} - \frac{31}{x^3})}{x^3(3 + \frac{1}{x^2} + \frac{25}{x^3})}$$

$$= \lim_{x\to\infty} \frac{(2x^2 + \frac{1}{x} - \frac{31}{x^3})}{(3 + \frac{1}{x^2} + \frac{25}{x^3})}$$

$$= \infty$$

46. <u>Example</u> Evaluate

$$\lim_{x\to\infty} \frac{\sqrt{2x^2 + x}}{x + 5}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2(2 + \frac{1}{x})}}{x(1 + \frac{5}{x})}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2 + \frac{1}{x}}}{x(1 + \frac{5}{x})}$$

$$= \lim_{x\to\infty} \frac{|x|\sqrt{2+\frac{1}{x}}}{x(1+\frac{5}{x})}$$

$$= \lim_{x\to\infty} \frac{x\sqrt{2+\frac{1}{x}}}{x(1+\frac{5}{x})}$$

$$= \lim_{x\to\infty} \frac{\sqrt{2+\frac{1}{x}}}{(1+\frac{5}{x})}$$

$$= \sqrt{2}$$

Note $\sqrt{x^2} = |x|$ Remember that $\sqrt{x^2}$ means the principal square root and is positive. Since the sign of x is unknown, we can guarantee a positive answer by using the absolute value of x. Also recall the definition of $|x|$ namely

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

So in the above example, since we were interested in

$$\lim_{x\to\infty}$$

we can restrict ourselves to positive values of x and hence

$$\sqrt{x^2} = |x| = x$$

47. Example Evaluate

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+x}}{x+5}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2(2+\frac{1}{x})}}{x(1+\frac{5}{x})}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{2+\dfrac{1}{x}}}{x(1+\dfrac{5}{x})}$$

$$= \lim_{x\to-\infty} \frac{|x|\sqrt{2+\dfrac{1}{x}}}{x(1+\dfrac{5}{x})}$$

$$= \lim_{x\to-\infty} \frac{-x\sqrt{2+\dfrac{1}{x}}}{x(1+\dfrac{5}{x})}$$

$$= \lim_{x\to-\infty} \frac{-\sqrt{2+\dfrac{1}{x}}}{(1+\dfrac{5}{x})}$$

$$= -\sqrt{2}$$

<u>Note</u>: In this example since we were interested in

$$\lim_{x\to-\infty}$$

we can restrict ourselves to negative values of x and hence

$$\sqrt{x^2} = |x| = -x$$

48. <u>Question</u> State the Squeeze Theorem for limits at infinity and illustrate with a diagram. (Figure 8)
<u>Answer</u> If $h(x) \leq f(x) \leq g(x)$ for all $x > k$ and if

$$\lim_{x\longrightarrow\infty} h(x) = L = \lim_{x\longrightarrow\infty} g(x)$$

then $\lim_{x\longrightarrow\infty} f(x)$ exists and is equal to L.

   (a) <u>Exercise</u> Give a similar definition for limits to negative infinity

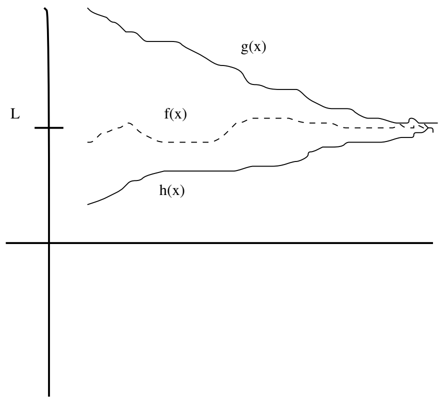

Figure 8: Illustrating the Squeeze Theorem for limits at infinity

49. Question State the definition of the term ”Horizontal Asymptote”
Answer The line y = L is a horizontal asymptote of the graph of f if

$$\lim_{x\longrightarrow\infty} f(x) = L \text{ or } \lim_{x\longrightarrow -\infty} = L$$

NOTE Unlike a vertical asymptote,the graph of f may cross a horizontal asymptote.

50. Example Examine

$$f(x) = \frac{\sqrt{2x^2+x}}{x+5}$$

for horizontal and vertical asymptotes.
Answer To find horizontal asymptotes, we must consider both

$$\lim_{x\to\infty} f(x) \text{ and } \lim_{x\to -\infty} f(x)$$

Since we have already done this in examples 46 and 47 the horizontal asymptotes are $y = \sqrt{2}$ and $y = -\sqrt{2}$. For a vertical asymptote, $x = -5$ would appear to be a good candidate. (numerator non-zero, denominator zero) but we must be certain that -5 is an open end point in the description of the domain of the function. In this case,it is since the domain is $(-\infty,-5)\bigcup(-5,-1/2]\bigcup[0,\infty)$ However,if the function had been

$$g(x) = \frac{\sqrt{2x^2+12x}}{x+5}$$

the domain would be $(-\infty, -6] \bigcup [0, \infty)$ and there would be no vertical asymptote for that function

(a) Exercise

  i. Establish the domains mentioned in exercise 50 for f(x) and g(x)

  ii. If it is available to you, use Matlab to graph f(x) and g(x)

(b) Exercise Examine the following functions for horizontal asymptotes

  i. $f(x) = \arctan x$

  ii. $f(x) = \dfrac{2x}{\sqrt{x^2+1}}$

  iii. f(x) = arcsec x

  iv. $f(x) = \dfrac{5}{2x-1}$

(c) Exercise Do any of the above functions have vertical asymptotes?

8 Definition of the derivative of a function

51. Question Give the Δ definition of the derivative of a function f.
Answer The **derivative** of f at x,$f'(x)$ is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided the limit exists.
Comment Remember that the phrase "provided the limit exists " means that you get the same answer as you allow Δx to approach 0 from the left and the right and the answer that you get is a finite number. When these conditions are satisfied, we say that f is **differentiable**.

52. Question. Using the above definition,find $f'(x)$ where
$f(x) = 2x^2 - 5x + 7$
Answer

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{[2(x+\Delta x)^2 - 5(x+\Delta x) + 7] - [2x^2 - 5x + 7]}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{[2x^2 + 4x\Delta x + 2(\Delta x)^2 - 5x - 5\Delta x + 7] - [2x^2 - 5x + 7]}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{4x\Delta x + 2(\Delta x)^2 - 5\Delta x}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} 4x + 2\Delta x - 5$$

$$f'(x) = 4x - 5$$

53. <u>Question</u>. Using the above definition, find $f'(x)$ where

$$f(x) = \frac{2}{3x - 4}$$

<u>Answer</u>

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\dfrac{2}{3(x + \Delta x) - 4} - \dfrac{2}{3x - 4}}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\dfrac{2[3x - 4] - 2[3(x + \Delta x) - 4]}{[3(x + \Delta x) - 4][3x - 4]}}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\dfrac{(6x - 8) - (6x + 6\Delta x) - 8)}{[3(x + \Delta x) - 4][3x - 4]}}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{-6\Delta x}{\Delta x[3(x + \Delta x) - 4][3x - 4]}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{-6}{[3(x + \Delta x) - 4][3x - 4]}$$

$$f'(x) = \frac{-6}{(3x - 4)^2}$$

54. Question. Using the above definition,find $f'(x)$ where $f(x) = \sqrt{x}$
Answer

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(\sqrt{x+\Delta x} - \sqrt{x})}{\Delta x} \frac{(\sqrt{x+\Delta x} + \sqrt{x})}{(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{1}{(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

