Math 2417 Student Notes #3A

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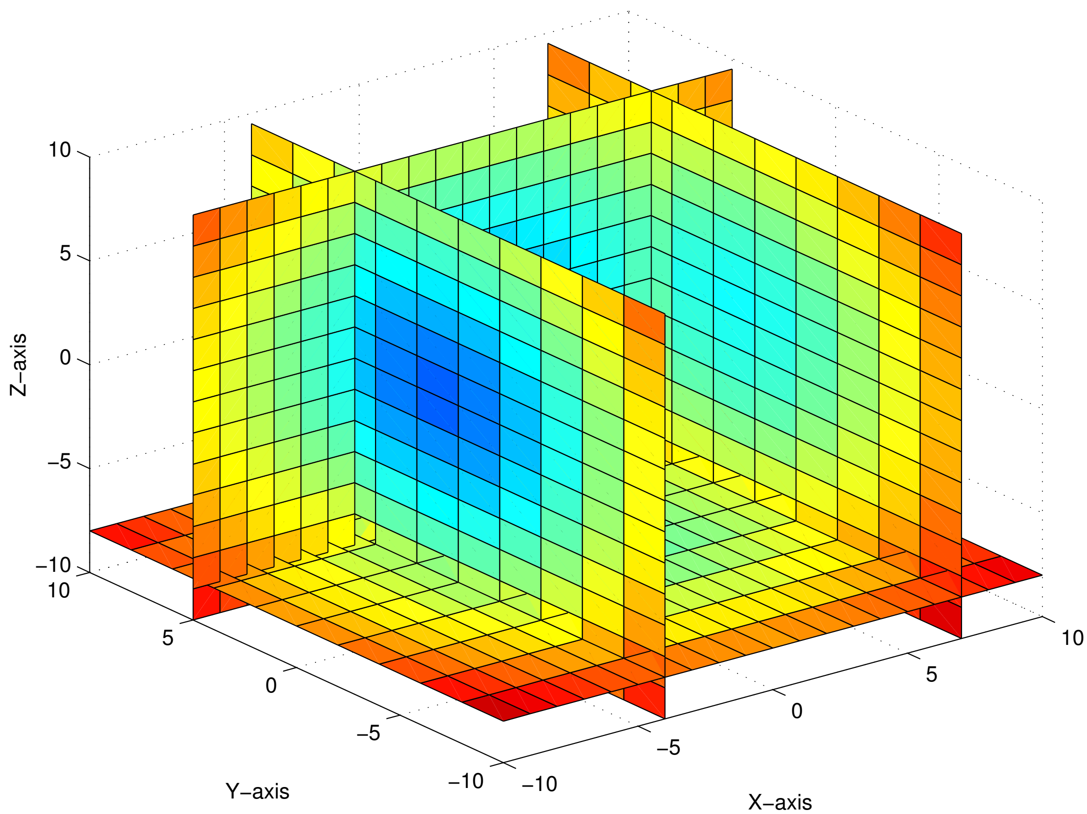

Figure 1: Cover Figure: Slices through the X = -4,X=7, Y=5 and Z= -8 planes

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1 Introduction

In this section of the Math 2417 notes, we examine the subject of integration and the various methods used to successfully evaluate an integral. Portions of these notes may be read as soon as you have completed §4.1 of your text book. However, there will be other parts which will refer to sections in Chapter 7 of your text book.

2 Basic Integration

Since integration is the reverse of differentiation,we can establish a rudimentary table of integrals by reversing the known derivatives. Table 1 on page 8 of these notes lists a few elementary integrals. Integrals leading to inverse trigonometric functions are given in Table 4 on page 22 of these notes. There are two common errors to which I would like to draw your attention.

ERROR 1: Some students believe that the integral of a product is the product of the integrals. This is WRONG.

For example $\int 2x\cos x\ dx$ is NOT equal to $x^2\sin x + C$

You may easily see this by differentiating the supposed answer using the product rule.

ERROR 2: Another mistake students make at this level is to believe that the integral of a quotient is the quotient of the integrals. Be very clear about this. The integral of a quotient is NOT the quotient of the integrals.

$\int \frac{4x^3+1}{3x^2}\,dx$ is NOT equal to $\frac{x^4+x}{x^3} +$ C. As explained below, the correct answer is $\int \frac{4x^3+1}{3x^2}\,dx = \frac{2}{3}x^2 - \frac{1}{3x} +$ C

1. Example $\int(3x^2+2x+7)\,dx$ This is a very simple problem. Integrating term by term, we have

$$\int (3x^2+2x+7)\,dx = x^3+x^2+7x+C$$

Remember to include the arbitrary constant namely $+C$

2. Example $\int \frac{x^3+x+1}{x^{1/2}} dx$
As a general rule, when there is a single term in the denominator of a rational expression, a good starting point is to divide out.

$$\begin{aligned} \int \frac{x^3+x+1}{x^{1/2}} dx &= \int (x^{5/2} + x^{1/2} + x^{-1/2} dx \\ &= \frac{2x^{7/2}}{7} + \frac{2x^{3/2}}{3} + 2x^{1/2} + C \end{aligned}$$

3. Example $\int \frac{x^3 - x^2 + 2x + 7}{x+1}\, dx$
This is an example of a general type of problem $\int \frac{P(x)}{Q(x)}\, dx$ where P(x) is a polynomial of degree greater than or equal to the degree of the polynomial Q(x). In this case , the first step is to divide.

$$\begin{aligned} \int \frac{x^3 - x^2 + 2x + 7}{x+1}\, dx &= \int (x^2 - 2x + 4 + \frac{3}{x+1}\, dx \\ &= \frac{x^3}{3} - x^2 + 4x + 3\ln|x+1| + C \end{aligned}$$

4. Example $\int x(x^3+1)\, dx$
Before integrating it is necessary to multiply out.

$$\begin{aligned} \int x(x^3+1)\, dx &= \int (x^4 + x)\, dx \\ &= \frac{x^5}{5} + \frac{x^2}{2} + C \end{aligned}$$

3 Integration By Substitution

In several previous math situations, the use of a substitution method has simplified the solution of the problem. For example, if you were asked to solve the following equation $2(2x-3)^2 - 7(2x-3) + 3 = 0$ you could multiply out,

Table 1: List of Integrals

$\int\ f(x)\ dx$	$F(x)+C$
$\int x^n\ dx$	$\dfrac{x^{n+1}}{n+1}+C$ $n\neq -1$
$\int \frac{1}{x}\ dx$	$\ln\|x\|+C$
$\int k\ dx$	$kx+C$
$\int \sin x\ dx$	$-\cos x+C$
$\int \cos x\ dx$	$\sin x+C$
$\int \sec^2 x\ dx$	$\tan x+C$
$\int \sec x \tan x\ dx$	$\sec x+C$
$\int \text{cosec}\ x \cot x\ dx$	$-\text{cosec}\ x+C$
$\int \text{cosec}^2 x\ dx$	$-\cot x+C$
$\int e^x\ dx$	e^x+C
$\int [f(x)\pm g(x)]\ dx$	$\int f(x)\ dx \pm \int g(x)\ dx$

gather terms and then solve the resulting quadratic equation. An alternative approach would be to make a substitution for the repeated term namely let $u = 2x - 3$ then the equation to be solved becomes $2u^2 - 7u + 3 = 0$. This is easily factored as $(2u - 1)(u - 3) = 0$ so $u = 1/2, 3$ Returning to x, we now have that $2x - 3 = 1/2$ or $2x - 3 = 3$ so that $x = 7/4, 3$ The substitution has made the problem a little longer but has made the solution easier to find. In an earlier section § 2.5 of your textbook, we met the chain rule for differentiation. Recall if given $y = \sqrt{x^3 + 1}$ then to find $\dfrac{dy}{dx}$ we let $u = x^3 + 1$ Then $y = \sqrt{u}$. The problem is now reduced to two very simple steps. $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$. Or in this case, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{u}} \dfrac{3x^2}{1}$ The answer must be given in terms of the original variable. $\dfrac{dy}{dx} = \dfrac{3x^2}{2\sqrt{x^3 + 1}}$ Integration by substitution is the reverse of this process. We will illustrate the method by considering a number of examples starting with an extremely simple one and moving to more complicated ones.

5. Example $\displaystyle\int \frac{3x^2}{2\sqrt{x^3 + 1}}\,dx$ The difficulty is not the square root but what is under the radical. In general, the overall aim of a substitution is to replace two or more symbols with a single symbol. This enables you to use the laws of exponents to simplify the expression or to transform the integral to a standard form. If we let $u = x^3 + 1$ then $\dfrac{du}{dx} = 3x^2$. Rewriting as a differential, $du = 3x^2 dx$. This is used to change the original integral to one involving the variable 'u' rather than 'x'.

   $\displaystyle\int \frac{3x^2}{2\sqrt{x^3 + 1}}\,dx$ becomes $\displaystyle\int \frac{du}{2\sqrt{u}} = \sqrt{u} + C$.
   Once again, you are reminded that you must express the answer in terms of the original variable.

$$\int \frac{3x^2}{2\sqrt{x^3 + 1}}\,dx = \sqrt{x^3 + 1} + C$$

   At this stage, the exercises are very simple and could be worked out without writing down the substitution. However, as the problems increase in complexity and require perhaps two or more substitutions in a single problem, it is important and good practice to always write down the substitutions. In example 5, everything was perfectly setup. No adjustments had to be made to complete the change from 'x' to 'u' . Usually this won't be the case.

6. Example $\int (3x+5)^4\,dx$ The complication in this integral is not the fourth power but the fact that it is $3x+5$ that is being raised to the fourth power. Let $u = 3x+5$, then $\frac{du}{dx} = 3$. Rewriting as a differential $du = 3dx$ or $\frac{du}{3} = dx$. Changing the original integral from "x" to the new variable "u", we have

$$\int (3x+5)^4\,dx = \frac{1}{3}\int u^4 du$$

This is now a simple integral namely $\frac{u^5}{15} + C$ Returning the answer to the original variable "x" , we have

$$\int (3x+5)^4\,dx = \frac{(3x+5)^5}{15} + C$$

You should verify your answer by differentiating the right hand side to obtain the original integrand.

7. Example $\int \frac{6x+12}{\sqrt{x^2+4x+5}}dx$
The complication is obvious here. It is not the radical but the expression under the radical. Let $u = x^2+4x+5$,then $\frac{du}{dx} = 2x+4$. Rewriting as a differential, $du = 2(x+2)dx$ or $\frac{du}{2} = (x+2)dx$. If this simple substitution is going to work, then the numerator must be expressed in terms of $(x+2)$

$$\int \frac{6x+12}{\sqrt{x^2+4x+5}}dx = 6\int \frac{x+2}{\sqrt{x^2+4x+5}}dx$$

Changing the original integral from "x" to the new variable "u", we have

$$\begin{aligned}
\int \frac{6x+12}{\sqrt{x^2+4x+5}}dx &= 6\int \frac{x+2}{\sqrt{x^2+4x+5}}dx \\
&= \frac{6}{2}\int \frac{du}{\sqrt{u}} \\
&= 6\sqrt{u} + C \\
&= 6\sqrt{x^2+4x+5} + C
\end{aligned}$$

8. Example $\int x\sqrt{1-x^2}\,dx$ .This is a straight forward substitution. Let $u = 1 - x^2$ Then $\frac{du}{dx} = -2x\;dx$ or $du = -2xdx$ so that $\frac{-du}{2} = xdx$
Changing the original integral from "x" to the new variable "u", we have

$$\begin{aligned}\int x\sqrt{1-x^2}\,dx &= \frac{-1}{2}\int u^{1/2}\,du \\ &= \frac{-1}{2}\frac{2}{3}u^{3/2} + C \\ &= \frac{-1}{3}(1-x^2)^{3/2} + C\end{aligned}$$

You should verify your answer by differentiation.

9. Example $\int x^2\sqrt{1-x^2}\,dx$ If we try the same method as used in Example 8, we run into trouble. To see this , repeat the above procedure. Let $u = 1 - x^2$ Then $\frac{du}{dx} = -2x\;dx$ or $du = -2xdx$ so that $\frac{-du}{2} = xdx$
Changing the original integral from "x" to the new variable "u", we have

$$\int x^2\sqrt{1-x^2}\,dx = \frac{-1}{2}\int x\;u^{1/2}\,du$$

Unfortunately there is an "extra" x that cannot be changed over to the new variable in a simple manner. This integral has to be solved using a trigonometric substitution.(See section 11)

10. Example $\int x^3\;\sqrt{1-x^2}\,dx$ If we let $u = 1 - x^2$. Then $\frac{du}{dx} = -2x\;\;dx$ or $du = -2xdx$ so that $\frac{-du}{2} = xdx$
Changing the original integral from "x" to the new variable "u", we have

$$\int x^3\;\sqrt{1-x^2}\,dx = \frac{-1}{2}\int x^2\;u^{1/2}\,du$$

Unlike the previous example,there is an "extra" x^2 that can be changed over to the new variable in a simple manner, namely, $x^2 = 1 - u$,using the substitution. Hence we can write

$$\begin{aligned}
\int x^3\sqrt{1-x^2}\,dx &= \frac{-1}{2}\int x^2\,u^{1/2}\,du \\
&= \frac{-1}{2}\int (1-u)\,u^{1/2}\,du \\
&= \frac{-1}{2}\int (u^{1/2}-u^{3/2})\,du \\
&= \frac{-1}{2}\left(\frac{2}{3}u^{3/2}-\frac{2}{5}u^{5/2}\right)+C \\
&= \frac{u^{5/2}}{5}-\frac{u^{3/2}}{3}+C \\
&= \frac{(1-x^2)^{5/2}}{5}-\frac{(1-x^2)^{3/2}}{3}+C \\
&= \frac{-(3x^2+2)(1-x^2)^{3/2}}{15}+C
\end{aligned}$$

11. Exercises

(a) Evaluate the following indefinite integrals

i. $\int (x+2)(x+1)^7\,dx$

ii. $\int (x+1)^2(x+2)^5\,dx$

iii. $\int \dfrac{(x+2)}{(x+3)^3}\,dx$

iv. $\int \dfrac{15x^2+5}{\sqrt{x^3+x+17}}\,dx$

v. $\int x^5\sqrt{x^2+1}\,dx$

(b) Solve the following differential equations

i. $\dfrac{dy}{dx}=\dfrac{6x^3+1}{(3x^4+2x+1)^2}$

ii. $\dfrac{dy}{dx}=\dfrac{x+1}{\sqrt{x^2+2x+5}}$

4 Integration of Trigonometric Functions I

Students often have trouble evaluating trigonometric integrals because they are unsure as to where to start. Lets begin this section by listing a few general questions that you might ask.

- Will a simple substitution help?
- Will a reduction to sine and cosine functions help?
- Will a trigonometric identity simplify the problem?
- Can the problem be transformed by means of a "Golden Rule"

Lets now consider some examples illustrating each item of the above list.

In table 1 , integrals of simple trigonometric functions were listed, e.g. $\int \cos x \; dx = \sin x + C$. Using the methods described in section 3 we may extend these entries to include problems that may be reduced to a simple form by means of a suitable substitution.

12. Example $\int \sin 5x \; dx$. If we let $u = 5x$, then $du = 5dx$ or $\frac{du}{5} = dx$ Therefore $\int \sin 5x \; dx$ can be transformed into $\frac{1}{5}\int \sin u \; du$ which equals $-\frac{1}{5}\cos u + C$ As we discussed in section 3, never leave your answer in terms of the new variable. Always give your answer in terms of the original variable. Hence $\int \sin 5x \; dx = -\frac{1}{5}\cos 5x + C$ This approach can be applied to all of the trigonometric functions. In fact, it leads to the revised version of portion of table 1 given as table 2 on page 14.