55. Exercises

(a) Using the above definition, find $f'(x)$ where
$f(x) = 4 - 2x - 3x^2$

(b) Using the above definition,find $f'(x)$ where $f(x) = \frac{1}{3x+2}$

(c) Using the above definition,find $f'(x)$ where $f(x) = \frac{1}{\sqrt{2x+1}}$

(d) Using the above definition,find $f'(x)$ where $f(x) = \sqrt{5x+2}$

(e) Using the above definition,find $f'(x)$ where
$f(x) = x^3 - 5x^2 + x - 8$

(f) Using the above definition,find $f'(x)$ where $f(x) = \sqrt{1-3x}$

(g) Using the above definition,find $f'(x)$ where $f(x) = 5x^2 - 7x + 3$

56. Question Using the above definition, find $f'(x)$ where $f(x) = \sin x$
Answer

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x}$$

Using the identity

$$\sin A - \sin B = 2 \cos\left(\frac{A + B}{2}\right) \sin\left(\frac{A - B}{2}\right)$$

$$\sin(x + \Delta x) - \sin(x) \text{ becomes } 2\cos\left(x + \frac{\Delta x}{2}\right) \sin\left(\frac{\Delta x}{2}\right)$$

$$\text{Hence } f'(x) = \lim_{\Delta x \to 0} \frac{2\cos\left(x + \frac{\Delta x}{2}\right) \sin(\frac{\Delta x}{2})}{\Delta x}$$

$$\text{or } f'(x) = \lim_{\Delta x \to 0} \frac{\cos(x + \frac{\Delta x}{2})}{1} \frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cos(x + \frac{\Delta x}{2})}{1} \lim_{\Delta x \to 0} \frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}}$$

$$f'(x) = \lim_{\Delta x \to 0} \cos(x + \frac{\Delta x}{2}) \text{ since } \lim_{\Delta x \to 0} \frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}} = 1$$

$$\text{Hence } f'(x) = \cos x$$

Note Look at page 112 of your text book for a different way of finding this derivative.

57. Question Using the above definition, find $f'(x)$ where $f(x) = \cos x$
Answer

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos(x)}{\Delta x}$$

Using the identity

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos(x+\Delta x) - \cos(x) \text{ becomes } -2\sin\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)$$

$$\text{Hence } f'(x) = \lim_{\Delta x\to 0}\frac{-2\sin\left(x+\frac{\Delta x}{2}\right)\sin(\frac{\Delta x}{2})}{\Delta x}$$

$$\text{or } f'(x) = \lim_{\Delta x\to 0}\frac{-\sin(x+\frac{\Delta x}{2})}{1}\frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}}$$

$$f'(x) = \lim_{\Delta x\to 0}\frac{-\sin(x+\frac{\Delta x}{2})}{1}\lim_{\Delta x\to 0}\frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}}$$

$$f'(x) = \lim_{\Delta x\to 0} -\sin(x+\frac{\Delta x}{2}) \text{ since } \lim_{\Delta x\to 0}\frac{\sin(\frac{\Delta x}{2})}{\frac{\Delta x}{2}} = 1$$

$$\text{Hence } f'(x) = -\sin x$$

58. <u>Example</u> Using the definition $f'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$, find $f'(x)$ where

$f(x) = \sin 8x$.

$$\begin{aligned} f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\sin 8(x+\Delta x)-\sin 8x}{\Delta x} \\ &= \lim_{\Delta x\to 0}\frac{\sin(8x+8\Delta x)-\sin 8x}{\Delta x} \end{aligned}$$

Using the identity

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \qquad (17)$$

, we have

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x\to 0}\frac{\sin(8x+8\Delta x)-\sin 8x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{2\cos(8x+4\Delta x)\sin(4\Delta x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}2\cos(8x+4\Delta x)\lim_{\Delta x\to 0}\frac{\sin(4\Delta x)}{\Delta x} \qquad (18)\\
&= \lim_{\Delta x\to 0}8\cos(8x+4\Delta x)\lim_{\Delta x\to 0}\frac{\sin(4\Delta x)}{4\Delta x} \qquad (19)\\
&= 8\cos 8x \qquad (20)
\end{aligned}
$$

Notice that to change from equation 18 to equation 19, we have multiplied numerator and denominator by 4 in order to satisfy the fundamental trig limit namely $\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha}=1$, that is the argument of the trig function must be exactly identical to the term in the denominator.Since the argument of the trig function was an even number, when we apply the trig identity 17, the 2 in the second term of the identity will cancel into the even argument of the trig function However if the argument of the trig function is odd , this will not happen and the multiplying factor at the end of the problem will have to be adjusted accordingly. The next example illustrates this point.

59. Example Using the definition $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$, find $f'(x)$ where

$f(x)=\sin 9x$.

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\sin 9(x+\Delta x)-\sin 9x}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\sin(9x+9\Delta x)-\sin 9x}{\Delta x}
\end{aligned}
$$

Using the identity

$$\sin A-\sin B=2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

, we have

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{\sin(9x + 9\Delta x) - \sin 9x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2\cos(9x + \frac{9\Delta x}{2})\sin(\frac{9\Delta x}{2})}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2\cos(9x + \frac{9\Delta x}{2}) \lim_{\Delta x \to 0} \frac{\sin(\frac{9\Delta x}{2})}{\Delta x} && (21) \\
&= \lim_{\Delta x \to 0} 9\cos(9x + \frac{9\Delta x}{2}) \lim_{\Delta x \to 0} \frac{\sin(\frac{9\Delta x}{2})}{\frac{9\Delta x}{2}} && (22) \\
&= 9\cos 9x && (23)
\end{aligned}$$

Notice that to change from equation 21 to equation 22, we have multiplied numerator and denominator by 9/2 in order to satisfy the restrictions of the fundamental trig limit.

(a) Exercise Using the above definition, find $f'(x)$ where $f(x) = \sin 5x$

(b) Exercise Using the above definition, find $f'(x)$ where $f(x) = \cos 7x$

(c) Exercise Using the above definition, find $f'(x)$ where $f(x) = \tan x$
Hint: Use the expansion for $\tan(A+B)$ given in your precalculus question list.

(d) Exercise Using the above definition, find $f'(x)$ where $f(x) = \sin^2 x$

(e) Exercise Using the above definition, find $f'(x)$ where $f(x) = \cos^2 x$

(f) Exercise Using the above definition, find $f'(x)$ where $f(x) = \cot x$
Hint: Use the expansion for $\cot(A + B)$ developed in your precalculus question list.

9 Differentiation Rules

While the use of the basic definition to obtain the derivative of a given function is a laudable task at the beginning of a Calculus course, it soon becomes an onerous task and needs to be replaced by the use of various rules which are proved by using the basic definition.

60. If $f(x) = c$ where c is a constant, then $f'(x) = 0$
Proof

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} \\ &= \lim_{\Delta x \to 0} 0 \\ &= 0 \end{aligned}$$

61. If $f(x) = cx$ where c is a constant, then $f'(x) = c$
Proof

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{c(x+\Delta x) - cx}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{cx + c\Delta x - cx}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{c\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} c \\ &= c \end{aligned}$$

62. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$
Proof While this rule is true in general, at the present time, we prove it only for the special case where n is a positive integer. Recall the binomial theorem discussed in elementary algebra , precalculus courses and also given to you on the left hand side of the back cover of your book.

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$$

If you have forgotten this theorem or perchance it has been overlooked in your previous mathematics experience, you should verify the formula by expanding

$(x+y)^2$, $(x+y)^3$, $(x+y)^4$, and show that the answers obtained agree with those obtained from basic algebraic manipulations.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta x^2 + \cdots + nx\Delta x^{n-1} + \Delta x^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta x^2 + \cdots + nx\Delta x^{n-1} + \Delta x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots + nx\Delta x^{n-2} + \Delta x^{n-1} \\
&= nx^{n-1}
\end{aligned}$$

63. If $f(x) = cx^n$ then $f'(x) = ncx^{n-1}$
A proof of this rule may be developed using the method of item 62 or using the product rule given in item 65

64. <u>Sum/Difference Rule</u> If f and g are differentiable functions and
if $h(x) = f(x) + g(x)$ then $h'(x) = f'(x) + g'(x)$
or if $h(x) = f(x) - g(x)$ then $h'(x) = f'(x) - g'(x)$
<u>Proof</u> Assume $h(x) = f(x) + g(x)$

$$\begin{aligned}
h'(x) &= \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - (f(x) + g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x) + g(x+\Delta x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(\frac{f(x+\Delta x) - f(x)}{\Delta x} + \frac{g(x+\Delta x) - g(x)}{\Delta x} \right) \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x)
\end{aligned}$$

A similar proof follows for the case $h(x) = f(x) - g(x)$. Note that the limits in the above proof are guaranteed to exist since we were told the f and g were differentiable.

65. The Product Rule If f and g are differentiable functions and

if $h(x) = f(x)g(x)$ then $h'(x) = f'(x)g(x) + f(x)g'(x)$

Proof Assume $h(x) = f(x)g(x)$

$$\begin{aligned}
h'(x) &= \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - g(x+\Delta x)f(x) + g(x+\Delta x)f(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)]g(x+\Delta x) + [g(x+\Delta x) - g(x)]f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)]g(x+\Delta x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{[g(x+\Delta x) - g(x)]f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)]}{\Delta x} \lim_{\Delta x \to 0} g(x+\Delta x) + \lim_{\Delta x \to 0} \frac{[g(x+\Delta x) - g(x)]}{\Delta x} \lim_{\Delta x \to 0} f(x) \\
&= f'(x)g(x) + g'(x)f(x)
\end{aligned}$$

Note that in the above proof, we used the trick of adding and subtracting the same quantity namely $g(x+\Delta x)f(x)$

66. The Quotient Rule If $h(x) = \frac{f(x)}{g(x)}$ and both f and g are differentiable, then

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\begin{aligned}
h'(x) &= \lim_{\Delta x \to 0} \frac{h(x+\Delta x) - h(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - g(x+\Delta x)f(x)}{g(x+\Delta x)g(x)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x) + f(x)g(x) - g(x+\Delta x)f(x)}{g(x+\Delta x)g(x)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x) - g(x+\Delta x)f(x) + f(x)g(x)}{g(x+\Delta x)g(x)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)]g(x) - [g(x+\Delta x) - g(x)]f(x)}{g(x+\Delta x)g(x)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\dfrac{[f(x+\Delta x) - f(x)]g(x)}{\Delta x} - \dfrac{[g(x+\Delta x) - g(x)]f(x)}{\Delta x}}{g(x+\Delta x)g(x)} \\
&= \frac{\lim\limits_{\Delta x \to 0} \dfrac{[f(x+\Delta x) - f(x)]g(x)}{\Delta x} - \lim\limits_{\Delta x \to 0} \dfrac{[g(x+\Delta x) - g(x)]f(x)}{\Delta x}}{\lim\limits_{\Delta x \to 0} g(x+\Delta x)g(x)} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}$$

Note that in the above proof, we used the trick of adding and sutracting the same quantity, namely $f(x)g(x)$

67. Using the alternative notation, we rewrite the product and quotient rules as follows
If $y = UV$ where U and V are functions of x, we have
$$\frac{dy}{dx} = U\frac{dV}{dx} + V\frac{dU}{dx}$$
If $y = \dfrac{U}{V}$ where U and V are functions of x, we have
$$\frac{dy}{dx} = \frac{V\dfrac{dU}{dx} - U\dfrac{dV}{dx}}{V^2}$$
When using the quotient rule, I strongly suggest that you write out the basic formula, and then set up an array containing the 4 quantities used in the

formula. Two of these quantities are known, namely the U and the V. Find the entries by differentiation of the known quantities.Then it becomes a simple matter of substitution into the formula and then, if possible, simplifying the result.

68. If $f(x) = \sin x$, then $f'(x) = \cos x$
<u>Proof</u> See item 56

69. If $f(x) = \cos x$, then $f'(x) = -\sin x$
<u>Proof</u> See item 57

70. If $f(x) = \tan x$, then $f'(x) = \sec^2 x$
To solve this problem, we rewrite $\tan x = \dfrac{\sin x}{\cos x}$ and use the quotient rule.

$$y = \tan x \text{ or } y = \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \frac{V\dfrac{dU}{dx} - U\dfrac{dV}{dx}}{V^2}$$

$$U = \sin x \qquad\qquad V = \cos x$$

$$\frac{dU}{dx} = \cos x \qquad\qquad \frac{dV}{dx} = -\sin x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ \frac{dy}{dx} &= \sec^2 x\end{aligned}$$

71. If $f(x) = \cot x$, then $f'(x) = -\text{cosec}^2\ x$
As a general rule in Calculus, if you know how to preform the stated operation for a trig function, then imitating the procedure will solve the same problem for the co-function. To solve this problem then, we rewrite $\cot x = \dfrac{\cos x}{\sin x}$ and

use the quotient rule.

$$y = \cot x \text{ or } y = \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = \frac{V\dfrac{dU}{dx} - U\dfrac{dV}{dx}}{V^2}$$

$$U = \cos x \qquad V = \sin x$$

$$\frac{dU}{dx} = -\sin x \qquad \frac{dV}{dx} = \cos x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sin x(-\sin x) - \cos x \sin x}{\sin^2 x} \\ &= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\ &= \frac{-1}{\sin^2 x} \\ \frac{dy}{dx} &= -\text{cosec}^2\, x \end{aligned}$$

72. If $f(x) = \sec x$, then $f'(x) = \sec x \tan x$

To solve this problem then, we rewrite $\sec x = \dfrac{1}{\cos x}$ and use the quotient rule.

$$y = \sec x \text{ or } y = \frac{1}{\cos x}$$

$$\frac{dy}{dx} = \frac{V\dfrac{dU}{dx} - U\dfrac{dV}{dx}}{V^2}$$

$$U = 1 \qquad V = \cos x$$

$$\frac{dU}{dx} = 0 \qquad \frac{dV}{dx} = -\sin x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos x(0) - (1)(-\sin x)}{\cos^2 x} \\ &= \frac{\sin x}{\cos^2 x} \\ \frac{dy}{dx} &= \sec x \tan x \end{aligned}$$

73. If $f(x) = \text{cosec} x$, then $f'(x) = -\text{cosec } x \cot x$

To solve this problem then, we rewrite $\text{cosec } x = \dfrac{1}{\sin x}$ and use the quotient rule.

$$y = \text{cosec } x \text{ or } y = \frac{1}{\sin x}$$

$$\frac{dy}{dx} = \frac{V\frac{dU}{dx} - U\frac{dV}{dx}}{V^2}$$

$$U = 1 \qquad\qquad V = \sin x$$

$$\frac{dU}{dx} = 0 \qquad\qquad \frac{dV}{dx} = \cos x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\sin x(0) - (1)(\cos x)}{\sin^2 x} \\ &= \frac{-\cos x}{\sin^2 x} \\ \frac{dy}{dx} &= -\text{cosec } x \cot x\end{aligned}$$

A summary of the derivatives proved up to this stage of your course is given in Table 1 on page 50. A summary of the differentiation rules is given in Table 2 on page 51

74. Exercises Differentiate the following functions

(a) $y = x^3 \cos x$

(b) $y = \dfrac{x^3 - 2}{x^3 + 4}$

(c) $y = \dfrac{x^2}{\tan x}$

(d) $y = \dfrac{\sin x}{x}$

(e) $f(x) = \sin^2 x + \cos^2 x$

Table 1: List of Derivatives

$f(x)$	$f'(x)$
c	0
x	1
cx	c
x^n	nx^{n-1}
cx^n	cnx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\text{cosec}^2$
$\sec x$	$\sec x \tan x$
$\text{cosec}\ x$	$-\text{cosec}\ x \cot x$

Table 2: List of Differentiation Rules

SUM/DIFFERENCE RULE $h(x) = f(x) \pm g(x)$ $y = U \pm V$	$h'(x) = f'(x) \pm g'(x)$ $\frac{dy}{dx} = \frac{dU}{dx} \pm \frac{dV}{dx}$
PRODUCT RULE $h(x) = f(x)g(x)$ $y = UV$	$h'(x) = f(x)g'(x) + f'(x)g(x)$ $\frac{dy}{dx} = U\frac{dV}{dx} + V\frac{dU}{dx}$
QUOTIENT RULE $h(x) = \frac{f(x)}{g(x)}$ $y = \frac{U}{V}$	$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ $\frac{dy}{dx} = \frac{V\frac{dU}{dx} - U\frac{dV}{dx}}{V^2}$
CHAIN RULE $h(x) = (f \circ g)(x)$ $y = f(u); u = g(x)$	$h\,'(x) = f\,'(g(x))g\,'(x)$ $\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$