    You are encouraged to learn these formulae so that you able to integrate simple trigonometric functions at sight. For example, you should be able to answer the following questions without recourse to table 2

13. Exercises

    (a) $\int \sin 17x \; dx$
    (b) $\int \cos 7x \; dx$
    (c) $\int \sec^2 3x \; dx$
    (d) $\int \cos(x/2) \; dx$
    (e) $\int \sin(\pi x/2) \; dx$

14. Example $\int x\cos(x^2) \; dx$. In this problem, the most complicated part of the integrand is the argument of the trigonometric function. If we make the sub-

Table 2: Revised list of Trigonometric Integrals

$\int \sin ax \; dx$	$-\dfrac{\cos ax}{a} + C$
$\int \cos ax \; dx$	$\dfrac{\sin ax}{a} + C$
$\int \sec^2 ax \; dx$	$\dfrac{\tan ax}{a} + C$
$\int \mathrm{cosec}^2 ax \; dx$	$-\dfrac{\cot ax}{a} + C$
$\int \mathrm{cosec}\, ax \cot ax \; dx$	$-\dfrac{\mathrm{cosec}\, ax}{a} + C$
$\int \sec ax \tan ax \; dx$	$\dfrac{\sec ax}{a} + C$

stitution $u = x^2$, then $du = 2x\ dx$ so that the integral becomes $\frac{1}{2}\int \cos u\ du$ which equals $\frac{1}{2}\sin u + C$. or $\frac{1}{2}\sin x^2 + C$

15. Example $\int \frac{\sin(1/x)}{x^2}\ dx$. Here, the major complication is the (1/x) in the argument of the trigonometric function. If we let $u = 1/x$, then $du = -(1/x^2)\ dx$ The integral now becomes $-\int \sin u\ du = \cos u + C$ Transforming back to the original variable, the answer is $\cos(1/x) + C$

16. Exercises

    (a) $\int x\sin(x^2+1)\ dx$
    (b) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\ dx$
    (c) $\int \frac{\cos(\pi/x)}{x^2}\ dx$
    (d) $\int \frac{\sec\sqrt{x}\tan\sqrt{x}}{\sqrt{x}}\ dx$

17. Example $\int \frac{\tan x}{\sec x}\ dx$ By changing every function to the proper combination of sines or cosines and simplifying,this problem reduces to a trivial case namely $\int \sin x\ dx$ which is a standard integral.

18. Exercises

    (a) $\int \cot x \sin x\ dx$
    (b) $\int \tan x \cos x\ dx$
    (c) $\int \frac{2}{\sec x\ \cot x}\ dx$
    (d) $\int \frac{\cot x}{\text{cosec}\ x}\ dx$
    (e) $\int \sin^2 x\ \text{cosec}\ x\ dx$

19. Example $\int(\sin^2 x + \cos^2 x)\, dx$ By means of the trigonometric identity, $\cos^2 x + \sin^2 x = 1$, this integral is easily evaluated as $x + C$

20. Exercises

    (a) $\displaystyle\int \tan^2 x + 1\ dx$

    (b) $\displaystyle\int \text{cosec}^2\ x - \cot^2 x\ dx$

    (c) $\int \cos 5x\ \ \cos 4x + \sin 5x\ \ \sin 4x\ dx$

21. Golden rules These "Golden Rules" are simply some automatic substitutions when dealing with certain combinations of trigonometric functions.The following "Golden Rules" will be useful.

    Rule 1 $\int F(a + b\sin x)\ \cos x\ dx$ Automatic substitution $u = a + b\sin x$

    Rule 2 $\int F(a + b\cos x)\ \sin x\ dx$ Automatic substitution $u = a + b\cos x$

    Rule 3 $\int F(a + b\tan x)\ \sec^2 x\ dx$ Automatic substitution $u = a + b\tan x$

    Rule 4 $\int F(a + b\cot x)\ \text{cosec}^2 x\ dx$ Automatic substitution $u = a + b\cot x$

    Rule 5 $\int F(a + b\sec x)\ \sec x \tan x\ dx$ Automatic substitution $u = a + b\sec x$

22. Example $\displaystyle\int \frac{\cos x}{\sin^5 x}\, dx$ If you rewrite this problem as $\displaystyle\int \frac{1}{\sin^5 x}\ \cos x\ dx$ you can see that it clearly falls into the category described in Rule 1 If we let $u = \sin x$,then $du = \cos x\ dx$ and the problem transforms into a simple case, namely $\displaystyle\int \frac{du}{u^5} = \frac{-1}{4u^4}\ + C = \frac{-1}{4\sin^4 x} + C$

23. Example $\displaystyle\int \frac{\cos x}{(3 + 2\sin x)}\, dx$ In each case mentioned in item # 21, the suggested substitutions should be interpreted liberally. The important thing is that the derivative of the substitution is the term not associated with the function. If you let u equal a linear function such as $u = 3 + 2\sin x, du = 2\cos x\ dx$ so that the integral transforms to $\displaystyle\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$ or $\displaystyle\frac{1}{2}\ln|3 + 2\sin x|$ In the present case since $3 + 2\sin x \geq 1$ we may omit the absolute value signs and write $\displaystyle\int \frac{\cos x}{(3 + 2\sin x)}\, dx = \frac{1}{2}\ln(3 + 2\sin x) + C$. When you do this, give a clear mathematical justification for taking this step.

24. Exercises

   (a) $\int \frac{\sin x}{\sqrt{(5+2\cos x)}}\, dx$

   (b) $\int \frac{\sec^2 x}{(4+7\tan x)^3}\, dx$

   (c) $\int \frac{\cos x}{(7-3\sin x)^3}\, dx$

4.1 Six Basic Trigonometric Integrals

25. Example $\int \sin x\ dx$
This is a simple anti derivative from table 1 $\int \sin x\ dx = -\cos x + C$

26. Example $\int \cos x\ dx$ Once again this integral may be evaluated from table 1 $\int \cos x\ dx = \sin x + C$

27. Example $\int \tan x\ dx$ Rewriting this problem in terms of sines and cosines we have $\int \tan x\ dx = \displaystyle\int \frac{\sin x}{\cos x}\, dx$. This is a simple example of Golden Rule # 2 namely $\int F(\cos x)\sin x\ dx$ If we let $u = \cos x$ then $-du = \sin x\ dx$ so that the integral reduces to $-\int \frac{dx}{u} = -\ln|\cos x| + C$ Using logarithim rules, an alternative answer would be $\ln|\sec x| + C$

28. Example $\int \cot x\ dx$ Imitating the procedure that we followed in Example 27
$\int \cot x\ dx = \displaystyle\int \frac{\cos x}{\sin x}\, dx$. This is a simple example of Golden Rule # 1 namely $\int F(a+b\sin x)\cos x\ dx$ If we let $u = \sin x$ then $du = \cos x\ dx$ so that the integral reduces to $\int \frac{dx}{u} = \ln|\sin x| + C$
Note If you can integrate a trigonometric function following a certain procedure, then usually imitating that same procedure will enable you to integrate the corresponding co-function. For example, if you know how to integrate $\tan^3 x$, then you should be able to integrate $\cot^3 x$

29. Example $\int \sec x\ dx$ This integral requires a tricky manipulation . Later, in section 12 we shall see another approach.

$$\int \sec x\ dx = \int \frac{(\sec x)(\sec x + \tan x)}{\sec x + \tan x}\ dx \text{ or}$$

$$\int \sec x\ dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\ dx$$

If you let $u = \sec x + \tan x$ then $du = (\sec x \tan x + \ sec^2 x)dx$ so that the integral changes to $\int \frac{du}{u} = \ln|u| + C$ or $\int \sec x\ dx = \ln|\sec x + \tan x| + C$

30. Example $\int \text{cosec}\ x\ dx$ We shall imitate the procedure used in example 29 See section 12 for another approach to this problem.

$$\int \text{cosec}\ x\ dx = \int \frac{(\text{cosec}\ x)(\text{cosec}\ x + \cot x)}{\text{cosec}\ x + \cot x}\ dx \text{ or}$$

$$\int \text{cosec}\ x\ dx = \int \frac{\text{cosec}^2 x + \text{cosec}\ x \cot x}{\text{cosec}\ x + \cot x}\ dx$$

If you let $u = \text{cosec}\ x + \cot x$ then $du = -(\text{cosec}\ x \cot x + \text{cosec}^2 x)dx$ so that the integral changes to $-\int \frac{du}{u} = -\ln|u| + C$ or

$\int \text{cosec}\ x\ dx = -\ln|\text{cosec}\ x + \cot x| + C$

The results from Examples $25 - 30$ are summarized in Table 3

4.2 Integrals of squares of trigonometric functions

The following trigonometric identities will be useful in this section

$$\cos 2x = 2\cos^2 x - 1 \quad (1)$$
$$\cos 2x = 1 - 2\sin^2 x \quad (2)$$
$$\tan^2 x + 1 = \sec^2 x \quad (3)$$
$$\cot^2 x + 1 = \text{cosec}^2 x \quad (4)$$

31. Example $\int \sin^2 x\ dx$ Using identity # 2, we can change this problem to $\int \frac{1 - \cos 2x}{2}\ dx$ which becomes $\frac{x}{2} - \frac{\sin 2x}{4} + C$

32. Example $\int \cos^2 x\ dx$ Using identity # 1, we can change this problem to $\int \frac{1 + \cos 2x}{2}\ dx$ which becomes $\frac{x}{2} + \frac{\sin 2x}{4} + C$

Table 3: Integrals of Six Trigonometric Functions

$\int \sin x \; dx$	$-\cos x + C$
$\int \cos x \; dx$	$\sin x + C$
$\int \tan x \; dx$	$-\ln\lvert\cos x\rvert + C$ $\ln\lvert\sec x\rvert$ + C
$\int \text{cosec } x \; dx$	$-\ln\lvert\text{cosec } x + \cot x\rvert + C$ $\ln\lvert\text{cosec } x - \cot x\rvert + C$ $\ln\lvert\tan(x/2)\rvert + C$
$\int \sec x \; dx$	$\ln\lvert\sec x + \tan x\rvert + C$ $-\ln\lvert\sec x - \tan x\rvert + C$
$\int \cot x \; dx$	$\ln\lvert\sin x\rvert + C$

33. Example $\int \tan^2 x\ dx$ Using identity # 3,we can change $\tan^2 x$ into $\sec^2 x - 1$ which is easy to integrate.Therefore
$\int \tan^2 x\ dx = \int \sec^2 x - 1\ dx = \tan x - x + C$

34. Example $\int \text{cosec}^2\ x\ dx$ This is a straight anti-derivative from table 1.

$$\int \text{cosec}^2 x\ dx = -\cot x + C$$

35. Example $\int \sec^2 x\ dx$ This is a straight anti-derivative from table 1.

$$\int \sec^2 x\ dx = \tan x + C$$

36. Example $\int \cot^2 x\ dx$ Imitating the procedure in Example 33, we use the identity # 4 to rewrite the problem.
$\int \cot^2 x\ dx = \int (\text{cosec}^2 x - 1)\ dx = -\cot x - x + C$

5 Integration of algebraic functions leading to Inverse Trigonometric functions

37. Preliminary Work Evaluate the following expressions without using your calculator.

    (a) $\arctan(-1)$
    (b) $\arctan(1)$
    (c) $\arctan(-\sqrt{3})$
    (d) $\text{arcsec}(2)$
    (e) $\text{arcsec}(-2)$
    (f) $\text{arcsec}\left(\frac{2\sqrt{3}}{3}\right)$
    (g) $\text{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$
    (h) $\arcsin\left(\frac{1}{2}\right)$
    (i) $\arcsin\left(-\frac{1}{2}\right)$

In table 4, three integrals are listed leading to inverse trigonometric functions. In this section, we show how it is possible to modify a variety of integrals into one of these three forms.

5.1 Algebraic Integrals leading to an arcsin function

In the following examples we try to convert the problem to an integral leading to an arcsin function. This will require the denominator to be in the form $\sqrt{1-u^2}$ and only numerical factors in the numerator.

38. Example $\displaystyle\int \frac{dx}{\sqrt{1-9x^2}}$

The difficulty here is the $9x^2$. If we replace $9x^2$ by u^2 or equivalently by letting $u = 3x$, it is possible to change this integral into one giving an inverse sine function for an answer. With $u = 3x$ and $du = 3dx$, we have the following.

$$\int \frac{dx}{\sqrt{1-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}}$$

The integral is now in standard form and may be easily evaluated as

$$\frac{1}{3}\arcsin u + C = \frac{1}{3}\arcsin(3x) + C$$

39. Example $\displaystyle\int \frac{dx}{\sqrt{4-9x^2}}$

In this example, we use the substitution $9x^2 = 4u^2$ or $3x = 2u$ or $x = \dfrac{2u}{3}$

Hence $dx = \dfrac{2du}{3}$. Making these substitutions, we have

$$\begin{aligned}\int \frac{dx}{\sqrt{4-9x^2}} &= \int \frac{2dt}{3\sqrt{4-4u^2}} \\ &= \int \frac{dt}{3\sqrt{1-u^2}} \\ &= \frac{1}{3}\arcsin u + C\end{aligned}$$

$\int \frac{dx}{\sqrt{1-x^2}}$	$\arcsin x + C$
$\int \frac{dx}{1+x^2}$	$\arctan x + C$
$\int \frac{dx}{x\sqrt{x^2-1}}$	$\text{arcsec}\lvert x\rvert + C$

Table 4: Algebraic Integrals and Inverse Trigonometric Functions

Hence the answer to the exercise may be written

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\arcsin\left(\frac{3x}{2}\right) + C$$

40. Example $\int \frac{dx}{\sqrt{6-x^2}}$

In this example, the difficulty arises from the presence of 6 rather than 1 under the radical sign. If we let $x^2 = 6u^2$ or $x = \sqrt{6}u$, then $dx = \sqrt{6}du$, the integral transforms as follows

$$\int \frac{dx}{\sqrt{6-x^2}} = \int \frac{\sqrt{6}\,du}{\sqrt{6-6u^2}}$$

After cancelling the numerical factor $\sqrt{6}$, this integral simplifies to a standard form, namely $\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C$.

Transforming back to the original variable, we have

$$\int \frac{dx}{\sqrt{6-x^2}} = \arcsin\left(\frac{x}{\sqrt{6}}\right) + C$$

41. Example $\int \frac{dx}{\sqrt{52+12x-3x^2}}$

In the previous examples, we have gradually increased the complexity of the problem considered. In this final example leading to an inverse sine function, first, we must complete the square.

$$\begin{aligned} 52+12x-3x^2 &= -3x^2+12x+52 \\ &= -3[x^2-4x-(52/3)] \\ &= -3[(x-2)^2-4-(52/3)] \\ &= -3(x-2)^2+64 \end{aligned}$$

$$\begin{aligned} \int \frac{dx}{\sqrt{52+12x-3x^2}} &= \int \frac{dx}{\sqrt{64-3(x-2)^2}} \\ &= \int \frac{8du}{\sqrt{3}\sqrt{64-64u^2}} \\ &= \int \frac{du}{\sqrt{3}\sqrt{1-u^2}} \\ &= \frac{1}{\sqrt{3}} \arcsin u + C \\ &= \frac{1}{\sqrt{3}} \arcsin \left(\frac{\sqrt{3}(x-2)}{8} \right) + C \end{aligned}$$

Here we have made the substitution $3(x-2)^2 = 64u^2$ or $x-2 = \frac{8u}{\sqrt{3}}$ leading to $dx = \frac{8}{\sqrt{3}}du$. At the end of the problem, we have solved the substitution equation for u, namely $u = \frac{\sqrt{3}(x-2)}{8}$

42. Exercises In the following exercises, practice converting the integral into a simple standard integral leading to an inverse sine function

(a) $\int \frac{dx}{\sqrt{1-5x^2}}$

(b) $\int \frac{dx}{\sqrt{10-x^2}}$

(c) $\int \frac{dx}{\sqrt{7-3x^2}}$

(d) $\int \frac{dx}{\sqrt{6+4x-x^2}}$

(e) $\displaystyle\int \frac{dx}{\sqrt{7+12x-2x^2}}$

(f) $\displaystyle\int \frac{4\ dx}{\sqrt{-x^2-8x}}$

(g) $\displaystyle\int \frac{x\ dx}{\sqrt{1-x^4}}$ Hint. Let $u = x^2$

5.2 Integrals leading to an arctan function

In these exercises, we try to convert the problem to an integral leading to an inverse tangent function. This will require the denominator to be in the form $1+u^2$ and only numerical factors in the numerator.