10 The Chain Rule I

75. Question Differentiate the following functions.

   (a) $y = \sqrt{6-5x^2}$
   (b) $y = \sin(5x^3)$
   (c) $y = \sec^4(2-6x^2)$
   (d) $y = \ln(36-x^4)$
   (e) $y = (\ln(2-x))^5$
   (f) $y = e^{1/x}$
   (g) $y = \arctan(x^3)$
   (h) $y = \arcsin^3(x^2)$
   (i) $y = [x^2 - \sqrt{2x^3+5}]^4$

Each of these questions require differentiation of complicated functions. The approach to solving such a problem involves the reduction of the compound problem into two or more simpler problems.
Lets consider 75a. Here the difficulty is the expression under the radical sign. Let $u = 6-5x^2$. Then

$$y = \sqrt{u} \text{ and } u = 6-5x^2 \text{are equivalent to the original statement}$$

Then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ or in this case $\frac{dy}{dx} = \frac{1}{2\sqrt{u}}(-10x)$ Never leave the answer with the variable "u"present. So rewriting, and simplifying ,we have

$$\frac{dy}{dx} = \frac{-5x}{\sqrt{6-5x^2}}$$

We can also look at this procedure using a composite function notation.

$$\text{If } h(x) = (f \circ g)(x), \text{ then } h'(x) = f'(g(x))g'(x)$$

In this case, $g(x) = 6-5x^2$ and $f(x) = \sqrt{x}$ In either case you should try to do these actions at sight and not rely on substitutions each time. In essence, you are working from the outside towards the inside. Or if you like, perform the global operation and then go inside. In this present example, the global operation is a square root. Look at it this way.

$$\frac{dy}{dx} = \frac{d}{d(\text{blank })}(\text{ blank })^{1/2} \text{ times } \frac{d}{dx}(\text{ blank })$$

In our case, blank $= 6 - 5x^2$, so that

$$\frac{dy}{dx} = \frac{1}{2(\text{ blank })^{1/2}}(-10x)$$

$$\frac{dy}{dx} = \frac{-5x}{\sqrt{6-5x^2}}$$

Lets consider 75b . Here the global function is the sine function. Inside we have $5x^3$

$$\text{Let } u = 5x^3\text{,then } y = \sin u$$

$$\frac{dy}{du} = \cos u \text{ and } \frac{du}{dx} = 15x^2$$

$$\text{Hence } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\text{Or } \frac{dy}{dx} = \cos u\,(15x^2)$$

Remember to return the answer to the original variable, namely x

$$\frac{dy}{dx} = 15x^2\cos(5x^3)$$

Now try to do this differentiation at sight. The derivative of the sine function is the cosine function with the argument unchanged. Then go inside and differentiate the argument , namely $5x^3$
For problem 75c, the global picture is a $[\text{blank1}]^4$ Inside that , we have a secant function while inside that we have an algebra function.

Differentiate from the outside in.

$$\frac{dy}{dx} = \frac{d}{d(\text{ blank1 })}[\text{ blank1 }]^4 \text{ times } \frac{d}{d(\text{ (blank2)}}\text{ (blank1) times } \frac{d}{dx}(\text{ (blank2))}$$

In this case, blank1 $= \sec(2-6x^2)$ and blank2 $= (2-6x^2)$

$$\frac{dy}{dx} = 4[\sec(2-6x^2)]^3 \text{ times } \sec(2-6x^2)\tan(2-6x^2) \text{ times } (-12x)$$

Cleaning up we have

$$\frac{dy}{dx} = -48x\sec^4(2-6x^2)\tan(2-6x^2)$$

An alternative approach would involve the following substitutions

$$\text{Let } u = 2-6x^2, \text{ then } v = \sec u \text{ and } y = v^4$$

$$\frac{dy}{dx} = \frac{dy}{dv}\,\frac{dv}{du}\,\frac{du}{dx}$$

$$\frac{dy}{dx} = 4v^3 \quad \sec u \tan u \quad (-12x)$$

Remember to return to the original variable " x "and simplifying, we have

$$\frac{dy}{dx} = -48x\sec^4(2-6x^2)\tan(2-6x^2)$$

In problem 75d, the outside function is the natural logarithm function . The derivative of this function is 1 over the entire argument . Then go inside and differentiate the argument.

$$\text{If } y = \ln(36-x^4) \text{ Then } \frac{dy}{dx} = \frac{1}{36-x^4} \quad (-4x^3)$$

$$\text{Cleaning up ,we have } \frac{dy}{dx} = \frac{-4x^3}{36-x^4}$$

For problem 75e, the global function is [blank $]^5$ where blank is a logarithm function with argument $2-x$

$$\frac{dy}{dx} = 5[\text{ blank }]^4 \frac{1}{2-x} \quad (-1)$$

Cleaning up , we have

$$\boxed{\frac{dy}{dx} = \frac{-5[\ln(2-x)]^4}{2-x}}$$

An alternative approach to problem 75e would be to let

$$u = 2-x\,; v = \ln u\,; \text{ with } y = v^5$$

$$\text{Then } \frac{dy}{dx} = \frac{dy}{dv}\,\frac{dv}{du}\,\frac{du}{dx}$$

$$= 5v^4 \, \frac{1}{u} \, (-1)$$

Replacing all of the variables in terms of the original variable x, we have

$$\frac{dy}{dx} = 5[\ln(2-x)]^4 \, \frac{1}{2-x} \, (-1)$$

$$\boxed{\frac{dy}{dx} = \frac{-5[\ln(2-x)]^4}{2-x}}$$

For problem 75f the easiest way to solve this exercise is to remember that the derivative of the exponential function is the same exponential function and then go inside this function and differentiate the argument. In the present case , if

$$y = e^{1/x} \quad \text{then } \frac{dy}{dx} = e^{1/x} \, \left(\frac{-1}{x^2}\right)$$

In the usual way of writing this expression, we move the algebraic part to the front so that

$$\boxed{\frac{dy}{dx} = \left(\frac{-1}{x^2}\right) e^{1/x}}$$

In problem 75g, the controlling function is the arctan function. Remember that the derivative of the arctan function is $1/[1 + (\text{argument})^2]$. Then go inside and differentiate the argument. So in the present case, we have

$$\text{If } y = \arctan(x^3) \ \text{ then } \frac{dy}{dx} = \frac{1}{1+(x^3)^2} \, (3x^2)$$

Cleaning up, we have,

$$\boxed{\frac{dy}{dx} = \frac{3x^2}{1+x^6}}$$

For problem 75h, the outside function is a $[\text{ blank}]^3$ Then as we go inside, we encounter an arcsin function. The derivative of this function is $1/\sqrt{1 - (\text{argument})^2}$. Finally go inside and differentiate the argument.

$$\text{If } y = [\arcsin(x^2)]^3, \ \text{Then } \frac{dy}{dx} = 3[\arcsin(x^2)]^2 \ \frac{1}{\sqrt{1-(x^2)^2}} \ 2x$$

Cleaning up , we have

$$\boxed{\frac{dy}{dx} = \frac{6x \, [\arcsin(x^2)]^2}{\sqrt{1-x^4}}}$$

For problem 75i, the outside function is a [blank $]^4$. Then as we go inside, we encounter algebraic functions including a radical. Rewrite the problem as

$$y = [\ x^2 - (2x^3+5)^{1/2}]^4$$

$$\text{Then } \frac{dy}{dx} = 4[\ x^2 - (2x^3+5)^{1/2}]^3\ [2x - (1/2)(2x^3+5)^{-1/2}(6x^2)]$$

$$\frac{dy}{dx} = 4[\ x^2 - \sqrt{2x^3+5}]^3\ \left[2x - \frac{6x^2}{2\sqrt{2x^3+5}}\right]$$

$$\boxed{\frac{dy}{dx} = 4[\ x^2 - \sqrt{2x^3+5}]^3\ \left[2x - \frac{3x^2}{\sqrt{2x^3+5}}\right]}$$

76. Exercises Find the first derivative of the following functions. It is possible that on the first reading of these notes, some of the functions may not have been studied at this time. Come back later and review all of these exercises.