43. Example $\displaystyle\int \frac{dx}{1+6x^2}$

If we let $u^2 = 6x^2$ or $u = \sqrt{6}\ x$ Then $du = \sqrt{6}\ dx$. Making these substitutions, we have

$$\begin{aligned}\int \frac{dx}{1+6x^2} &= \frac{1}{\sqrt{6}}\int \frac{du}{1+u^2}\\ &= \frac{1}{\sqrt{6}}\arctan u + C\\ &= \frac{1}{\sqrt{6}}\arctan(\sqrt{6}x) + C\end{aligned}$$

44. Example $\displaystyle\int \frac{dx}{9+x^2}$
If we let $x^2 = 9u^2$ or $x = 3u$ then $dx = 3du$ Making these substitutions, we have

$$\begin{aligned}\int \frac{dx}{9+x^2} &= \int \frac{3du}{9+9u^2}\\ &= \int \frac{du}{3(1+u^2)}\\ &= \frac{1}{3}\arctan u + C\\ &= \frac{1}{3}\arctan(x/3) + C\end{aligned}$$

45. $\displaystyle\int \frac{dx}{9+4x^2}$

In this problem, we must convert the denominator into the form $1+u^2$. We can accomplish this by the following substitution $9u^2 = 4x^2$ or $x = \dfrac{3u}{2}$ leading to $dx = \dfrac{3du}{2}$. Changing the variable in the original problem, we have

$$\begin{aligned}
\int \frac{dx}{9+4x^2} &= \int \frac{3du}{2(9+9u^2)} \\
&= \int \frac{du}{6(1+u^2)} \\
&= \frac{1}{6} \arctan u + C \\
&= \frac{1}{6} \arctan\left(\frac{2x}{3}\right) + C
\end{aligned}$$

Note that we have returned the answer to the original variable x by solving the substitution equation for u namely $u = \dfrac{2x}{3}$.

46. $\displaystyle\int \frac{dx}{16+25(x-2)^2}$

If we make the substitution $25(x-2)^2 = 16u^2$, we can convert the denominator into the form $1+u^2$. From the substitution, we obtain $5(x-2) = 4u$ or $x = \dfrac{4u}{5} + 2$ leading to $dx = \dfrac{4du}{5}$. Making these substitutions into the original problem, we have

$$\begin{aligned}
\int \frac{dx}{16+25(x-2)^2} &= \int \frac{4du}{5(16+16u^2)} \\
&= \int \frac{du}{20(1+u^2)} \\
&= \frac{1}{20} \arctan u + C \\
&= \frac{1}{20} \arctan\left(\frac{5(x-2)}{4}\right) + C
\end{aligned}$$

Note that we have returned the answer to the original variable x by solving the substitution equation for u namely $u = \dfrac{5(x-2)}{4}$.

47. $\displaystyle\int \frac{dx}{9x^2+18x+13}$

We begin by completing the square

$$\begin{aligned} 9x^2+18x+13 &= 9\left[x^2+2x+\frac{13}{9}\right] \\ &= 9\left[(x+1)^2-1+\frac{13}{9}\right] \\ &= 9(x+1)^2-9+13 \\ &= 9(x+1)^2+4 \end{aligned}$$

In the integral we need to convert $9(x+1)^2+4$ into the form $1+u^2$ by an appropriate substitution. Let $9(x+1)^2 = 4u^2$ or $x+1 = \dfrac{2u}{3}$ Hence $dx = \dfrac{2du}{3}$ Making these substitutions, we have

$$\begin{aligned} \int \frac{dx}{9x^2+18x+13} &= \int \frac{dx}{9(x+1)^2+4} \\ &= \int \frac{2du}{3(4u^2+4)} \\ &= \int \frac{du}{6(u^2+1)} \\ &= \frac{1}{6}\arctan\left(\frac{3(x+1)}{2}\right)+C \end{aligned}$$

In the above work, we have returned the answer to the original variable by solving the substitution equation for u namely $u = \dfrac{3(x+1)}{2}$

48. <u>Exercises</u> In the following exercises convert the given integrand into a form leading to an arctan function.

(a) $\displaystyle\int \frac{dx}{1+16x^2}$

(b) $\displaystyle\int \frac{dx}{64+x^2}$

(c) $\displaystyle\int \frac{dx}{81+25x^2}$

(d) $\displaystyle\int \frac{dx}{x^2+6x+13}$

(e) $\displaystyle\int \frac{dx}{4x^2+16x+25}$

(f) $\displaystyle\int \frac{e^x dx}{e^{2x}+2e^x+5}$

(g) $\displaystyle\int \frac{dx}{\sqrt{x}(1+x)}$ Hint. Let $x = t^2$

5.3 Integrals giving an arcsec function.

In the following examples, we try to convert the problem to an integral leading to an inverse secant function. This will require the denominator to be in the form $u\,\sqrt{u^2-1}$ and the numerator to contain only numerical factors.

49. $\displaystyle\int \frac{dx}{x\sqrt{x^2-9}}$
In this problem, the difficulty is the 9 under the radical sign. We need to convert to the form $u^2 - 1$. Let $x^2 = 9u^2$ or $x = 3u$ leading to $dx = 3du$. Making the sustitutions into the integral, we have

$$\begin{aligned}
\int \frac{dx}{x\sqrt{x^2-9}} &= \int \frac{3du}{3u\sqrt{9u^2-9}} \\
&= \int \frac{du}{3u\sqrt{u^2-1}} \\
&= \frac{\text{arcsec}|u|}{3} + C \\
&= \frac{1}{3}\text{arcsec}\left(\frac{|x|}{3}\right) + C
\end{aligned}$$

As is required, we give the answer in terms of the original variable by solving the substitution equation for u namely $u = \dfrac{x}{3}$

50. $\displaystyle\int \frac{dx}{x\sqrt{9x^2-1}}$
In this exercise, we need to transform the denominator into the form $u^2 - 1$.

We can achieve this by making the substitution $u^2 = 9x^2$ or $u = 3x$ or $dx = \dfrac{du}{3}$

$$\begin{aligned}
\int \frac{dx}{x\sqrt{9x^2-1}} &= \int \frac{du}{3(u/3)\sqrt{u^2-1}} \\
&= \int \frac{du}{u\sqrt{u^2-1}} \\
&= \text{arcsec}|\text{u}| + C \\
&= \text{arcsec}|\text{3x}| + C
\end{aligned}$$

51. $\displaystyle\int \frac{dx}{x\sqrt{4x^2-49}}$

As before, we need to convert the denominator into the form $u^2 - 1$. We can accomplish this by the substitution $4x^2 = 49u^2$ or $x = \dfrac{7u}{2}$leading to $dx = \dfrac{7du}{2}$ Making these substitutions into our problem, we have

$$\begin{aligned}
\int \frac{dx}{x\sqrt{4x^2-49}} &= \int \frac{7du}{2(7u/2)\sqrt{49u^2-49}} \\
&= \int \frac{7du}{49u\sqrt{u^2-1}} \\
&= \frac{1}{7}\text{arcsec}|u| + C \\
&= \frac{1}{7}\text{arcsec}\left|\frac{2x}{7}\right| + C
\end{aligned}$$

As in the previous examples, the answer is given in terms of the original variable x by solving the substitution equation $u = \dfrac{2x}{7}$

52. $\displaystyle\int \frac{dx}{(x-3)\sqrt{(x-3)^2-4}}$

We begin by letting $(x-3)^2 = 4t^2$ or $x - 3 = 2t$. Hence we may write $x = 2t + 3$ so that $dx = 2dt$. Making these substitutions into the integral, we have

$$\begin{aligned}\int \frac{dx}{(x-3)\sqrt{(x-3)^2-4}} &= \int \frac{2dt}{2t\sqrt{4t^2-4}} \\ &= \int \frac{dt}{2t\sqrt{t^2-1}} \\ &= \frac{1}{2}\text{arcsec}|t| + C \\ &= \frac{1}{2}\text{arcsec}\left|\frac{(x-3)}{2}\right| + C\end{aligned}$$

As in the previous examples, we have expressed the answer in terms of the original variable by solving the substitution equation for t namely $t = \dfrac{(x-3)}{2}$.

53. $\displaystyle\int \frac{dx}{(3x-3)\sqrt{2x^2-4x-7}}$
In this exercise, we must begin by completing the square under the radical sign.

$$\begin{aligned}2x^2-4x-7 &= 2\left[x^2-2x-\frac{7}{2}\right] \\ &= 2\left[(x-1)^2-1-\frac{7}{2}\right] \\ &= 2(x-1)^2-2-7 \\ &= 2(x-1)^2-9\end{aligned}$$

To convert the integral into a standard form, we make the substitution $2(x-1)^2 = 9u^2$ or $x-1 = \dfrac{3u}{\sqrt{2}}$ leading to $dx = \dfrac{3du}{\sqrt{2}}$. Substituting into the integral, we have

$$\begin{aligned}
\int \frac{dx}{(3x-3)\sqrt{2x^2-4x-7}} &= \int \frac{dx}{(3x-3)\sqrt{2(x-1)^2-9}} \\
&= \int \frac{3du}{3\sqrt{2}(\frac{3u}{\sqrt{2}})\sqrt{9u^2-9}} \\
&= \frac{1}{9}\int \frac{du}{u\sqrt{u^2-1}} \\
&= \frac{1}{9}\text{arcsec}|u| + C \\
&= \frac{1}{9}\text{arcsec}\left|\frac{\sqrt{2}(x-1)}{3}\right| + C
\end{aligned}$$

As in the previous exercises, the answer has been expressed in terms of the original variable x by solving the substitution equation for u namely
$u = \dfrac{\sqrt{2}(x-1)}{3}$

54. <u>Exercises</u> In the following exercises, use the above techniqes to evaluate the given integral

    (a) $\displaystyle\int \frac{dx}{4x\sqrt{x^2-25}}$

    (b) $\displaystyle\int \frac{dx}{x\sqrt{16x^2-1}}$

    (c) $\displaystyle\int \frac{dx}{x\sqrt{49x^2-25}}$

    (d) $\displaystyle\int \frac{dx}{(2x-10)\sqrt{(x-5)^2-16}}$

    (e) $\displaystyle\int \frac{dx}{(x-7)\sqrt{3x^2-42x+122}}$

    (f) $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}$ Hint. Let $t = x^2$

55. Examine the above worked examples very carefully. Learn to develop " recognition factors" to determine when you can and when you cannot apply this approach to solving an integration problem. Consider the following questions

(a) $\displaystyle\int \frac{x\,dx}{\sqrt{1-6x^2}}$

(b) $\displaystyle\int \frac{dx}{\sqrt{1-6x^2}}$

(c) $\displaystyle\int \frac{x^2dx}{\sqrt{1-6x^2}}$

(d) $\displaystyle\int \frac{x^3dx}{\sqrt{1-6x^2}}$

Only one of these exercises would be solved using the methods developed in this section. Two of the exercises would use a simple algebraic substitution $u = 1 - 6x^2$ and one of the integrations requires the method of §7.4 Identify each of these cases and evaluate all of them except the integral requiring §7.4 techniques.

5.4 Absolute value sign in answer to $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

In table 4, we state that $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}} = \operatorname{arcsec} \mid x \mid + C$. The requirement of the absolute sign is a result of our definition. Other definitions may not require the absolute value sign. In figure 2, we show the graph of $y = \dfrac{1}{x\sqrt{x^2-1}}$. Clearly the answer to $\displaystyle\int_{\frac{2\sqrt{3}}{3}}^{2} \frac{dx}{x\sqrt{x^2-1}}$ should be positive while $\displaystyle\int_{-2}^{\frac{-2\sqrt{3}}{3}} \frac{dx}{x\sqrt{x^2-1}}$ should give a negative answer.

$$\begin{aligned}
\int_{\frac{2\sqrt{3}}{3}}^{2} \frac{dx}{x\sqrt{x^2-1}} &= \Big[\operatorname{arcsec} \mid x \mid \Big]_{\frac{2\sqrt{3}}{3}}^{2} \\
&= \operatorname{arcsec} \mid 2 \mid - \operatorname{arcsec} \left|\frac{2\sqrt{3}}{3}\right| \\
&= \pi/3 - \pi/6 \\
&= \pi/6
\end{aligned}$$

The answer, $\pi/6$, is positive as expected. Of course ,here the absolute value signs played no role since the integration limits were positive. Now lets consider the other case.

$$\begin{aligned}
\int_{-2}^{\frac{-2\sqrt{3}}{3}} \frac{dx}{x\sqrt{x^2-1}} &= \Big[\text{arcsec} \mid x \mid \Big]_{-2}^{\frac{-2\sqrt{3}}{3}} \\
&= \text{arcsec}\left|\frac{-2\sqrt{3}}{3}\right| - \text{arcsec}|-2| \\
&= \text{arcsec}\ 2\sqrt{3}/3 - \text{arcsec}\ 2 \\
&= \pi/6 - \pi/3 \\
&= -\pi/6
\end{aligned}$$

As required, this answer is negative. However, if one is careless and omits the absolute value signs, we have an incorrect answer as shown below.

$$\begin{aligned}
\int_{-2}^{\frac{-2\sqrt{3}}{3}} \frac{dx}{x\sqrt{x^2-1}} &= \Big[\text{arcsec}\ x\Big]_{-2}^{\frac{-2\sqrt{3}}{3}} \\
&= \text{arcsec}\left(\frac{-2\sqrt{3}}{3}\right) - \text{arcsec}(-2) \\
&= 5\pi/6 - 2\pi/3 \\
&= \pi/6
\end{aligned}$$

This positive answer is incorrect as you may easily observe from an examination of figure 2. Remember, when you examine another book, there may be a different definition for arcsec$x$ and the absolute value sign may not be required.

6 Integration by Parts

This is a very powerful method of integration. It allows us to integrate a variety of functions. First let us recall the product rule. If $y = U\ V$ where U and V are functions of x , then

$$\frac{dy}{dx} = U\ \frac{dV}{dx} + V\ \frac{dU}{dx}$$

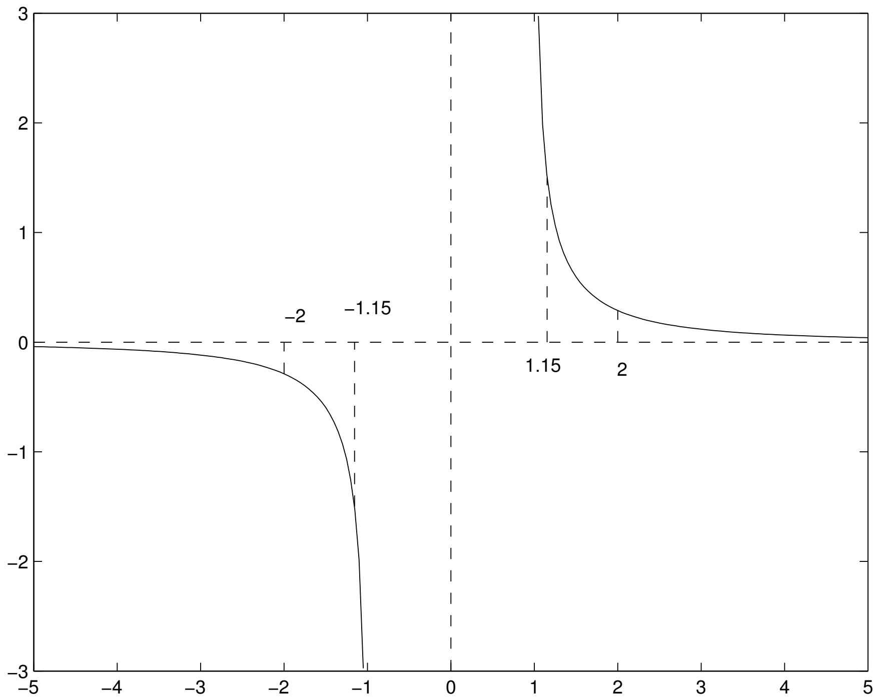


Figure 2: Graph of $y = \dfrac{1}{x\sqrt{x^2 - 1}}$

Rearranging , we have

$$U \frac{dV}{dx} = \frac{dy}{dx} - V \frac{dU}{dx}$$

Integrating both sides with respect to x, we have

$$\int U \frac{dV}{dx}\, dx = \int \frac{dy}{dx}\, dx - \int V \frac{dU}{dx}\, dx$$

Lets consider the term $\int \frac{dy}{dx}\, dx$ First we are asked to differentiate y with respect to x and then we are asked to integrate the result with respect to x. Obviously we have travelled a complete circle. The answer to this statement is simply "y" . Hence we can write

$$\int U \frac{dV}{dx}\, dx = y - \int V \frac{dU}{dx}\, dx$$

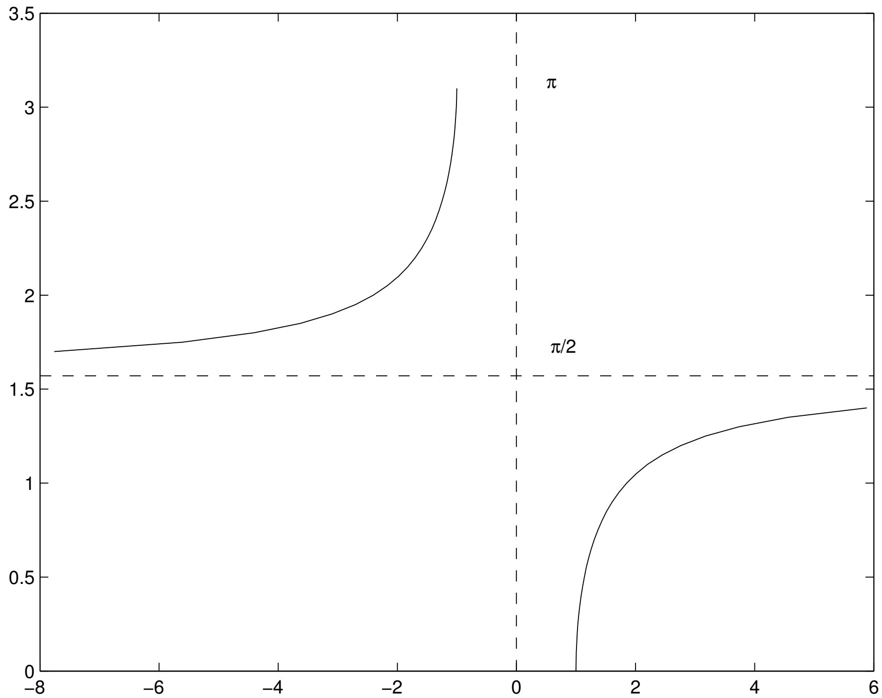


Figure 3: Graph of $y = \text{arcsec } x$ Domain $(-\infty < x \leq -1) \cup (1 \leq x < \infty)$
Range ; $(0 \leq x < \pi/2) \cup (\pi/2 < x \leq \pi)$

However, since $y = U\ V$, we can write

$$\boxed{\int U\ \frac{dV}{dx}\ dx = U\ V - \int V\ \frac{dU}{dx}\ dx}$$

This is the formula for Integration By Parts. The integral on the left is complicated so we change it to the one on the right side. Hence,as you do Integration By Parts problems, develop the habit of asking the very important question, "Is the Integral on the right side simpler than the one on the left side"? Have you made progress? If the answer is "NO', then either you should not be using this method or you are making errors in applying the method.Hopefully the answer will be yes , namely you have made progress. At the very least, there should be no increase in the complexity of the integrand. We shall see that in some cases (Examples 63 and 65), while there may appear to be a lack of progress, there certainly is not an increase in complexity of the integrand. Now you should develop your recognition factors. What types of integration problems may be solved using this technique? The following list is a general guide to the types of integrands which might be evaluated by this method. However it should not be regarded as all inclusive.

56. Integrands usually suitable for Integration by Parts

- Mixed Products
- Logarithms
- Inverse trigonometric functions
- Certain Trigonometric functions
- Hyperbolic and Inverse Hyperbolic Functions

Once you have established that the problem is best solved by Integration by Parts, I suggest that you write out the basic formula,namely

$$\int U \frac{dV}{dx} dx = U V - \int V \frac{dU}{dx} dx$$

Then set up an array containing the 4 quantities involved in the formula.

$\boxed{U =}$	$V =$
$\frac{dU}{dx} =$	$\boxed{\frac{dV}{dx} =}$

Usually, but as we shall see, not always, the integral on the left side is a product of two functions. For most integration by parts problems, you begin by selecting one of the functions as the " U " in the formula. Whatever remains is set equal to $\frac{dV}{dx}$ From the "U" selection, the term $\frac{dU}{dx}$ is generated. Likewise the "V" term is obtained by integrating the

$\frac{dV}{dx}$ term. Note that the original integrand is the product of the two terms in the boxed entries. An arbitrary constant is not required when obtaining the "V" term. No additional information is obtained by introducing a constant at this stage. (See Example 59) One interesting way of choosing the "U" term is to use the mnemonic ILATE to suggest the order of selection where

I; stands for an inverse function

L; stands for a logarithm function

A; stands for an algebraic function

T; stands for a trigonometric function

E; stands for an exponential function

The logic behind these suggestions is clear. Differentiating an inverse or a logarithmic function immediately gives an algebraic function. At the other end, differentiating an exponential does not lead to any simplification while integrating an exponential function does not complicate the problem. Lets consider an example from each type mentioned in item # 56.

57. Example $\int xe^{-x}\,dx$ As a first step, write out the integration by parts formula.

$$\int U\,\frac{dV}{dx}\,dx = U\,V - \int V\,\frac{dU}{dx}\,dx$$

Then write out the array containing the quantities involved in the formula and fill in the boxed entries.Then generate the remaining two entries.

$$\boxed{U = x} \qquad\qquad V = -e^{-x}$$

$$\frac{dU}{dx} = 1 \qquad\qquad \boxed{\frac{dV}{dx} = e^{-x}}$$

Substituting into the basic formula, we have

$$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx$$

Always ask the question"Have you made progress?" "Is the integral on the right side simpler than the one on the left side?"If the answer is "YES",then you are probably on the correct path for a solution to the problem. If the answer is "NO", then almost certainly you have made a mistake in setting up the problem or perhaps have applied the wrong technique to the problem. In any case, STOP and re-evaluate your position. Note that in the ILATE mnemonic, "A" comes before "E" thereby suggesting that the algebraic factor "x" be chosen as the assignment for "U". In the present case, the answer to our question is yes ,so we can procede.

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx \text{ or}$$

$$\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

58. Example $\int xe^{-x}\,dx$ In this example which is the same as discussed in Example 57, we make the incorrect choice for"U" and explore the consequences.

$$\int U\,\frac{dV}{dx}\,dx = U\,V - \int V\,\frac{dU}{dx}\,dx$$

$$\boxed{U = e^{-x}} \qquad\qquad V = \frac{x^2}{2}$$

$$\frac{dU}{dx} = -e^{-x} \qquad\qquad \boxed{\frac{dV}{dx} = x}$$

Substituting these values into the basic formula, we have

$$\int xe^{-x}\,dx = e^{-x}\,\frac{x^2}{2} - \int \frac{-x^2e^{-x}}{2}\,dx$$

$$\text{or}\quad \int xe^{-x}\,dx = \frac{x^2e^{-x}}{2} + \frac{1}{2}\int x^2e^{-x}\,dx$$

If we ask the question " Have you made progress? ", clearly there should be a negative response. You should immediately stop and examine the processes whereby you arrived at this impasse. If you have made an incorrect choice for "U", it should be immediately obvious after the first application of the parts formula.

59. Example $\int x^2 \ln x\,dx$ Here there is a natural restriction on the domain as $\ln x$ is defined only for $x > 0$ in the real number system. In this example, we

examine the effect of including an arbitrary constant C_1 to the "V" term. As before, write out the integration by parts formula.

$$\int U \; \frac{dV}{dx} \; dx = U \; V - \int V \; \frac{dU}{dx} \; dx$$

Then write out the array containing the quantities involved in the formula and fill in the boxed entries.Then generate the remaining two entries.

$$\boxed{U = \ln x} \qquad\qquad V = \frac{x^3}{3} + C_1$$

$$\frac{dU}{dx} = \frac{1}{x} \qquad\qquad \boxed{\frac{dV}{dx} = x^2}$$

Substituting into the basic formula, we have

$$\int x^2 \; \ln x \; dx = \left(\frac{x^3}{3} + C_1\right) \; \ln x - \int \left(\frac{x^3}{3} + C_1\right) \; \frac{1}{x} \; dx$$

$$\int x^2 \; \ln x \; dx = \frac{x^3 \; \ln x}{3} + C_1 \; \ln x - \int \frac{x^3}{3x} \; dx - \int \frac{C_1}{x} \; dx$$

or

$$\int x^2 \; \ln x \; dx = \frac{x^3 \; \ln x}{3} + C_1 \; \ln x - \frac{x^3}{9} - C_1 \; \ln x$$

or

$$\int x^2 \; \ln x \; dx = \frac{x^3 \; \ln x}{3} - \frac{x^3}{9} + C$$

Notice that the addition of the arbitrary constant C_1 contributed nothing to the final answer. However don't forget to include the arbitrary constant C at the end of the problem. Also note that we need not include absolute value signs around x in the $\ln x$ term as $x > 0$

60. Example $\int \ln x \;\; dx$ Note, we shall assume that $x > 0$ As before , write out the integration by parts formula.

$$\int U \; \frac{dV}{dx} \; dx = U \; V - \int V \; \frac{dU}{dx} \; dx$$

Then write out the array containing the quantities involved in the formula and fill in the boxed entries.Then generate the remaining two entries.

$$\boxed{U = \ln x} \qquad\qquad V = x$$

$$\frac{dU}{dx} = \frac{1}{x} \qquad\qquad \boxed{\frac{dV}{dx} = 1}$$

Notice that we have set $\dfrac{dV}{dx} = 1$ This is a common procedure in integrating logarithmic or inverse trigonometric functions. Substituting back into the basic formula, we have

$$\int \ln x \; dx = x \ln x - \int x \; \frac{1}{x} \; dx$$

$$\int \ln x \; dx = x \ln x - \int 1 \; dx$$

or

$$\int \ln x \; dx = x \ln x - x + C$$

Don't forget to include the arbitrary constant C at the end of the problem. Also note that we need not include absolute value signs around x in the $\ln x$ term as $x > 0$ Verify that the derivative of $x \ln x - x + C$ is $\ln x$.

61. Example $\int \arcsin x \; dx$ As in the previous example, the starting point is to put a one in front of the $\arcsin x$ This approach applies to the integration of all inverse trigonometric functions.

As before, write out the integration by parts formula.

$$\int U \; \frac{dV}{dx} \; dx = U \; V - \int V \; \frac{dU}{dx} \; dx$$

Then write out the array containing the quantities involved in the formula and fill in the boxed entries.Then generate the remaining two entries.

$$\boxed{U = \arcsin x} \qquad\qquad V = x$$

$$\frac{dU}{dx} = \frac{1}{\sqrt{1 - x^2}} \qquad\qquad \boxed{\frac{dV}{dx} = 1}$$

Substituting into the basic formula, we have

$$\int \arcsin x \; dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \; dx$$

$$\int \arcsin x \; dx = x \arcsin x + \sqrt{1-x^2} + C$$

where the integral on the right side has been evaluated by a simple substitution (Let $t = 1 - x^2$) You should verify the answer by differentiating the right hand side and verifying that it indeed does simplify to the left hand side.

62. Example $\int x^2 \cos x \; dx$ Write out the integration by parts formula.

$$\int U \; \frac{dV}{dx} \; dx = U \; V - \int V \; \frac{dU}{dx} \; dx$$

Then write out the array containing the quantities involved in the formula and fill in the boxed entries.Then generate the remaining two entries. This time we select $U = x^2$. Notice that this agrees with the ILATE mnemonic.

$$\boxed{U = x^2} \qquad\qquad V = \sin x$$

$$\frac{dU}{dx} = 2x \qquad\qquad \boxed{\frac{dV}{dx} = \cos x}$$

Substituting into the basic formula, we have

$$\int x^2 \cos x \; dx = x^2 \sin x - \int 2x \sin x \; dx \tag{5}$$

While the integral on the right hand side is simpler than the one one the left hand side, it still requires some manipulation. In fact, we need to apply the integration by parts technique a second time, namely this time to the problem $\int 2x \sin x \; dx$.

$$\int U \; \frac{dV}{dx} \; dx = U \; V - \int V \; \frac{dU}{dx} \; dx$$

$$\boxed{U = 2x} \qquad\qquad V = -\cos x$$

$$\frac{dU}{dx} = 2 \qquad\qquad \boxed{\frac{dV}{dx} = \sin x}$$

Substituting into the basic formula, we have

$$\int 2x \sin x \; dx = -2x \cos x - \int 2(-\cos x) \; dx$$

$$\text{or} \quad \int 2x \sin x \; dx = -2x \cos x + \int 2 \cos x \; dx$$

$$\int 2x \sin x \; dx = -2x \cos x + 2 \sin x$$

Substituting this result into equation 5 we have

$$\int x^2 \cos x \; dx = x^2 \sin x - (-2x \cos x + 2 \sin x)$$

$$\text{That is} \quad \int x^2 \cos x \; dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$$

63. <u>Example</u> $\int e^x \sin x \;\; dx$ This is of the classic Integration by Parts form, namely a "mixed Product". In this case, there is no preferred selection for "U". One choice is just as good as another.

$$\int U \, \frac{dV}{dx} \; dx = U \; V - \int V \, \frac{dU}{dx} \; dx$$

$$\boxed{U = \sin x} \qquad\qquad V = e^x$$

$$\frac{dU}{dx} = \cos x \qquad\qquad \boxed{\frac{dV}{dx} = e^x}$$

Substituting into the basic formula, we have

$$\int e^x \sin x \; dx = e^x \sin x - \int e^x \cos x \; dx \tag{6}$$

Now we must apply the Integration by Parts formula to $\int e^x \cos x \; dx$ .However we must be consistent.Since we selected U = trigonometric function in the first part of the problem, we must do the same thing in this part.

$$\boxed{U = \cos x} \qquad\qquad V = e^x$$

$$\frac{dU}{dx} = -\sin x \qquad\qquad \boxed{\frac{dV}{dx} = e^x}$$

Substituting into the basic formula, we have

$$\int e^x \cos x \; dx = e^x \cos x + \int e^x \sin x \; dx \tag{7}$$

Equation 7 could be substituted into equation 6 to obtain the answer to our problem. However we are now in a position to get the answer to two problems, namely $\int e^x \sin x \; dx$ and $\int e^x \cos x \; dx$
From equation 6 we have,

$$\int e^x \sin x \; dx + \int e^x \cos x \; dx = e^x \sin x \tag{8}$$

From equation 7 we have,

$$-\int e^x \sin x \; dx + \int e^x \cos x \; dx = e^x \cos x \tag{9}$$

We now have two equations and two unknowns. Solve for each by addition or subtraction. If we add equations 8 and 9, we obtain

$$2\int e^x \cos x \; dx = e^x \sin x + e^x \cos x$$

$$\boxed{\int e^x \cos x \; dx = \frac{e^x \sin x + e^x \cos x}{2} + C}$$

If we subtract equation 9 from equation 8, we obtain

$$2\int e^x \sin x \; dx = e^x \sin x - e^x \cos x$$

$$\boxed{\int e^x \sin x \; dx = \frac{e^x \sin x - e^x \cos x}{2} + C}$$

64. Example $\int \sec^3 x \; dx$ Start by rewriting the problem $\int \sec x \sec^2 x \; dx$ This time, your first choice is $\frac{dV}{dx} = \sec^2 x$ Whatever is left over goes to the "U" term.This is done since $\sec^2 x$ is easy to integrate. Write out the integration by parts formula.

$$\int U \frac{dV}{dx} \, dx = U\,V - \int V \frac{dU}{dx} \, dx$$

Then write out the array containing the quantities involved in the formula and fill in the boxed entries.Then generate the remaining two entries.

$$\boxed{U = \sec x} \qquad\qquad V = \tan x$$

$$\frac{dU}{dx} = \sec x \tan x \qquad\qquad \boxed{\frac{dV}{dx} = \sec^2 x}$$

Substituting into the basic formula, we have

$$\begin{aligned}
\int \sec^3 x \; dx &= \sec x \tan x - \int \tan x \sec x \tan x \; dx \\
\int \sec^3 x \; dx &= \sec x \tan x - \int \tan^2 x \sec x \; dx \\
\int \sec^3 x \; dx &= \sec x \tan x - \int (\sec^2 x - 1) \sec x \; dx \\
\int \sec^3 x \; dx &= \sec x \tan x - \int \sec^3 x \; dx + \int \sec x \; dx \\
2\int \sec^3 x \; dx &= \sec x \tan x + \int \sec x \; dx \\
2\int \sec^3 x \; dx &= \sec x \tan x + \ln|\sec x + \tan x|
\end{aligned}$$