   (a) $y = \sqrt[5]{x^2 - \cos(4x^3)}$
   (b) $y = (2x+5)^3(7-3x^2)^5$
   (c) $y = \dfrac{2x+3}{\sqrt{5x+2}}$
   (d) $y = \arcsin^5(1-x^2)$
   (e) $y = \sqrt{(3x+1)^3 + (2x+5)^7}$
   (f) $y = x^5\ e^{1-x^2}$
   (g) $y = (x^2+2x+5)^{7/2}$
   (h) $y = \sqrt{x^3 + \sin^2(5x^2)}$
   (i) $y = \sqrt[3]{\sin(5x)}$
   (j) $y = \sin(\sqrt[3]{5x})$
   (k) $y = \sin(5\sqrt[3]{x})$
   (l) $y = \sin(1/x);\ x \neq 0$

   The answers to these exercises will be found in item number 95 on page 72

11 Rolle's Theorem, Mean Value Theorem

77. Question State Rolle's Theorem

    Answer Let f be continuous on the closed interval [a,b] and differentiable on the open interval (a,b). If

    $$f(a) = f(b)$$

    then there is at least one number c in (a,b) such that $f'(c) = 0$

    Note: When answering this question be certain to include all of the conditions namely

    (a) Continuous on the closed interval [a,b]
    (b) Differentiable on the open interval (a,b)
    (c) If $f(a) = f(b)$ then
    (d) there is at least one number c in the open interval (a,b)
    (e) such that $f'(c) = 0$

78. Question State the Mean Value Theorem

    Answer If f is continuous on the closed interval [a,b] and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that

    $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

    Note: When answering this question be certain to include all of the conditions namely

    (a) Continuous on the closed interval [a,b]
    (b) Differentiable on the open interval (a,b)
    (c) there is at least one number c in the open interval (a,b)
    (d) such that

    $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

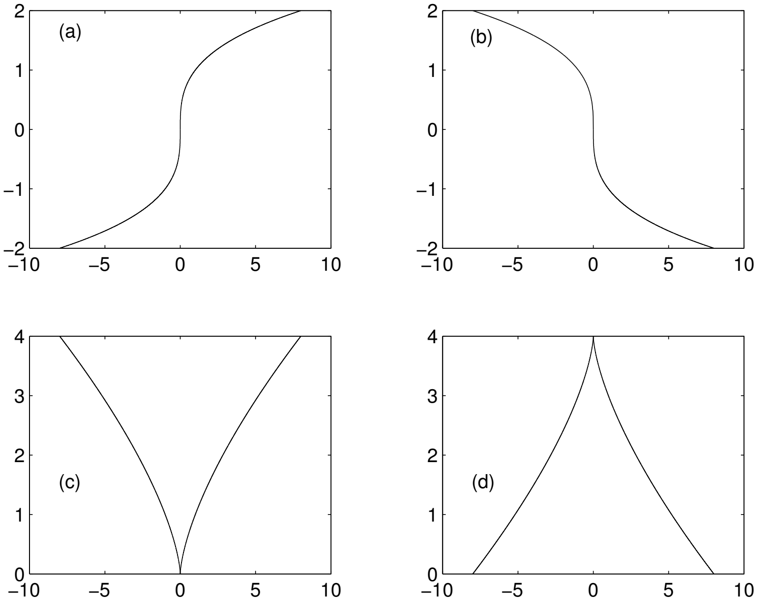

Figure 9: Graphs showing vertical tangents and cusps. $(a) f(x) = x^{1/3}$, $(b) f(x) = -x^{1/3}$, $(c) f(x) = x^{2/3}$, $(d) f(x) = 4 - x^{2/3}$

12 Cusps, Vertical Tangent Lines

The simple functions $f(x) = x^{1/3}$ and $f(x) = x^{2/3}$ are used to examine the first, second, and third derivative test results at either vertical tangents or cusps. From figure 9, it should be clear that the graphs in (a) and (b) have a point of inflection at $x = 0$ whereas graphs in (c) and (d) have relative extrema at $x = 0$. In all of the above cases, there are vertical tangent lines at $x = 0$.

$$\begin{array}{ll}
f(x) = x^{1/3}; & f(x) = x^{2/3} \\
f'(x) = (1/3)x^{-2/3}; & f'(x) = (2/3)x^{-1/3} \\
f''(x) = -(2/9)x^{-5/3}; & f''(x) = -(2/9)x^{-4/3} \\
f'''(x) = (10/27)x^{-8/3}; & f'''(x) = (8/27)x^{-7/3}
\end{array}$$

For graphs (a) and (b), $x = 0$ is a critical number since $f'(x)$ is undefined at this point but $f(0)$ exists. For graph (a) $f'(x)$ is positive to the left and right of the critical number, that is, there is no change in the sign of the first derivative and consequently, there is no extrema at that point. A similar comment applies to graph (b) except in this case the derivative is negative on

both sides of the critical number.
For (a) $\lim\limits_{x\to 0} f'(x) = \infty$ whereas for (b) $\lim\limits_{x\to 0} f'(x) = -\infty$
Hence we can say that there is a vertical tangent at these points.

For graphs (c) and (d), $x = 0$ is a critical number since $f'(x)$ is undefined at this point but $f(0)$ exists. For graph (c) $f'(x)$ is negative on the left and positive on the right of the critical number, that is, there is a relative minimum at that point. A similar comment applies to graph (d) except in that case the derivative is positive on the left and negative on the right of the critical number, that is there is a relative maximum at that point.
For (c) $\lim\limits_{x\to 0^+} f'(x) = \infty$ while $\lim\limits_{x\to 0^-} f'(x) = -\infty$

For (d) $\lim\limits_{x\to 0^+} f'(x) = -\infty$ while $\lim\limits_{x\to 0^-} f'(x) = \infty$
There are vertical tangents at $x = 0$ but clearly these graphs are not smooth at $x = 0$. We say that there are cusps at $x = 0$ for these graphs. If $|\lim\limits_{x\to c} f'(x)| = \infty$, then there is a vertical tangent at $x = c$. Also note the difference between a corner and a cusp . If a graph has a corner at $x = c$, then at least one of the one sided limits for $f'(x)$ as x approaches c will have a finite value . There is no tangent line at a corner. Consider the graph of $f(x) = |x|$. At $x = 0$, there is no tangent line. In this case, $\lim\limits_{x\to 0^-} f'(x) = -1$ while $\lim\limits_{x\to 0^+} f'(x) = 1$

For (a) $f''(x)$ is undefined at $x = 0$. This is a candidate for a point of inflection since f(0) exists. $f''(x)$ is positive to the left of $x = 0$ and negative to the right of this point indicating a change in concavity and hence a point of inflection at $x = 0$. In this case, we can write $\lim\limits_{x\to 0} f'''(x) = \infty$ For (c) $f''(x)$ is undefined at $x = 0$. This is a candidate for a point of inflection since $f(0)$ exists. Since we have already concluded that for this graph there is a relative minima at this point, it cannot be a point of inflection. However lets look at how the tests handle this case. $f''(x)$ is negative to the left of $x = 0$ and negative to the right of this point indicating no change in concavity and hence $x = 0$ is not a point of inflection. What about the third derivative test for this case? We can write $\lim\limits_{x\to 0^+} f'''(x) = \infty$ and $\lim\limits_{x\to 0^-} f'''(x) = -\infty$ so that we cannot write a description of the nonexistence of the statement $\lim\limits_{x\to 0} f'''(x)$ that is we cannot say that this limit goes to ∞ or $-\infty$.

Hence the third derivative test for a point of inflection should be interpreted as follows; If $f'''(c) \neq 0$ or $\lim_{x \to c} f'''(x) = \infty$ or $\lim_{x \to c} f'''(x) = -\infty$, then there is a point of inflection at $x = c$

13 Fundamental Theorem of Calculus

79. Question State the Fundamental Theorem of Calculus
Answer If f is continuous on a closed interval $[a, b]$ and F is any antiderivative of f on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note In Math 2419 we will remove the restriction that a and b are finite and that f(x) is continuous on the interval $[a, b]$

80. Question List some of the important properties of the definite integral
Answer

    (a) If a function f is continuous on the closed interval [a,b] then f is integrable on [a,b]

    (b) If f is defined at x = a , then $\int_a^a f(x)dx = 0$

    (c) If f is integrable on $[a.b]$, then $\int_b^a f(x)dx = -\int_a^b f(x)dx$

    (d) If f is integrable on the three closed intervals determined by a,b,and c, then

    $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

    (e) If f and g are integrable on [a,b] then

    $$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

    (f) If f is integrable on [a,b] and k is a constant then

    $$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

(g) If f is integrable and nonnegative on the closed interval $[a, b]$,then

$$\int_a^b f(x)dx \geq 0$$

(h) If f and g are integrable on the closed interval [a,b] and $f(x) \leq g(x)$ for every x in [a,b] then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

81. Example Evaluate $\int_{-2}^{3} (x^2 + 2x + 1)\ dx$

Note : At this stage of your course, the given integrals will satisfy the requirements of item number 79. Solving this problem is a two step process. First, you must find an antiderivative,F(x). Then evaluate $F(b) - F(a)$ or in our case $F(3) - F(-2)$.

$$\begin{aligned}
\int_{-2}^{3} (x^2 + 2x + 1)\ dx &= \left[\frac{x^3}{3} + x^2 + x\right]_{-2}^{3} \\
&= (9 + 9 + 3) - (-\frac{8}{3} + 4 - 2) \\
&= 21 + \frac{8}{3} - 4 + 2 \\
&= \frac{65}{3}
\end{aligned}$$

Note the careful use of parentheses in the second step in order to minimize the chances of making a mistake. Always solve one difficulty at a time. In the above exercise, you have the difficulty of raising a negative number to an odd/even power as well as the additional difficulty of distributing a negative one across several terms. You will notice that I substituted the negative numbers into the expression and obtained those answers before trying to distribute the negative sign across the expression.

82. Exercises Evaluate the following definite integrals

(a) $\int_{-1}^{2} (x^3 - 2x^2 + x - 5)\ dx$

(b) $\int_{-2}^{-1} \frac{(x^3+2x^2+1)}{x^2}\,dx$

(c) $\int_{-\pi/3}^{-\pi/6} \sin x\;dx$

(d) $\int_{-\pi/4}^{-\pi/6} \cos x\;dx$

(e) $\int_{-\pi/4}^{-\pi/6} \sec^2 x\;dx$

14 Second Fundamental Theorem of Calculus

83. Question State the Second Fundamental Theorem of Calculus
Answer If f is continuous on an open interval I containing a, then for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x) \tag{24}$$

Alternate statement If f is continuous on an open interval I containing a, then for every x in the interval,

$$F(x) = \int_a^x f(t)dt \tag{25}$$

$$\text{and}\quad F'(x) = f(x) \tag{26}$$

84. Question Find $F'(x)$ where $F(x) = \int_3^x \sqrt{t^3+1}\,dt$
Answer Question is set up exactly according to the Theorem.
$f(t) = \sqrt{t^3+1}$ so the answer is $f(x) = \sqrt{x^3+1}$

85. Question Find

$$\frac{d}{dx}\left[\int_x^5 \frac{1}{t^5+7}\,dt\right]$$

Answer The question is not set up exactly according to the Theorem. The variable x must be the upper integration limit. Rewrite the integral

$$\frac{d}{dx}\left[\int_x^5 \frac{1}{t^5+7}\,dt\right] = -\frac{d}{dx}\left[\int_5^x \frac{1}{t^5+7}\,dt\right]$$

$$= -\frac{1}{x^5+7}$$

Comment In this case $f(t) = 1/(t^5+7)$ so $f(x) = 1/(x^5+7)$

Note: The following theorem is an extension of the Second Fundamental Theorem using the Chain rule for differentiation.

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)dt\right] = g'(x)f(g(x)) = g'(x)(f \circ g)(x)$$

Alternatively if

$$F(x) = \int_a^{g(x)} f(t)\,dt \text{ then } \; F'(x) = g'(x)\,(f(g(x))$$

86. Question Find

$$\frac{d}{dx}\left[\int_2^{x^3} \sin(t^5)\,dt\right]$$

Answer

$$\frac{d}{dx}\left[\int_2^{x^3} \sin(t^5)\,dt\right] = 3x^2\sin(x^{15})$$

87. Question $F(x) = \displaystyle\int_{\cos x}^{\sin x} \sqrt{t^4+5}\,dt$ Find $F'(x)$

Answer

$$F(x) = \int_{\cos x}^{\sin x} \sqrt{t^4+5}\,dt$$

$$F(x) = \int_{\cos x}^{a} \sqrt{t^4+5}\,dt + \int_a^{\sin x} \sqrt{t^4+5}\,dt$$

$$F(x) = -\int_a^{\cos x} \sqrt{t^4+5}\,dt + \int_a^{\sin x} \sqrt{t^4+5}\,dt$$

Hence $F'(x) = \sin x\sqrt{\cos^4 x+5} + \cos x\sqrt{\sin^4 x+5}$

(a) Exercise $F(x) = \displaystyle\int_{x^2}^{x^5} \frac{t^2}{t^6+1}\,dt$. Find $F'(x)$

(b) Exercise Find

$$\frac{d}{dx}\int_{\sec x}^{\tan x} \frac{t}{\sqrt{t^3+7}}\, dt$$

(c) Exercise Find

$$\frac{d}{dx}\int_{x^4}^{5} \tan^6 t^2\, dt$$

(d) Exercise $F(x) = \displaystyle\int_{\sqrt{x}}^{1/x} t^6 + t^4 + 1\ dt$. Find $F'(x)$

15 Logarithmic and Exponential Functions

15.1 Logarithmic Functions

In your previous studies, for example in precalculus, you have encountered the logarithm function and the associated basic logarithm rules. This function and the associated basic rules are extremely important in this course. As a reminder, these rules are now summarized. Although these rules are valid for logarithms of any base, we have a special interest in the natural logarithm function and so these rules are expressed in terms of that function.

$$\begin{aligned}
\ln(ab) &= \ln a + \ln b & (27)\\
\ln\left(\frac{a}{b}\right) &= \ln a - \ln b & (28)\\
\ln(a^n) &= n \ln a & (29)\\
\ln 1 &= 0 & (30)\\
\ln e &= 1 & (31)
\end{aligned}$$

In §5.1 of your textbook, the logarithm function is defined in terms of a definite integral. All of the above rules may be derived from this definition using the properties listed in item# 80

$$\ln x = \int_1^x \frac{1}{t}\, dt \tag{32}$$

From the graph of $y = \ln x$ on page 65, we are able to deduce certain properties of the logarithm function.