Note that we have used the result from item# 29 for $\int \sec x \; dx$ It is typical for integrals of secants raised to odd powers to require the answer for all lower odd powers.(See item# 83 on page 56 of these notes.) Simplifying,we have

$$\boxed{\int \sec^3 x \; dx = \frac{\sec x \tan x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C} \qquad (10)$$

65. <u>Example</u> $\int \cos 5x \sin 3x \ \ dx$ Write out the integration by parts formula.

$$\int U \frac{dV}{dx} \ dx = U \ V - \int V \ \frac{dU}{dx} \ dx$$

Then write out the array containing the quantities involved in the formula and fill in the boxed entries.Then generate the remaining two entries. In this problem, it does not matter which function you select as the "U". However , once you have made the choice, you must be consistent.

$$\boxed{U = \sin 3x} \qquad\qquad V = \frac{\sin 5x}{5}$$

$$\frac{dU}{dx} = 3\cos 3x \qquad\qquad \boxed{\frac{dV}{dx} = \cos 5x}$$

Substituting into the basic parts formula, we have

$$\int \cos 5x \sin 3x \ \ dx = \frac{\sin 3x \sin 5x}{5} - \frac{3}{5}\int \sin 5x \cos 3x \ dx \tag{11}$$

Now we have to apply the integration by parts formula to $\int \sin 5x \cos 3x \ \ dx$. In doing so, be consistent. That is in the first attempt, we let "U" = the trigonometric function with the argument "3x". We must do the same for this next part.

$$\boxed{U = \cos 3x} \qquad\qquad V = \frac{-\cos 5x}{5}$$

$$\frac{dU}{dx} = -3\sin 3x \qquad\qquad \boxed{\frac{dV}{dx} = \sin 5x}$$

Substituting into the parts formula, we obtain

$$\int \sin 5x \cos 3x \ \ dx = -\frac{\cos 3x \cos 5x}{5} - \frac{3}{5}\int \sin 3x \cos 5x \ dx \tag{12}$$

Substituting equation 12 into equation 11 we obtain

$$\int \cos 5x \sin 3x \ \ dx = \frac{\sin 3x \sin 5x}{5} - \frac{3}{5}\left(\frac{-\cos 3x \cos 5x}{5} - \frac{3}{5}\int \sin 3x \cos 5x \ dx\right)$$

$$\int \cos 5x \sin 3x \; dx = \frac{\sin 3x \sin 5x}{5} + \frac{3\cos 3x \cos 5x}{25} + \frac{9}{25}\int \sin 3x \cos 5x \; dx$$

$$\frac{16}{25}\int \cos 5x \sin 3x \quad dx = \frac{\sin 3x \sin 5x}{5} + \frac{3\cos 3x \cos 5x}{25}$$

$$\int \cos 5x \sin 3x \quad dx = \frac{5\sin 3x \sin 5x}{16} + \frac{3\cos 3x \cos 5x}{16} + C$$

66. Exercises

(a) $\int xe^x \; dx$ (b) $\int x^2 e^{-x} \; dx$

(c) $\int x \sec^2 x \; dx$ (d) $\int x \sec x \tan x \; dx$

(e) $\int x \mathrm{cosec}^2 x \; dx$ (f) $\int \arctan x \; dx$

(g) $\int \mathrm{cosec}^3 x \; dx$ (h) $\int x \mathrm{cosec} x \cot x \; dx$

(i) $\int \arctan\left(\frac{x}{4}\right) \; dx$ (j) $\int \arcsin\left(\frac{x}{2}\right) \; dx$

(k) $\int \sin\sqrt{t} \; dt$. Hint. Let $x = \sqrt{t}$ (l) $\int x^3 \ln(x+1) \; dx$

7 Integration involving $\ln x$

67. In this section, we examine various integration techniques which are successful when the integrand involves a logarithm function.

$\int \ln x \; dx$ This problem has already been discussed in § 6 on page 38 of these notes. The method is integration by parts. The following integrals can be evaluated in a similar manner.

(a) $\int \ln(17x+9) \; dx$ Hint;try substitution for $(17x+9)$

(b) $\int \ln\sqrt{x} \; dx$ Hint: use logarithm rules

(c) $\int \ln(2x+9)^2 dx$

(d) $\int \sin(\ln x)dx$ try parts twice

(e) $\int x\ln(x^2+9)dx$ try substitution

68. If the integrand has both a logarithm and a one over x factor, a simple substitution will usually work.That is ,if the integral is of the form $\int \frac{F(a+b\ln x)}{x}\,dx$, then the substitution $u = a + b\ln x$ will usually work.

(a) $\int \frac{(5+\ln x)^2}{x}dx$

(b) $\int \frac{dx}{x\sqrt{7+2\ln x}}$

(c) $\int \frac{dx}{x(\ln x)^3}$

Each of the above integrals may be solved by a simple substitution. consider the following definite integral

$\int_1^{e^7} \frac{dx}{x(1+\ln x)^{2/3}}$

We begin by making the substitution $u = 1+\ln x$ giving $\frac{du}{dx} = \frac{1}{x}$ or $du = \frac{dx}{x}$. Furthermore, when $x = 1, u = 1$ and when $x = e^7, u = 8$

$$\begin{aligned}
\int_1^{e^7} \frac{dx}{x(1+\ln x)^{2/3}} &= \int_1^8 \frac{du}{u^{2/3}} \\
&= 3\left[u^{1/3}\right]_1^8 \\
&= 3(2-1) \\
&= 3
\end{aligned}$$

69. The following integrals may be evaluated using the method of integration by parts including repeated applications.

(a) $\int x^2 \ln x\ dx$

(b) $\int \frac{\ln x}{x^{1/2}}$

(c) $\int (\ln x)^2 \ dx$

(d) $\int x (\ln x)^2 \ dx$

8 Integration involving e^x

9 Integration of Inverse Trigonometric functions

10 Integration involving Trigonometric functions II

In this section, we continue our study of the integration of trigonometric functions. It would be a good idea to review section 4 on page 12 of these notes.

10.1 Integration of sines or cosines

10.1.1 $\int \sin^m x \ dx$ or $\int \cos^m x \ dx$ where m is positive, odd integer

The starting point in solving these integrals is to rewrite the problem so that it is a product of an even power and a first power of the same trigonometric function. The application of the trigonometric identity $\sin^2 x + \cos^2 x = 1$ enables you to convert an even power of one of these functions to an even power of the other function. Hence the integral is converted to the form of a "Golden Rule " as discussed in item 21.

70. Example $\int \sin^5 x \ dx$

$$\int \sin^5 x \ dx = \int \sin^4 x \ \sin x \ dx$$

$\int \sin^m x \, dx$				
Case	Substitution	Differential	Identity	Reference
(i) m positive odd	$u = \cos x$	$-du = \sin x \, dx$	$\sin^2 x = 1 - \cos^2 x$	item # 70
(ii) m positive even	Reduce powers of Trigonometric	by application identities	$\cos^2 x = \frac{1 + \cos 2x}{2}$ $\sin^2 x = \frac{1 - \cos 2x}{2}$	item# 72

Table 5: Integrals of powers of sine functions

$\int \cos^m x \, dx$				
Case	Substitution	Differential	Identity	Reference
(i) m positive odd	$u = \sin x$	$du = \cos x \, dx$	$\cos^2 x = 1 - \sin^2 x$	item # 71
(ii) m positive even	Reduce powers of Trigonometric	by application identities	$\cos^2 x = \frac{1 + \cos 2x}{2}$	item# 73

Table 6: Integrals of powers of cosine functions

Since $\sin^2 x = 1 - \cos^2 x$, we can rewrite the integral as

$$\int \sin^5 x \; dx = \int (1 - \cos^2 x)^2 \; \sin x \; dx$$

This integral is now of the form of rule # 2 in item 21

$$\int (1 - \cos^2 x)^2 \; \sin x \; dx = -\int (1 - u^2)^2 \; du \text{ where } u = \cos x \text{ and } du = -\sin x \; dx$$

Thus the trigonometric integral is transformed to a simple algebraic integral, namely $-\int (1 - 2u^2 + u^4) \; du = -u + \frac{2u^3}{3} - \frac{u^5}{5} + C$
Transforming back to the original variable, we have

$$\int \sin^5 x \; dx = -\cos x + \frac{2\cos^3 x}{3} - \frac{\cos^5 x}{5} + C$$

71. <u>Example</u> $\int \cos^3 x \; dx$

$$\int \cos^3 x \; dx = \int \cos^2 x \; \cos x \; dx$$

Since $\cos^2 x = 1 - \sin^2 x$, we can rewrite the integral as

$$\int \cos^3 x \; dx = \int (1 - \sin^2 x) \; \cos x \; dx$$

This integral is now of the form of rule # 1 in item 21

$$\int (1 - \sin^2 x) \; \cos x \; dx = \int (1 - u^2) \; du \text{ where } u = \sin x \text{ and } du = \cos x \; dx$$

Thus the trigonometric integral is transformed to a simple algebraic integral, namely $\int (1 - u^2) \; du = u - \frac{u^3}{3} + C$
Transforming back to the original variable, we have

$$\int \cos^3 x \; dx = \sin x - \frac{\sin^3 x}{3} + C$$

10.1.2 $\int \sin^m x\, dx$ or $\int \cos^m x\, dx$ where m is positive, even integer

72. Example $\int \sin^4 x\, dx$

$$
\begin{aligned}
\int \sin^4 x\, dx &= \int \sin^2 x \sin^2 x\, dx \\
&= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1-\cos 2x}{2}\right) dx \\
&= \int \frac{(1-\cos 2x)^2}{4}\, dx \\
&= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx \qquad (13)
\end{aligned}
$$

The identity given in equation 1 namely

$$\cos 2x = 2\cos^2 x - 1$$

is valid as long as there is a two to one ratio in the arguments of the trigonometric functions. So that it is possible to write

$$\cos 4x = 2\cos^2 2x - 1 \qquad (14)$$

Substituting equation 14 into equation 13, we have

$$
\begin{aligned}
\int \sin^4 x\, dx &= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx \\
&= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2}\right) dx \\
&= \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8}\right) + C \\
&= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C
\end{aligned}
$$

$$\boxed{\int \sin^4 x\, dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C} \qquad (15)$$

73. <u>Example</u> $\int \cos^4 x\, dx$

$$\begin{aligned}
\int \cos^4 x\, dx &= \int \cos^2 x \cos^2 x\, dx \\
&= \int \left(\frac{1+\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx \\
&= \int \frac{(1+\cos 2x)^2}{4}\, dx \\
&= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\, dx \qquad (16)
\end{aligned}$$

Once again, use equation 14 to reduce the power of $\cos^2 2x$. Substituting into equation 16, we have

$$\begin{aligned}
\int \cos^4 x\, dx &= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx \\
&= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2}\right) dx \\
&= \frac{1}{4}\left(\frac{3x}{2} + \sin 2x + \frac{\sin 4x}{8}\right) + C \\
&= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C
\end{aligned}$$

$$\boxed{\int \cos^4 x\, dx = \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C} \qquad (17)$$

74. <u>Exercise</u>

(a) Using the results given in equations 15 and 17, evaluate $\int (\cos^4 x - \sin^4 x)\, dx$.

(b) Evaluate $\int (\cos^4 x - \sin^4 x)\, dx$ without evaluating $\int \cos^4 x\, dx$ or $\int \sin^4 x\, dx$

$\int \sin^m x \cos^n x \; dx$				
Case	Substitution	Differential	Identity	Reference
(i) m positive odd	$u = \cos x$	$-du = \sin x \; dx$	$\sin^2 x = 1 - \cos^2 x$	item # 70
(ii) n positive odd	$u = \sin x$	$du = \cos x dx$	$\cos^2 x = 1 - \sin^2 x$	item# 71
(iii)m positive even and n positive even	Reduce to either sine	powers of or cosine alone	$\sin^2 x = 1 - \cos^2 x$ $\cos^2 x = 1 - \sin^2 x$	item # 78

Table 7: Integrals of products of sines and cosines

10.1.3 $\int \sin^m x \; \cos^n x \; dx$ where m is positive, odd integer

75. Example $\int \sin^5 x \cos^2 x \; dx$ m = 5 = odd; n = 2 = even

$$\begin{aligned}\int \sin^5 x \cos^2 x \; dx &= \int \sin^4 x \cos^2 x \sin x \; dx \\ &= \int (1 - \cos^2 x)^2 \cos^2 x \sin x \; dx \\ &= \int (1 - 2\cos^2 x + \cos^4 x) \cos^2 x \sin x \; dx \\ &= \int (\cos^2 x - 2\cos^4 x + \cos^6 x) \sin x \; dx \\ &= -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C\end{aligned}$$

The integral was evaluated by a simple $u = \cos x$ substitution

10.1.4 $\int \sin^m x \ \cos^n x \ dx$ where n is positive, odd integer

76. Example $\int \sin^2 x \cos^3 x \ dx$ m = 2 = even; n = 3 = odd

$$\begin{aligned}
\int \sin^2 x \cos^3 x \ dx &= \int \sin^2 x \cos^2 x \cos x \ dx \\
&= \int \sin^2 x (1 - \sin^2 x) \cos x \ dx \\
&= \int (\sin^4 x - \sin^6 x) \cos x \ dx \\
&= \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C
\end{aligned}$$

Note: If both m and n are positive odd integers, either of the above methods may be used.

77. Example $\int \sin^3 x \cos^3 x \ dx$ m = 3 = odd; n = 3 = odd

(a)

$$\begin{aligned}
\int \sin^3 x \cos^3 x \ dx &= \int \sin^3 x \cos^2 x \cos x \ dx \\
&= \int \sin^3 x (1 - \sin^2 x) \cos x \ dx \\
&= \int (\sin^3 x - \sin^5 x) \cos x \ dx \\
&= \frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C
\end{aligned}$$

The integral was evaluated by a simple $u = \sin x$ substitution

(b)

$$\begin{aligned}
\int \sin^3 x \cos^3 x \ dx &= \int \sin^2 x \cos^3 x \sin x \ dx \\
&= \int (1 - \cos^2 x) \cos^3 x \sin x \ dx \\
&= \int (\cos^3 x - \cos^5 x) \sin x \ dx \\
&= -\frac{\cos^4 x}{4} + \frac{\cos^6 x}{6} + C
\end{aligned}$$

The integral was evaluated by a simple $u = \cos x$ substitution

10.1.5 $\int \sin^m x \quad \cos^n x \quad dx$ where both m and n are positive, even integers

78. Example $\int \sin^2 x \cos^2 x\ dx$ m = 2 = even ; n = 2 = even This is the simplest possible case for both m and n even. It could be further simplified by using the double angle formula $\sin 2x = 2 \sin x \cos x$ However let us treat it in a more general fashion, namely changing the integrand so that only one trigonometric function appears, either sine or cosine.

$$\begin{aligned}
\int \sin^2 x \cos^2 x\ dx &= \int \sin^2 x(1 - \sin^2 x)\ dx \\
&= \int \sin^2 x - \sin^4 x)\ dx \\
&= \frac{x}{2} - \frac{\sin 2x}{4} - \left(\frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32}\right) + C \\
&= \frac{x}{8} - \frac{\sin 4x}{32} + C
\end{aligned}$$

Note In this example, we used the results obtained in Example 31 and equation 15. In this case, it would have been equally efficient to convert to cosines since both of these results are available. Always convert to the trigonometric function that is going to give the lowest powers of the trigonometric function and the simplest algebra in expanding $(1 - \cos^2 x)^n$ or $(1 - \sin^2 x)^n$

79. Exercise Show that you get the same answer to Example 78 by converting to cosine functions

10.1.6 $\int \sin ax \quad \cos bx\ dx$; $\int \sin ax \quad \sin bx\ dx$; $\int \cos ax \quad \cos bx\ dx$

Note that the integrand involves a product of sines and cosines having different arguments. These integrals may be solved using the method of Integration by Parts (see Example 65). However the use of an appropriate trigonometric identity offers a simpler approach.