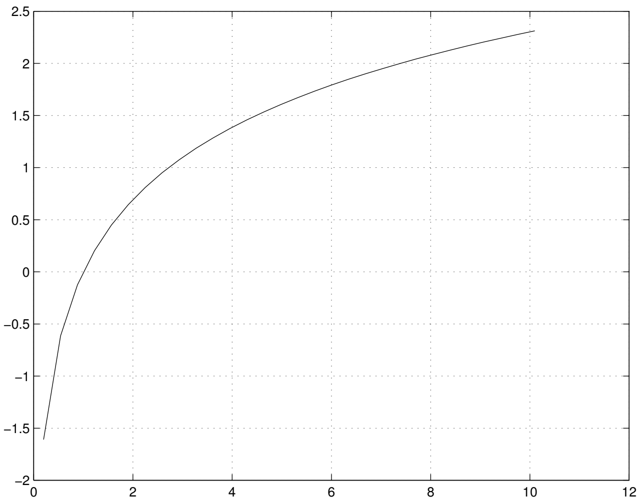

Figure 10: The Natural Logarithm Function ; $y = \ln x$

- The domain is $(0, \infty)$ while the range is $(-\infty, \infty)$
- $\lim\limits_{x \to 0^-} \ln x = -\infty$
- $\lim\limits_{x \to \infty} \ln x = \infty$
- The graph of $y = \ln x$ is always increasing and
- The graph of $y = \ln x$ is always concave down

88. Exercise Prove Rule# 27 using the definition for $\ln x$ given above. (32)

$$\begin{aligned} \ln(ab) &= \int_1^{ab} \frac{1}{t}\,dt \\ \text{or } \int_1^{ab} \frac{1}{t}\,dt &= \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt \end{aligned}$$

The first integral on the right hand side is of the correct form. Now consider $\int_a^{ab} \frac{1}{t}\,dt$. Let $t = au$ Then $dt = adu$. When $t = a, u = 1$ and when $t = ab, u = b$ so that the integral becomes

$$\int_a^{ab} \frac{1}{t}\,dt = \int_1^{b} \frac{1}{u}\,du$$

Hence

$$\begin{aligned}\ln(ab) &= \int_1^{ab} \frac{1}{t}\,dt \\ \ln(ab) &= \int_1^{a} \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt \\ \ln(ab) &= \int_1^{a} \frac{1}{t}\,dt + \int_1^{b} \frac{1}{u}\,du \\ \ln(ab) &= \ln a + \ln b\end{aligned}$$

89. Exercise Prove Rule# 28 using the definition for $\ln x$ given above. (32)

$$\begin{aligned}\ln\left(\frac{a}{b}\right) &= \int_1^{\frac{a}{b}} \frac{1}{t}\,dt \\ \text{or } \int_1^{\frac{a}{b}} \frac{1}{t}\,dt &= \int_1^{a} \frac{1}{t}\,dt + \int_a^{\frac{a}{b}} \frac{1}{t}\,dt\end{aligned}$$

The first integral on the right hand side is of the correct form. Now consider $\int_a^{\frac{a}{b}} \frac{1}{t}\,dt$. Let $t = \frac{au}{b}$ Then $dt = \frac{a}{b}du$. When $t = a, u = b$ and when $t = \frac{a}{b}, u = 1$ so that the integral becomes

$$\int_a^{\frac{a}{b}} \frac{1}{t}\,dt = \int_b^{1} \frac{1}{u}\,du = -\int_1^{b} \frac{1}{u}\,du$$

Hence

$$\begin{aligned}
\ln\left(\frac{a}{b}\right) &= \int_1^{\frac{a}{b}} \frac{1}{t}\,dt \\
\ln\left(\frac{a}{b}\right) &= \int_1^{a} \frac{1}{t}\,dt + \int_a^{\frac{a}{b}} \frac{1}{t}\,dt \\
\ln\left(\frac{a}{b}\right) &= \int_1^{a} \frac{1}{t}\,dt + \int_b^{1} \frac{1}{u}\,du \\
\ln\left(\frac{a}{b}\right) &= \int_1^{a} \frac{1}{t}\,dt - \int_1^{b} \frac{1}{u}\,du \\
\ln\left(\frac{a}{b}\right) &= \ln a - \ln b
\end{aligned}$$

90. Exercise Prove Rule# 29 using the definition for $\ln x$ given above. (32)

$$\ln(a^n) = \int_1^{a^n} \frac{1}{t}\,dt$$

Let $t = u^n$. Then $dt = nu^{n-1}du$ When $t = 1, u = 1$ and when $t = a^n, u = a$

$$\begin{aligned}
\int_1^{a^n} \frac{1}{t}\,dt &= \int_1^{a} \frac{1}{u^n} nu^{n-1}\,du \\
\int_1^{a^n} \frac{1}{t}\,dt &= \int_1^{a} \frac{n}{u}\,du \\
\int_1^{a^n} \frac{1}{t}\,dt &= n\int_1^{a} \frac{1}{u}\,du \\
\ln(a^n) &= n\ln a
\end{aligned}$$

91. Exercises Without using your calculator, evaluate the following expressions;

   (a) $\ln(e^3)$
   (b) $(\ln e^2)^3$
   (c) $\ln\sqrt{e}$
   (d) $\dfrac{3}{\ln e^4}$

   The answers to these questions will be found in item # 96 on page 73

92. Example Given $f(x) = \ln x$ find $f'(x)$

Answer Using the definition for $\ln x$ given in equation 32 and also the second fundamental theorem, (item 83, equations 25 and 26), we have immediately $f'(x) = \dfrac{1}{x}$

$$f(x) = \ln x = \int_1^x \frac{1}{t}\, dt$$

$$f'(x) = \frac{d}{dx} \int_1^x \frac{1}{t}\, dt = \frac{1}{x}$$

$$\boxed{\text{If } y = \ln x,\ \text{Then } \frac{dy}{dx} = \frac{1}{x} \text{ and } \frac{d^2y}{dx^2} = -\frac{1}{x^2}}$$

Note; All differentiation Rules Apply. All of the previous rules and procedures can be applied to this function.

93. Example Differentiate the following functions

(a) $y = \ln(3x + 5)$

(b) $y = \ln(3 - 5x)$

(c) $y = x^3 \ln(6x)$

(d) $y = \dfrac{\ln x}{x}$

(e) $y = \ln(\sqrt{6x^2 + 5})$

(f) $y = \ln\left(\dfrac{6x + 7}{5x^3 + 4}\right)$

(g) $y = \ln\left(\dfrac{(3x + 7)^5(4x^2 + 9)^3}{(2x + 9)^7}\right)$

(h) $x^2 - 3\ln y + y^2 = 10$

(i) $y = \dfrac{\sqrt[3]{7x + 9}(4x + 13)^7}{(5x + 17)^6\sqrt{3x + 19}}$

Answer Question 93a is an example of a simple chain rule problem.
Let $u = 3x + 5$, so that $y = \ln u$. Then $\dfrac{du}{dx} = 3$

$$\text{Hence} \frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

$$\text{or } \frac{dy}{dx} = \frac{1}{u}\,3$$

. Always return the expression to the original variable.Hence if

$$\boxed{y = \ln(3x+5)}$$

$$\boxed{\text{then } \frac{dy}{dx} = \frac{3}{3x+5}}$$

This is the slow way of answering the question. After some practice, you should be able to do it at sight. Remember that the derivative of the natural logarithm function is the reciprocal of the argument. Then go inside and differentiate the argument. So if $y = \ln(3x+5)$, then

$$\frac{dy}{dx} = \left(\frac{1}{3x+5}\right) 3 \text{ or } \frac{3}{3x+5}$$

Question 93b is almost identical to the previous question. However in this case, be careful of the negative sign arising from the derivative of $3 - 5x$ So if

$$\boxed{y = \ln(3-5x)}$$

,

$$\boxed{\text{then } \frac{dy}{dx} = \frac{-5}{3-5x}}$$

Question 93c is an example of a product rule.

$$\text{If } y = x^3 \ln(6x)$$

$$\text{then } \frac{dy}{dx} = 3x^2 \ln(6x) + x^3 \frac{1}{6x} 6$$

or

$$\frac{dy}{dx} = x^2[3\ln(6x) + 1]$$

Note that we have factored the answer as completely as possible. This is good practice just in case the differentiation process is just the first step in solving a problem. For example, you might be asked to find the critical points of a function. In such a problem, the next step would be to set the derivative equal

to 0 and solve the resulting equation.
Question 93d is an example of a quotient rule.

$$\begin{aligned} y &= \frac{\ln x}{x} \\ \frac{dy}{dx} &= \frac{x\frac{1}{x} - \ln x}{x^2} \\ \frac{dy}{dx} &= \frac{1 - \ln x}{x^2} \end{aligned}$$

In general, if you can use logarithm rules in an expression, do so before differentiating. Questions 93e- 94 are examples where this procedure may be very effective. Application of suitable logarithm rules will simplify the expression before any calculus is used.

$$\begin{aligned} y &= \ln(\sqrt{6x^2+5}) \\ y &= \ln(6x^2+5)^{1/2} \\ y &= \frac{1}{2}\ln(6x^2+5) \text{ using equation 29} \\ \frac{dy}{dx} &= \frac{1}{2}\frac{12x}{6x^2+5} \\ \frac{dy}{dx} &= \frac{6x}{6x^2+5} \end{aligned}$$

For question 93f the application of a logarithm rule (equation 28) will simplify the expression.

$$\begin{aligned} y &= \ln\left(\frac{6x+7}{5x^3+4}\right) \\ y &= \ln(6x+7) - \ln(5x^3+4) \\ \frac{dy}{dx} &= \frac{6}{6x+7} - \frac{15x^2}{5x^3+4} \end{aligned}$$

Question 93g clearly should not be attempted by brute force.It is obvious from simply eyeballing the problem that some method is required that will greatly reduce the amount of unnecessary labor. Here is a classic case of simplification through the application of logarithim rules. Using logarithm rules, we have

$$\begin{aligned} y &= \ln\left(\frac{(3x+7)^5(4x^2+9)^3}{(2x+9)^7}\right) \\ y &= 5\ln(3x+7)+3\ln(4x^2+9)-7\ln(2x+9) \\ \frac{dy}{dx} &= \frac{15}{3x+7}+\frac{24x}{4x^2+9}-\frac{14}{2x+9} \end{aligned}$$

Question 93h involves implicit differentiation. It is not possible to solve for y in terms of x. Following the process used in implicit differentiation, we differentiate both sides with respect to x. Whenever you see an "x" variable, differentiate using the usual rules.When you differentiate a "y" variable, do so using the usual rules but also multiply your result by $\frac{dy}{dx}$

$$\begin{aligned} x^2-3\ln y+y^2 &= 10 \\ 2x-3\frac{1}{y}\frac{dy}{dx}+2y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}\left(2y-\frac{3}{y}\right) &= -2x \\ \frac{dy}{dx}\left(\frac{2y^2-3}{y}\right) &= -2x \\ \frac{dy}{dx} &= \frac{-2xy}{2y^2-3} \\ \text{or } \frac{dy}{dx} &= \frac{2xy}{3-2y^2} \end{aligned}$$

In questions 93e - 93h, a natural logarithm was present and so the complicated function could be simplified by the immediate application of logarithm rules before differentiating. In question 93i, the given function is very complicated but a logarithm function is not present. Clearly,if possible, you should avoid at all costs a brute force differentiation. In these types of problems, the first step is to take the natural logarithm of both sides. Then apply logarithm rules before differentiating implicitly.

$$\begin{aligned}
y &= \frac{\sqrt[3]{7x+9}\,(4x+13)^7}{(5x+17)^6\sqrt{3x+19}} \\
\ln y &= \ln\left(\frac{\sqrt[3]{7x+9}\,(4x+13)^7}{(5x+17)^6\sqrt{3x+19}}\right) \\
\ln y &= \ln(7x+9)^{1/3} + \ln(4x+13)^7 - \ln(5x+17)^6 - \ln(3x+19)^{1/2} \\
\ln y &= \frac{\ln(7x+9)}{3} + 7\ln(4x+13) - 6\ln(5x+17) - \frac{\ln(3x+19)}{2} \\
\frac{1}{y}\frac{dy}{dx} &= \frac{7}{3(7x+9)} + \frac{28}{4x+13} - \frac{30}{5x+17} - \frac{3}{2(3x+19)} \\
\frac{dy}{dx} &= \left[\frac{7}{3(7x+9)} + \frac{28}{4x+13} - \frac{30}{5x+17} - \frac{3}{2(3x+19)}\right] y \\
\frac{dy}{dx} &= \left[\frac{7}{3(7x+9)} + \frac{28}{4x+13} - \frac{30}{5x+17} - \frac{3}{2(3x+19)}\right] \frac{\sqrt[3]{7x+9}\,(4x+13)^7}{(5x+17)^6\sqrt{3x+19}}
\end{aligned}$$

94. Exercises Find the first derivative of the following functions.

    (a) $y = \ln 7x$

    (b) $y^2 = \ln(xy)$

    (c) $xy = \ln(x^2 + y^2)$

    (d) $y = [\ln(x + \sqrt{x^2 - 1})\,]^3$

    (e) $y = \ln\left[\dfrac{\sqrt{6 - x^2}}{(5x^2 - 7)^4}\right]$

    (f) $y = \dfrac{(x^3 + 5)^4}{\sqrt[3]{2x^3 - 5}\,(4x^7 + 5)^2}$

    The answers to these exercises will be found in item number 97 on page 73

16 Answers to Selected Exercises

95. Answers for exercise 76

    (a) (76a); $\dfrac{dy}{dx} = \dfrac{[2x + 12x^2\sin(4x^3)]}{5\,[x^2 - \cos(4x^3)]^{4/5}}$

(b) (76b); $\dfrac{dy}{dx} = (42 - 78x^2 - 150x)(2x+5)^2(7-3x^2)^4$

(c) (76c);$\dfrac{dy}{dx} = \dfrac{10x - 7}{2(5x+2)^{3/2}}$

(d) (76d); $\dfrac{dy}{dx} = \dfrac{-10x \arcsin^4(1-x^2)}{\sqrt{2x^2 - x^4}}$

(e) (76e); $\dfrac{dy}{dx} = \dfrac{9(3x+1)^2 + 14(2x+5)^6}{2\sqrt{(3x+1)^3 + (2x+5)^7}}$

(f) (76f); $\dfrac{dy}{dx} = x^4 e^{1-x^2}(5 - 2x^2)$

(g) (76g); $\dfrac{dy}{dx} = 7(x+1)(x^2+2x+5)^{5/2}$

(h) (76h); $\dfrac{dy}{dx} = \dfrac{3x^2 + 10x\sin(10x^2)}{\sqrt{x^3 + \sin^2(5x^2)}}$

(i) (76i); $\dfrac{dy}{dx} = \dfrac{5\cos(5x)}{3\ (\sin(5x))^{2/3}}$

(j) (76j); $\dfrac{dy}{dx} = \dfrac{5\ \cos(\sqrt[3]{5x})}{3(5x)^{2/3}} = \dfrac{5^{1/3}\cos(\sqrt[3]{5x})}{3\ x^{2/3}}$

(k) (76k); $\dfrac{dy}{dx} = \dfrac{5\cos(5\sqrt[3]{x})}{3x^{2/3}}$

(l) (76l) $\dfrac{dy}{dx} = \dfrac{-1}{x^2}\ \cos(1/x)$

96. Answers for exercise 91

    (a) (91a) 3

    (b) (91b) 8

    (c) (91c) $\dfrac{1}{2}$

    (d) (91d)$\dfrac{3}{4}$

97. Answers for exercise 94

    (a) (94a); $\dfrac{dy}{dx} = \dfrac{1}{x}$

(b) (94b); $\dfrac{dy}{dx} = \dfrac{y}{x(2y^2-1)}$

(c) (94c); $\dfrac{dy}{dx} = \dfrac{2x - x^2y - y^3}{x^3 + xy^2 - 2y}$

(d) (94d); $\dfrac{dy}{dx} = \dfrac{3[\ln(x+\sqrt{x^2-1})\ ]^2}{\sqrt{x^2-1}}$

(e) (94e); $\dfrac{dy}{dx} = \dfrac{-x}{6-x^2} - \dfrac{40x}{5x^2-7}$

(f) (94f); $\dfrac{dy}{dx} = \left(\dfrac{12x^2}{x^3+5} - \dfrac{2x^2}{2x^3-5} - \dfrac{56x^6}{4x^7+5}\right)\ \left(\dfrac{(x^3+5)^4}{\sqrt[3]{2x^3-5}\ (4x^7+5)^2}\right)$

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August 11, 2005

The time is 13h 12min.