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)] \tag{18}$$

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)] \tag{19}$$

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)] \tag{20}$$

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)] \tag{21}$$

80. Example $\int \sin 3x \cos 5x \; dx$

$$\begin{aligned}\int \sin 3x \cos 5x \; dx &= \frac{1}{2}\int [\sin 8x + \sin(-2x)] \; dx \\ &= \frac{1}{2}\int [\sin 8x - \sin(2x)] \; dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C\end{aligned}$$

81. Exercise Reconcile the answers obtained in Examples 65 and 80

10.2 Integration of secant functions raised to even/odd powers

10.2.1 $\int \sec^m x \; dx$ where m is positive, even integer

82. Example $\int \sec^6 x \; dx$ m = 6 = even

$$\begin{aligned}\int \sec^6 x \; dx &= \int \sec^4 x \sec^2 x \; dx \\ &= \int (\tan^2 x + 1)^2 \sec^2 x \; dx\end{aligned}$$

Let $u = \tan x$, then$\dfrac{du}{dx} = \sec^2 x$

$$\begin{aligned}
\int (\tan^2 x + 1)^2 \sec^2 x \; dx &= \int (u^2+1)^2 \; du \\
&= \int (u^4 + 2u^2 + 1) \; du \\
&= \frac{u^5}{5} + \frac{2u^3}{3} + u
\end{aligned}$$

Returning this expression to trigonometric functions, we have

$$\int \sec^6 x \; dx = \frac{\tan^5 x}{5} + \frac{2\tan^3 x}{3} + \tan x + C$$

10.2.2 $\int \sec^m x \; dx$ where m is positive, odd integer

83. Example $\int \sec^5 x \; dx$ Integrals of odd powers of secants use the method of Integration by Parts. Rewriting the problem, we have $\int \sec^3 x \sec^2 x \; dx$. As before, write out the integration by parts formula.

$$\int U \; \frac{dV}{dx} \; dx = U \; V - \int V \; \frac{dU}{dx} \; dx$$

Then write out the array containing the quantities involved in the formula and fill in the boxed entries.Then generate the remaining two entries.

$$\boxed{U = \sec^3 x} \qquad\qquad V = \tan x$$

$$\frac{dU}{dx} = 3\sec^3 x \tan x \qquad\qquad \boxed{\frac{dV}{dx} = \sec^2 x}$$

$$\begin{aligned}
\int \sec^5 x \; dx &= \sec^3 x \tan x - 3\int \sec^3 x \tan^2 x \; dx \\
\int \sec^5 x \; dx &= \sec^3 x \tan x - 3\int \sec^3 x(\sec^2 x - 1) \; dx \\
\int \sec^5 x \; dx &= \sec^3 x \tan x - 3\int \sec^5 x + 3\int \sec^3 x \; dx \\
4\int \sec^5 x \; dx &= \sec^3 x \tan x + 3\int \sec^3 x \; dx \qquad (22)
\end{aligned}$$

$\int \sec^3 x \ dx$ has been evaluated previously.(Equation 10) Substituting this result into equation 22, we have

$$4 \int \sec^5 x \ dx = \sec^3 x \tan x + 3 \left(\frac{\sec x \tan x}{2} + \frac{\ln |\sec x + \tan x|}{2} \right)$$

$$\boxed{\int \sec^5 x \ dx = \frac{\sec^3 x \tan x}{4} + \frac{3 \sec x \tan x}{8} + \frac{3 \ln |\sec x + \tan x|}{8} + C}$$

10.3 Integration of Tangent functions raised to even/odd powers

The method involves the development of a "Reduction" formula . When applied to any integral involving positive powers of tangent, it will result in the reduction of the power by two. Eventually the integral will reduce to either one involving tangent or one in ivolving the square of tangent. In either case, the end result is a trivial integral.

$$\int \tan^m x \ dx = \int \tan^{m-2} x \ \tan^2 x \ dx = \int \tan^{m-2} x \ (\sec^2 x - 1) \ dx$$

$$\text{That is } \int \tan^m x \ dx = \int \tan^{m-2} x \ \sec^2 x \ dx - \int \tan^{m-2} x \ \ dx$$

$$\text{or } \int \tan^m x \ dx = \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x \ \ dx$$

The first integral is trivial and is a simple example of 'Golden Rule # 3'. Repeated application of this process leads to $\int \tan x \ dx$ or to $\int \tan^2 x \ dx$ depending on whether m is odd or even. Both of these integrals have been discussed previously; examples 27 and 33. Don't memorize this formula. Understand the approach to solving the problem.

10.3.1 $\int \tan^m x \ dx$ where m is positive, even integer

84. <u>Example</u> $\int \tan^6 x \ dx$

$$\int \tan^6 x \ dx = \int \tan^4 x \ \tan^2 x \ dx$$

$$\int \tan^6 x\ dx = \int \tan^4 x\ (\sec^2 x - 1)\ dx$$

$$\int \tan^6 x\ dx = \int \tan^4 x\ \sec^2 x\ dx - \int \tan^4 x\ dx$$

$$\int \tan^6 x\ dx = \int \tan^4 x\ \sec^2 x\ dx - \int \tan^2 x\ \tan^2 x\ dx$$

$$\int \tan^6 x\ dx = \int \tan^4 x\ \sec^2 x\ dx - \left(\int \tan^2 x\ (\sec^2 x - 1)\ dx\right)$$

$$\int \tan^6 x\ dx = \int \tan^4 x\ \sec^2 x\ dx - \left(\int \tan^2 x\ \sec^2 x\ dx - \int \tan^2 x\ dx\right)$$

$$\int \tan^6 x\ dx = \int \tan^4 x\ \sec^2 x\ dx - \left(\int \tan^2 x\ \sec^2 x\ dx - \left\{\int (\sec^2 x - 1)\ dx\right\}\right)$$

$$\int \tan^6 x\ dx = \frac{\tan^5 x}{5} - \left(\frac{\tan^3 x}{3} - \{\tan x - x\}\right) + C$$

$$\int \tan^6 x\ dx = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C$$

10.3.2 $\int \tan^m x\ dx$ where m is positive, odd integer

85. Example $\int \tan^5 x\ dx$

$$\int \tan^5 x\ dx = \int \tan^3 x\ \tan^2 x\ dx$$

$$\int \tan^5 x\ dx = \int \tan^3 x\ (\sec^2 x - 1)\ dx$$

$$\int \tan^5 x\ dx = \int \tan^3 x\ \sec^2 x\ dx - \int \tan^3 x\ dx$$

$$\int \tan^5 x\ dx = \int \tan^3 x\ \sec^2 x\ dx - \int \tan x\ \tan^2 x\ dx$$

$\int \tan^m x \sec^n x \, dx$				
Case	Substitution	Differential	Identity	Reference
(i) n positive even	$u = \tan x$	$du = \sec^2 x \, dx$	$\sec^2 x = 1 + \tan^2 x$	item # 86
(ii) m positive odd	$u = \sec x$	$du = \sec x \tan x dx$	$\tan^2 x = \sec^2 x - 1$	item# 87
(iii) m positive even and n positive odd	Change integrand to powers of secant alone	Integration by Parts required	$\tan^2 x = \sec^2 x - 1$	item # 88

Table 8: Integrals of products of tangents and secants

$$\int \tan^5 x \, dx = \int \tan^3 x \ \sec^2 x \, dx - \left(\int \tan x \, (\sec^2 x - 1) \, dx \right)$$

$$\int \tan^5 x \, dx = \int \tan^3 x \ \sec^2 x \, dx - \left(\int \tan x \ \sec^2 x \, dx - \int \tan x \, dx \right)$$

$$\int \tan^5 x \, dx = \frac{\tan^4 x}{4} - \left(\frac{\tan^2 x}{2} - \{-\ln|\cos x|\} \right) + C$$

$$\int \tan^5 x \, dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} - \ln|\cos x| + C$$

10.4 Integrals of products of tangents and secants

A summary of the various substitutions is given in Table 8

10.4.1 $\int \tan^m x \ \sec^n x \ dx$ where n is positive, even integer

86. Example $\int \tan^4 x \sec^4 x \ dx$ m = 4 = even; n = 4 = even

$$\begin{aligned}
\int \tan^4 x \sec^4 x &= \int \tan^4 x \sec^2 x \sec^2 x \ dx \\
&= \int \tan^4 x (\tan^2 x + 1) \sec^2 x \ dx \\
&= \int (\tan^6 x + \tan^4 x) \sec^2 x \ dx \\
&= \frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + C
\end{aligned}$$

The latter integral is obtained by a simple $u = \tan x$ substitution. Note that the power of the tangent function is unimportant here. It is the even power of the secant function which enables this method to work.

10.4.2 $\int \tan^m x \ \sec^n x \ dx$ where m is positive, odd integer

87. Example $\int \tan^3 x \sec^3 x \ dx$ m = 3 = odd; n = 3 = odd

$$\begin{aligned}
\int \tan^3 x \sec^3 x &= \int \tan^2 x \sec^2 x \sec x \tan x \ dx \\
&= \int (\sec^2 x - 1) \sec^2 x \ \sec x \tan x \ dx
\end{aligned}$$

Let $u = \sec x$ then $\dfrac{du}{dx} = \sec x \tan x$

$$\int (\sec^2 x - 1) \sec^2 x \sec x \tan x \ dx = \int (u^2 - 1) u^2 du = \frac{u^5}{5} - \frac{u^3}{3} + C.$$

Replacing the "u" variable by $\sec x$ we have

$$\int \tan^3 x \sec^3 x = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

Note that the power of the secant function is unimportant here. It is the odd power of the tangent function which enables this method to work.

10.4.3 $\int \tan^m x \ \sec^n x \ dx$ where m is a positive even integer and n is a positive odd integer

88. Example $\int \tan^4 x \sec^3 x \ dx$ m = 4 = even; n = 3 = odd
The previous methods will fail for this problem.

$$\int \tan^4 x \sec^3 x \ dx \quad = \quad \int \tan^4 x \sec x \sec^2 x \ dx$$

Once we have isolated the $\sec^2 x$, we need to convert everything else to a function of $\tan x$. This is not possible since the secant function is raised to an odd power. The relevant trigonometric identity requires even powers, not odd powers. Alternatively we can try the second approach

$$\int \tan^4 x \sec^3 x \ dx \quad = \quad \int \tan^3 x \sec^2 x \sec x \tan x \ dx$$

In this approach, once we have isolated the $\sec x \tan x$, everything else has to be converted to a function of secants. As before this is not possible since the tangent function is raised to an odd power. The relevant trigonometric identity requires even powers, not odd powers. To solve this problem we must convert everything to powers of secants. Unfortunately, the secants will be raised to odd powers and therefore will require the technique of Integration by Parts

$$\begin{aligned}
\int \tan^4 x \sec^3 x \ dx &= \int (\sec^2 x - 1)^2 \sec^3 x \ dx \\
&= \int (\sec^4 x - 2\sec^2 x + 1) \sec^3 x \ dx \\
&= \int (\sec^7 x - 2\sec^5 x + \sec^3 x) \ dx
\end{aligned}$$

We won't pursue this problem any further. The method is obvious(see item 64 and 83)
Note: In some cases the problem may be solved using more than one method. The answers may appear to be different but can be reconciled. The following example illustrates this point.

89. Example $\int \tan^3 x \sec^4 x \ dx$

(a)

$$\begin{aligned}\int \tan^3 x \sec^4 x\ dx &= \int \tan^3 x \sec^2 x \sec^2 x\ dx\\ &= \int \tan^3 x(\tan^2 x + 1)\sec^2 x\ dx\\ &= \int (\tan^5 x + \tan^3 x)\sec^2 x\ dx\\ &= \frac{\tan^6 x}{6} + \frac{\tan^4 x}{4} + C\end{aligned}$$

(b)

$$\begin{aligned}\int \tan^3 x \sec^4 x\ dx &= \int \tan^2 x \sec^3 x \sec x \tan x\ dx\\ &= \int (\sec^2 x - 1)\sec^3 x \sec x \tan x\ dx\\ &= \int (\sec^5 x - \sec^3 x)\sec x \tan x\ dx\\ &= \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C\end{aligned}$$

11 Integration of algebraic functions using trigonometric substitutions

In previous sections, we have transformed complicated agebraic integrals into standard forms by means of an algebraic substitution.

For example

$$\int \frac{x}{\sqrt{1-x^2}}\ dx$$

may be solved by maing a simple substitution, namely $u = 1 - x^2$ However, if we make a slight change to the problem, this substitution wont work. For example

$$\int \frac{x^2}{\sqrt{1-x^2}}\ dx$$

cannot be solved this way. Using a suitable trigonometric substitution will often change the problem into a tractable form. We will look at three examples and then discuss the method in general, including the development of recognition factors.

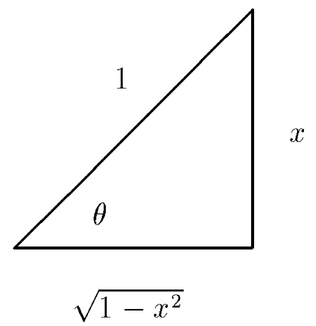
90. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

Let $x = \sin\theta; -\pi/2 \le \theta \le \pi/2$

Then $\dfrac{dx}{d\theta} = \cos\theta$ or $dx = \cos\theta\ d\theta$. Putting these substitutions into the integral, we have

$$\begin{aligned}
\int \frac{x^2}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\ d\theta \\
&= \int \frac{\sin^2\theta\cos\theta\ d\theta}{\sqrt{\cos^2\theta}} \\
&= \int \frac{\sin^2\theta\cos\theta}{\cos\theta} \\
&= \int \sin^2\theta d\theta \\
&= \int \frac{1-\cos 2\theta}{2}\,d\theta \\
&= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C
\end{aligned}$$

At the beginning of the exercise, we changed from the variable x to the variable θ. Now we have to reverse the process. Initially, we looked at the substitution from left to right, namely $x = \sin\theta$. Now we should consider it in the reverse order, $\sin\theta = \dfrac{x}{1}$. From this statement, we must find expressions for any other trigonometric expressions in the answer. This may be accomplished by using trigonometric identities and/or a right triangle. If $\sin\theta = x, -\pi/2 \le \theta \le \pi/2$, then $\theta = \arcsin x$. Also, we have the trigonometric identity, $\sin 2\theta = 2\sin\theta\cos\theta$. In the evaluation of the integral, we have $\sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ since $-\pi/2 \le \theta \le \pi/2$. From figure 4, we find that the third side is $\sqrt{1-x^2}$. Notice that it is of the same algebraic form (not necessarily raised to the same power) as the radical in the original integral. From the triangle, we conclude that $\cos\theta = \sqrt{1-x^2}$. Hence the answer to the exercise may be



$\sin\theta = x/1$; $\cos\theta = \sqrt{1-x^2}$

Figure 4: Right triangle for Example 90; $\int \frac{x^2}{\sqrt{1-x^2}}dx$

written

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^2}}\,dx &= \frac{\theta}{2} - \frac{\sin 2\theta}{4} + C \\ &= \frac{\arcsin x}{2} - \frac{2x\sqrt{1-x^2}}{4} + C \\ &= \frac{\arcsin x - x\sqrt{1-x^2}}{2} + C\end{aligned}$$

91. <u>Example</u> $\int_0^{\sqrt{3}/2} \frac{dx}{(1-x^2)^{3/2}}$
We begin by making the substitution $x = \sin\theta \quad -\pi/2 \le \theta \le \pi/2$ Hence $\frac{dx}{d\theta} = \cos\theta$ or $dx = \cos\theta \; d\theta$ Making these substitutions into the original integral, we have , when $x = 0, \sin\theta = 0$ or $\theta = 0.$; when $x = \sqrt{3}/2, \sin\theta =$

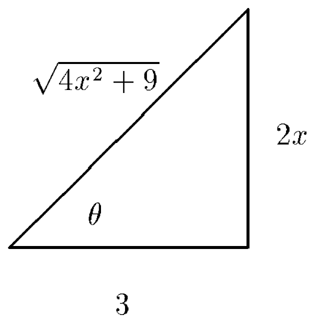
$\sqrt{3}/2$ or $\theta = \pi/3$

$$\begin{aligned}
\int_0^{\sqrt{3}/2} \frac{dx}{(1-x^2)^{3/2}} &= \int_0^{\pi/3} \frac{\cos\theta d\theta}{(1-\sin^2\theta)^{3/2}} \\
&= \int_0^{\pi/3} \frac{\cos\theta d\theta}{(\cos^2\theta)^{3/2}} \\
&= \int_0^{\pi/3} \frac{\cos\theta d\theta}{\cos^3\theta} \\
&= \int_0^{\pi/3} \frac{d\theta}{\cos^2\theta} \\
&= \int_0^{\pi/3} \sec^2\theta d\theta \\
&= [\tan\theta]_0^{\pi/3} \\
&= \tan(\pi/3) - \tan 0 \\
&= \sqrt{3}
\end{aligned}$$

92. $\int \frac{\sqrt{(4x^2+9)}}{x^4}\,dx$. In this case, we have a tangent substituion. Let $4x^2 = 9\tan^2\theta$ or $2x = 3\tan\theta$; $-\pi/2 < \theta < \pi/2$. Finally, we can write the substitution as $x = \frac{3\tan\theta}{2}$ giving $dx = \frac{3\sec^2\theta\ d\theta}{2}$. Putting these substituions into our exercise, we have

$$\begin{aligned}
\int \frac{\sqrt{(4x^2+9)}}{x^4}\,dx &= \int \frac{\sqrt{(9\tan^2\theta+9)}(3/2)\sec^2\theta\ \ d\theta}{(81\tan^4\theta)/16} \\
&= \int \frac{8}{9}\frac{\sqrt{\sec^2\theta}\sec^2\theta d\theta}{\tan^4\theta} \\
&= \int \frac{8}{9}\frac{\sec^3\theta}{\tan^4\theta}d\theta \\
&= \int \frac{8}{9}\frac{\cos\theta}{\sin^4\theta}\ d\theta \\
&= -\frac{8}{27}\frac{1}{\sin^3\theta} + C \\
&= -\frac{(4x^2+9)^{3/2}}{27x^3} + C
\end{aligned}$$

In the above work, we initially found the answer in terms of θ. We have to return this anwer to the original variable x. This is done by reversing the original substitution, namely $x = \frac{3}{2}\tan\theta$. We now look at this in terms of $\tan\theta$ so that $\tan\theta = \frac{2x}{3}$ from this statement, we must find expressions for any other trigonometric expressions in the answer. We can do this by using trigonometric identities and /or a right triangle,(see Figure 5). Hence $\sin\theta = \frac{2x}{\sqrt{4x^2+9}}$



$$\tan\theta = 2x/3\ ;\sin\theta = \frac{2x}{\sqrt{4x^2+9}}$$

Figure 5: Right triangle for Example 92; $\int \frac{\sqrt{4x^2+9}}{x^4}dx$

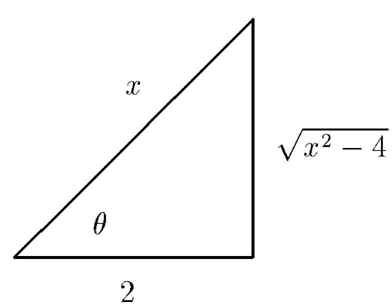
93. Example $\quad \int \frac{dx}{x^2\sqrt{x^2-4}}$

Let $x = 2\sec\theta; 0 \leqslant \theta < \frac{\pi}{2}$

Then dx $=2\sec\theta\tan\theta d\theta$ substituting into the original problem, we have.

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2-4}} &= \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\sqrt{4\sec^2\theta-4}}\,d\theta \\
&= \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\sqrt{4(\sec^2\theta-1)}}\,d\theta \\
&= \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta\sqrt{4\tan^2\theta}}\,d\theta \\
&= \int \frac{2\sec\theta\tan\theta}{4\sec^2\theta 2\tan\theta}\,d\theta \\
&= \frac{1}{4}\int \frac{1}{\sec\theta}d\theta \\
&= \frac{1}{4}\int \cos\theta d\theta \\
&= \frac{1}{4}\sin\theta + C
\end{aligned}$$

Note that in the above work, there is the simplification $\sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ since $0 \leq \theta < \frac{\pi}{2}$. As in the previous examples, we must give the



$$\sin\theta = \frac{\sqrt{x^2-4}}{x}$$

Figure 6: Right triangle for Example 93, $\int \frac{dx}{x^2\sqrt{x^2-4}}$

answer in terms of the original variable, x. The original substitution was $x = 2sec\theta$. Rewriting in terms of θ,we have $\sec\theta = \frac{x}{2}$ or $\cos\theta = \frac{2}{x}$. Other trigonometric functions may be found either by the use if trigonometric identities or the use of the right triangle from figure 6, we have $\sin\theta = \dfrac{\sqrt{x^2-4}}{x}$

Hence

$$\int \frac{dx}{x^2\sqrt{x^2-4}} = \frac{\sqrt{x^2-4}}{4x} + C$$

Check your answer by differentiation. Note that once again, the third side in the right triangle is of the same algebraic form as that in the original integral.

12 Integration of Rational algebraic functions

94. In this section we consider integration of rational functions, that is integrals of the form $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$ where Px and Q(x) are polynomials. In this analysis, we shall assume that the degree of P(x) is less than the degree of Q(x). If this is not the case, then the first step is to divide Q(x) into P(x). Some problems of this type have been encountered previously and have been solved by simple substitution methods.

    (a) $\displaystyle\int \frac{6x+2}{3x^2+2x+7}dx$

    (b) $\displaystyle\int \frac{x^3+7x^2+11x+8}{x+5}dx$

    (c) $\displaystyle\int \frac{12x+30}{x^2+5x+9}dx$

    The above three exercises may be solved by methods discussed previously. We now consider exercises which cannot be handled by these simple procedures. The method is purely algebraic and involves separating the rational function into its component parts. The procedure is called the method of partial fractions.

    In all partial fraction problems we begin with two essential steps

    (a) The degree of the denominator must be greater than the degree of the numerator. If not, divide to give a proper fraction

    (b) Factor the denominator as completely as possible

95. <u>Case 1</u> The denominator is a product of two or more linear factors, none repeated.

(a) $\int \frac{x-16}{x^2+3x-10}dx$
In this exercise, a simple substitution won't work. The degree of the numerator is less than the degree of the denominator so we begin by factoring the denominator

$$\frac{x-16}{x^2+3x-10} = \frac{x-16}{(x+5)(x-2)}$$

At this stage of the process, it is probably best to concentrate on the algebra and omit the integration symbols. We begin by rewriting the problem

$$\frac{x-16}{(x+5)(x-2)} = \frac{A}{x-5} + \frac{B}{x-2}$$

We need to find the constants A and B which make this a true statement. Notice that the numerator in one of these components is the full term one degree less than the denominator. In the present case, since the denominator is a linear term, we require only a constant in the numerator. This is true for both components in this separation process.

$$\begin{aligned} \frac{x-16}{(x+5)(x-2)} &= \frac{A}{x+5} + \frac{B}{x-2} \\ \frac{x-16}{(x+5)(x-2)} &= \frac{A(x-2)+B(x+5)}{(x+5)(x-2)} \qquad (23) \end{aligned}$$

Equation 23 is an identity, true for all values of x . Since the denominators are equal, the numerators must be equal. That is;

$$x-16 = A(x-2)+B(x+5) \qquad (24)$$

At this stage in the solution of the problem, there are two possible approaches. The first method uses the theorem that two polynomials are equal if and only if the coefficients of like powers are equal. To use this method, we must mulitply out the right hand side, gather terms and then solve a system of equations. In this simple problem, this step involves only

two equations and two unknowns. However, as we shall see in subsequent problems, the solution process can become complicated very quickly. A second method relies on the fact that equation 24 is an identity, true for all values of x , and so we are able to obtain solutions immediately.
<u>Method 1</u>

$$\begin{aligned} x-16 &= A(x-2)+B(x+5) \\ x-16 &= Ax-2A+Bx+5B \\ x-16 &= x(A+B)+(-2A+5B) \end{aligned}$$

Setting the coefficients of the x terms equal and the same for the constant terms, we have the following system of equations.

$$\begin{aligned} 1 &= A+B \\ -16 &= -2A+5B \end{aligned}$$

The solution is straight forward. $B=-2$ and $A=3$ Hence the original integral now becomes the sum/difference of two trivial integrals, namely

$$\begin{aligned} \int \frac{x-16}{x^2+3x-10}dx &= \int \frac{x-16}{(x+5)(x-2)}dx \\ &= \int \frac{A}{x+5}+\frac{B}{x-2}dx \\ &= \int \frac{3}{x+5}-\frac{2}{x-2}dx \\ &= 3\ln|x+5|-2\ln|x-2|+K \end{aligned}$$

<u>Method 2</u> Begins with the same equation as method 1, namely equation 24

$$x-16=A(x-2)+B(x+5)$$

By cunningly selecting value for x, we can write down solutions for A and B
If we let $x=2$, then it follows that $-14=7B$ or $B=-2$
If we let $x=-5$, then $-21=-7A$ or $A=3$ as before.
Clearly, even with this simple problem, method 2 involves fewer algebraic steps and is less subject to error. In some of the exercises to follow, it may not be possible to completely solve the problem using method 2. However, its use will reduce the number of simultaneous equations to be solved and this is always a desirable outcome.

(b) $\int \frac{35x+5}{(x-2)(x+3)(x-7)}dx$ Here the denominator is already fully factored.

Following the method given in the previous Example, we have

$$\begin{aligned}\frac{35x+5}{(x-2)(x+3)(x-7)} &= \frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{x-7}\\ \frac{35x+5}{(x-2)(x+3)(x-7)} &= \frac{A(x+3)(x-7)+B(x-2)(x-7)+C(x-2)(x+3)}{(x-2)(x+3)(x-7)}\end{aligned}$$

Since the denominators are equal, we can equate the numerators

$$35x+5 = A(x+3)(x-7)+B(x-2)(x-7)+C(x-2)(x+3)$$

setting $x=7$, we have $250=50C$ or $C=5$
setting $x=2$, we have $75=-25A$ or $A=-3$
setting $x=-3$, we hae $-100=50B$ or $B=-2$

We recommend that you solve this exercise using method 1 and compare the relative merits of each method. The calculus part of the problem is trivial.

$$\begin{aligned}\int \frac{35x+5}{(x-2)(x+3)(x-7)}dx &= \int \frac{A}{x-2}+\frac{B}{x+3}+\frac{C}{x-7}dx\\ &= \int \frac{-3}{x-2}-\frac{2}{x+3}+\frac{5}{x-7}dx\\ &= -3\ln|x-2|-2\ln|x+3|+5\ln|x-7|+K\\ &= \ln\left|\frac{(x-7)^5}{(x+3)^2(x-2)^3}\right|+K\end{aligned}$$

96. <u>Exercises</u>

(a) $\int \frac{4}{x^3-x}dx$

(b) $\int \frac{48x-24}{(x-5)(x+1)(x+7)}dx$

97. Case 2 ; Denominator is a product of two or more linear factors including repeated factors.

Example $\int \frac{3-2x}{x(x+1)^2}dx$

For the present, lets concentrate on the algebra. Recall in junior high school algebra, if we were asked to combine the following fractions

$$\frac{1}{x} + \frac{1}{x+1} + \frac{1}{(x+1)^2}$$

we would begin by noting that the LCD is $x(x+1)^2$. The same result would be found if we were combining the fractions

$$\frac{1}{x} + \frac{1}{(x+1)^2}$$

Therefore, when we are trying to separate this fraction into its component parts, we must allow for the possible presence of an $(x+1)$ factor as well as the $(x+1)^2$ factor. In general, we must allow for every power up to and including the power of the repeated factor. Additional examples are given below. In the present case, we have

$$\begin{aligned} \frac{3-2x}{x(x+1)^2} &= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \\ &= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2} \end{aligned}$$

equating the numerators, we have

$$3 - 2x = A(x+1)^2 + Bx(x+1) + Cx \qquad (25)$$

By substituting appropriate values for x into equation 25, we can find immediate solutions for two of the constants. Substituting $x=0$ into equation 25, we have $3 = A$. Substituting $x=-1$ into equation 25, we have $5 = -C$ or $C = -5$. Because of the repeated factor, no other value of x will give an immediate solution for the third constant, B. However equation 25 is an identity and we may substitute any convenient value for x into it. In this example, since we have not used $x = 1$ previously, we shall try this value.

$$1 = 4A + 2B + C$$

Using the previously determined values of 3 for A and -5 for C we have $1 = 12 + 2B - 5$ or $B = -3$ The original integration problem now becomes

$$\begin{aligned} \int \frac{3-2x}{x(x+1)^2} &= \int \left(\frac{3}{x} - \frac{3}{x+1} - \frac{5}{(x+1)^2} \right) dx \\ &= 3 \ln |x| - 3 \ln |x+1| + \frac{5}{x+1} + K \end{aligned}$$

98. The following examples illustrate the set up for the separation into component parts when repeated linear factors are involved.

$$\begin{aligned} \int \frac{f(x)}{x^2(x-2)^3} dx &= \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-2)} + \frac{D}{(x-2)^2} + \frac{E}{(x-2)^3} \right) dx \\ \int \frac{f(x)}{x^3(x+1)^2} dx &= \int \left(\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} + \frac{E}{(x+1)^2} \right) dx \end{aligned}$$

99. Example $\int \frac{4x+1}{x^3(x+1)} dx$. Following the above procedures, we have

$$\begin{aligned} \frac{4x+1}{x^3(x+1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x+1} \\ &= \frac{Ax^2(x+1) + Bx(x+1) + C(x+1) + Dx^3}{x^3(x+1)} \end{aligned}$$

Equating the numerators, we have

$$4x + 1 = Ax^2(x+1) + Bx(x+1) + C(x+1) + Dx^3 \quad (26)$$

Substituting $x = 0$, we have an immediate solution for C, namely $C = 1$ Substituting $x = -1$, we also obtain the value for D, namely $D = 3$. No other values of x will give immediate solutions for the other constants A, and B. If we now substitute two values for x, we will obtain two equations with two unknowns to solve. However, this is still preferable to solving 4 equation and 4 unknowns. Substituting $x = 1$ into equation 26, we have

$$\begin{aligned} 5 &= 2A + 2B + 2C + D \\ 5 &= 2A + 2B + 2 + 3 \\ 0 &= 2A + 2B \quad (27) \end{aligned}$$

In the above work, we have used the previously determined values for the constants C and D. Substituting $x = 2$ into equation 26, we have

$$\begin{aligned} 9 &= 12A + 6B + 3C + 8D \\ 9 &= 12A + 6B + 3 + 24 \\ -18 &= 12A + 6B \\ -3 &= 2A + B \end{aligned} \tag{28}$$

Solving equations 27 and 28, we find that $A = -3$ and $B = 3$

We now return to the calculus part of the exercise

$$\begin{aligned} \int \frac{4x+1}{x^3(x+1)}dx &= \int \left(\frac{-3}{x} + \frac{3}{x^2} + \frac{1}{x^3} + \frac{3}{x+1} \right) dx \\ &= -3 \ln|x| - \frac{3}{x} - \frac{1}{2x^2} + 3 \ln|x+1| + K \\ &= 3 \ln \left| \frac{x+1}{x} \right| - \frac{3}{x} - \frac{1}{2x^2} + K \end{aligned}$$

100. <u>Case 3</u> Denominator is a product of a linear factor and a quadratic factor which cannot be further factored in the real number system.

<u>Example</u>

$$\int \frac{5x^2+5x+2}{x(x^2+1)}dx$$

As before, lets begin by doing the algebraic separation into component parts.

$$\begin{aligned} \frac{5x^2+5x+2}{x(x^2+1)} &= \frac{A}{x} + \frac{Bx+C}{x^2+1} \\ \frac{5x^2+5x+2}{x(x^2+1)} &= \frac{A(x^2+1) + x(Bx+C)}{x(x^2+1)} \end{aligned}$$

As we indicated earlier, in the method of partial fractions, for the numerator, we write the full term, one degree less than the denominator. Since the denominator is a quadratic in this case, we need a linear term in the numerator. Hence, the $Bx + C$ term is required in the numerator of the $x^2 + 1$ factor. Equating the numerators, we have

$$5x^2 + 5x + 2 = A(x^2+1) + x(Bx+C)$$

By setting $x = 0$, we have an immediate solution for A, namely $A = 2$ To find the other two constants, we must substitute convenient values for x. In this case, try $x = 1$ and $x = -1$
for $x = 1$, we have $12 = 2A + B + C$ or $B + C = 8$, using the known value for A. For $x = -1$, we have $2 = 2A + B - C$ or $B - C = -2$ Hence we must solve the following system of equations

$$\begin{aligned} B + C &= 8 \qquad (29) \\ B - C &= -2 \qquad (30) \end{aligned}$$

adding equations 29 and 30, we find $B = 3$ and hence $C = 5$.

Returning to the calculus problem, we have

$$\begin{aligned} \int \frac{5x^2 + 5x + 2}{x(x^2+1)} dx &= \int \frac{A}{x} + \frac{Bx + C}{x^2+1} dx \\ &= \int \frac{2}{x} + \frac{3x+5}{x^2+1} dx \\ &= \int \frac{2}{x} + \frac{3x}{x^2+1} + \frac{5}{x^2+1} dx \\ &= 2\ln|x| + \frac{3}{2}\ln(x^2+1) + 5\arctan x + K \end{aligned}$$

101. Example $\int \frac{3x^2+7}{(x-1)(x^2+4)} dx$ Following the method give above, we have

$$\begin{aligned} \frac{3x^2+7}{(x-1)(x^2+4)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+4} \\ &= \frac{A(x^2+4) + (x-1)(Bx+C)}{(x-1)(x^2+4)} \end{aligned}$$

As before, equating the numerators, we have

$$3x^2 + 7 = A(x^2+4) + (x-1)(Bx+C)$$

Substituting x = 1, we have an immediate solution for A, namely $10 = 5A$ or $A = 2$. Notice that by starting with this substitution, we eliminate B and C from the equation and so can get a solution for A. Our next choice will be $x = 0$ which will remove B from the equation, leaving only A and C. Since we already know A, we can find C. Substituting $x = 0$ we have $7 = 4A - C$ or

$C = 8 - 7$ or $C = 1$ where we have used the previously found value for A of 2. Now we must substitute in any convenient value for x no previously used, so lets try $x = 2$. This will give an equation in A, B, and C but since we already know A and C, we can determine B

For $x = 2$, we have $19 = 8A + 2B + C$ or $19 = 16 + 2B + 1$ or $B = 1$
Returning to the calculus part of the problem, we have

$$\begin{aligned}
\int \frac{3x^2+7}{(x-1)(x^2+4)}dx &= \int \frac{2}{x-1} + \frac{x+1}{x^2+4}dx \\
&= \int \frac{2}{x-1} + \frac{x}{x^2+4} + \frac{1}{x^2+4}dx \\
&= 2\ln|x-1| + \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K
\end{aligned}$$

102. Example $\displaystyle\int \frac{(x+4)}{x(x^2+4)}dx$

We recognize this problem as a partial fraction problem, the case of a linear factor and a non-reducible quadratic factor

$$\frac{x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\frac{x+4}{x(x^2+4)} = \frac{A(x^2+4) + x(Bx+C)}{x(x^2+4)}$$

Setting the numerators equal, we have $x + 4 = A(x^2 + 4) + x(Bx + C)$

Due to the $x(Bx+C)$ factor, setting $x = 0$ will yield a value for A but no other x value will give a value for B or C. Compare this equation to say

$x + 4 = A(x^2 + 4) + (x - 5)(Bx + C)$

Here a value of $x = 5$ will give a solution for A and a value of $x = 0$ will give an easy solution for C, so in our case, it is probably easier to multiply out, gather terms and solve equations.

$x + 4 = Ax^2 + 4A + Bx^2 + Cx$

$0 = A + B$ (coefficients of x^2)

$1 = C$ (coefficients of x)

$4 = 4A$ (constants)

Hence $A = 1$, $B = -1$ and $C = 1$

$$\begin{aligned}
\int \frac{x+4}{x(x^2+4)}dx &= \int \frac{1}{x}dx + \int \frac{-x+1}{x^2+4}dx \\
&= \int \frac{1}{x}dx - \int \frac{x}{x^2+4}dx + \int \frac{1}{x^2+4}dx \\
&= \ln|x| - \tfrac{1}{2}\ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + K
\end{aligned}$$

103. Case 4 Denominator is a product of a linear factor and a repeated quadratic factor.

Example $\displaystyle\int \frac{3x^4+3x^3+2x^2-3x+3}{(x-1)(x^2+1)^2}\,dx$

$$\begin{aligned}
\frac{3x^4+3x^3+2x^2-3x+3}{(x-1)(x^2+1)^2} &= \frac{A}{(x-1)} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \\
&= \frac{A(x^2+1)^2 + (x-1)(x^2+1)(Bx+C) + (x-1)(Dx+E)}{(x-1)(x^2+1)^2}
\end{aligned}$$

Equating the numerators, we have

$$\begin{aligned}
&3x^4+3x^3+2x^2-3x+3 \\
&\quad = A(x^2+1)^2 + (x-1)(x^2+1)(Bx+C) + (x-1)(Dx+E) \qquad (31)
\end{aligned}$$

Clearly this problem could lead to some messy algebra. However, a common sense approach may give an easy solution. First of all, by substituting $x = 1$, we can get an immediate solution for A, namely $8 = 4A$ or $A = 2$

However, any other substitution will not give an immediate solution. A careful examination of equation 31 indicates that the x^4 coefficients only involve A and B. Hence, if we multipy out the right hand side and gather coefficients of similar powers, we should be able to find B. At the very worst, this will lead to 3 equations and 3 unkowns, a considerable improvement on 5 equations and 5 unknowns.

Multipying out the terms on the right hand side of equation 31 we have

$$\begin{aligned} &3x^4+3x^3+2x^2-3x+3 \\ &\qquad = A(x^2+1)^2+(x-1)(x^2+1)(Bx+C)+(x-1)(Dx+E) \end{aligned}$$

$$\begin{aligned} &= A(x^4+2x^2+1)+(x^3+x-x^2-1)(Bx+C) \\ &\qquad +(Dx^2+Ex-Dx-E) \end{aligned}$$

$$\begin{aligned} &= Ax^4+2Ax^2+A+Bx^4+Bx^2-Bx^3-Bx \\ &\qquad +Cx^3+Cx-Cx-Cx^2-C+Dx^2+Ex-Dx-E \end{aligned}$$

From the coefficients of x^4, we have $3=A+B$. Since $A=2$, we conclude that $B=1$. Equating the coefficients of x^3 we have $3=-B+C$ giving $C=4$. Equating the coefficients of the x^2 terms, we have $2=2A+B-C+D$ giving $D=1$. Equating the constant terms, we have $3=A-C+E$ giving $E=-5$. We have now completed the separation process in a relatively painless way. The integral now becomes

$$\begin{aligned} &\int \frac{3x^4+3x^3+2x^2-3x+3}{(x+1)(x^2+1)^2}\,dx \\ &\qquad = \int \left(\frac{2}{x-1}+\frac{x+4}{x^2+1}+\frac{x-5}{(x^2+1)^2}\right)\,dx \\ &\qquad = \int \frac{2}{x-1}dx+\int \frac{x}{x^2+1}dx+\int \frac{4}{x^2+1}dx \\ &\qquad\qquad +\int \frac{x}{(x^2+1)^2}dx-\int \frac{5}{(x^2+1)^2}dx \end{aligned}$$

Evaluate each of these integrals independently.

$\displaystyle\int \frac{2}{x-1}dx = 2\ln|x-1|+K_1$

$\displaystyle\int \frac{x}{x^2+1}dx = \frac{1}{2}\ln(x^2+1)+K_2.$

Note, since x^2+1 is always > 0, we can omit the absolute values signs in the $\ln(x^2+1)$ expression.

$\int \frac{4}{x^2+1} dx = 4 \arctan x + K_3$

$\int \frac{x}{(x^2+1)^2} dx$ This integral may be evaluated by a simple substitution, namely $u = x^2+1$ leading to $du = 2x dx$. Hence, the answer is $\frac{-1}{2(x^2+1)} + K_4$

The final integral requires a trigonometric substitution. Let $x = \tan\theta$ or $dx = \sec^2\theta d\theta$

$$\begin{aligned}
\int \frac{5}{(x^2+1)^2} &= \int \frac{5\sec^2\theta}{\tan^2\theta+1)^2}\, d\theta \\
&= \int \frac{5\sec^2\theta}{(\sec^2\theta)^2}\, d\theta \\
&= \int \frac{5}{\sec^2\theta}\, d\theta \\
&= \int 5\cos^2\theta\ d\theta
\end{aligned}$$

This integral has been evaluated previously in § 4.2, example 32 on page 18 of these notes.

$$\begin{aligned}
\int \cos^2\theta d\theta &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} \\
&= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}
\end{aligned}$$

From the original substitution $x = \tan\theta$, we have $\sin\theta = \frac{x}{\sqrt{1+x^2}}$ and $\cos\theta = \frac{1}{\sqrt{1+x^2}}$ Also $\theta = \arctan x$. Substituting back into the integral, we have

$$\begin{aligned}
\int \frac{5}{x^2+1)^2} &= \int 5\cos^2\theta d\theta \\
&= \frac{5\theta}{2} + \frac{\sin\theta\cos\theta}{2} + K_5 \\
&= \frac{5}{2}\left(\arctan x + \frac{x}{1+x^2}\right) + K_5
\end{aligned}$$

Finally, we have the answer to the original problem.

$$\int \frac{3x^4+3x^3+2x^2-3x+3}{(x-1)(x^2+1)^2}dx$$

$$= 2\ln|x-1| + \frac{1}{2}\ln(x^2+1) + 4\arctan x - \frac{1}{2(x^2+1)}$$

$$- \frac{5}{2}\left(\arctan x + \frac{x}{1+x^2}\right) + K$$

$$= 2\ln|x-1| + \frac{1}{2}\ln(x^2+1) + \frac{3}{2}\arctan x - \frac{5x+1}{2(x^2+1)} + K$$

The reader should verify this answer by differentiating the right hand side and show that the derivative simplifies to the integrand on the left hand side.

13 catch all

104. Example

$$\int_2^3 \frac{2x-3}{\sqrt{4x-x^2}}dx$$

Complete the square for the expression under the radical

$$4x - x^2 = 4 - (x-2)^2$$

$$\int_2^3 \frac{2x-3}{\sqrt{4-(x-2)^2}}dx$$

Let $x - 2 = 2t$

Changing the integral entirely in terms of the new variable "t" we have

$$\int_0^{1/2} \frac{2(4t+1)}{\sqrt{4-4t^2}}dt$$

$$\int_0^{1/2} \frac{4t+1}{\sqrt{1-t^2}}dt$$

$$\int_0^{1/2} \frac{4t}{\sqrt{1-t^2}}dt + \int_0^{1/2} \frac{1}{\sqrt{1-t^2}}dt$$

$$-4\left[\sqrt{1-t^2}\right]_0^{1/2}+\left[\arcsin t\right]_0^{1/2}$$

$$-4\left\{\frac{\sqrt{3}}{2}-1\right\}+\{\arcsin(1/2)-\arcsin(0)\}$$

$$4-2\sqrt{3}+\frac{\pi}{6}$$

14 Indeterminate Forms

105. In earlier sections, we met a few "indeterminate " forms, e.g. in §1.3, there is a worked example #6 on page 60

$$\lim_{x\to 1}\frac{x^3-1}{x-1}$$

which is described as a $\frac{0}{0}$ problen. In general , if the problem is of the form $\lim_{x\to a}\frac{P(x)}{Q(x)}$ where P(x) and Q(x) are polynomials in x and $\lim_{x\to a}P(x)=0$ and $\lim_{x\to a}Q(x)=0$ then, the indeterminacy may be removed by cancelling the $(x-a)$ factor which is guaranteed to be present by the fundamental theorem of algebra,"If $P(\alpha)=0,(x-\alpha)$ is a factor". In the same section on page 62, there is an example

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$$

which requires rationalization of the numerator before the factor x can be cancelled. However, neither of these techniques would work for the following problem

$$\lim_{x\to 1}\frac{\arctan x-(\pi/4)}{x-1}$$

Also in this section , we evaluated some trigonometric limits using the fundamental trigonometric limit $\lim_{x\to 0}\frac{\sin x}{x}=1$.
In §3.5, we evaluated limits of the form

$$\lim_{x\to\infty}\frac{2x^3+7x^2-5x+83}{5x^3-25x^2+19x-253}$$

which is described as $\dfrac{\infty}{\infty}$ type problem. In this case, we factored out the highest power of x leading to reciprocals in x. We then took advantage of the known limits, $\lim_{x\to\infty} \dfrac{1}{x^n} = 0$ where n is a positive number. In this problem, the answer is 2/5. However, this technique will not work in the following case.

$$\lim_{x\to\infty} \frac{\ln(x^2+5)}{\ln(x+3)}$$

In future work, we have to guess intelligently what is likely to happen. In this case, try to guess what this limit will be. Remember, that this guess is NOT an answer to the problem. Clearly we need some additional tools to handle in a routine , mechanical manner, a wide variety of limit problems. First, let us summarize the various types of indeterminate forms

(a) $\dfrac{0}{0}$; $\dfrac{\infty}{\infty}$

(b) $0 \cdot \infty$

(c) $1^\infty, \infty^0, 0^0, [0^\infty = 0,\ 0^{-\infty} = \infty]$

(d) $\infty - \infty$

Remember that these statements are short-hand ways of writing limit statements.

14.1 $\dfrac{0}{0}$ type

106. L'Hôpitals Rule ,$\dfrac{0}{0}$ case. Let f and g be functions that are differentiable on an open interval (a,b) containing c , except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a,b) except possible at c itself. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ so that $\lim_{x\to c} \dfrac{f(x)}{g(x)}$ produces the indeterminate form $\dfrac{0}{0}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite).
Note: In the above rule, the following limit statements may be substituted for the limit statement $\lim_{x\to c}$

(a) $\lim_{x \to c^+}$

(b) $\lim_{x \to c^-}$

(c) $\lim_{x \to \infty}$

(d) $\lim_{x \to -\infty}$

provided they lead to the indeterminate form $\frac{0}{0}$ In applying L'Hôpitals Rule, be aware that the right hand side is a quotient of derivatives and not the derivative of a quotient. Also be on the lookout for simplification through trigonometric identities or algebraic manipulation. You may apply L'Hôpitals Rule as long as you have the indeterminate form $\frac{0}{0}$ but dont apply the procedure if it is no longer an indeterminate form $\frac{0}{0}$. (Shortly ,we shall extend L'Hôpitals Rule, to cover the indeterminate form $\frac{\infty}{\infty}$)

107. First let us apply L'Hôpitals rule to some examples we could have worked from techniques given in §1.3

Example. Evaluate the following limits

(a) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

(b) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

Solution 107a

$$\begin{aligned} \lim_{x \to 1} \frac{x^3 - 1}{x - 1} &= \lim_{x \to 1} \frac{3x^2}{1} \\ &= 3 \end{aligned}$$

Solution 107b

$$\begin{aligned} \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} &= \lim_{x \to 4} \frac{\frac{1}{2\sqrt{x}}}{1} \\ &= \frac{1}{4} \end{aligned}$$

108. Example Evaluate the following limit

$$\lim_{x\to 0}\frac{e^{2x}-1}{x}$$

Solution First, we see that the problem leads to an indeterminate form, $\frac{0}{0}$ and so we can apply L'Hôpitals rule.

$$\begin{aligned}\lim_{x\to 0}\frac{e^{2x}-1}{x} &= \lim_{x\to 0}\frac{2e^{2x}}{1}\\ &= 2\end{aligned}$$

109. Example Evaluate the following limit

$$\lim_{x\to 0}\frac{6e^x-3x^2+x^3-6x-6}{x^3}$$

First we determine that this limit is an indeterminate form, namely a $\frac{0}{0}$ type. Hence we can apply L'Hôpitals rule.

$$\begin{aligned}\lim_{x\to 0}\frac{6e^x-3x^2+x^3-6x-6}{x^3} &= \lim_{x\to 0}\frac{6e^x-6x+3x^2-6}{3x^2}\ ;\ \frac{0}{0}\\ &= \lim_{x\to 0}\frac{6e^x-6+6x}{6x}\ ;\ \frac{0}{0}\\ &= \lim_{x\to 0}\frac{6e^x+6}{6}\\ &= 2\end{aligned}$$

110. Example Evaluate the following limit.

$$\lim_{x\to 0^+}\frac{\tan x}{x^2}$$

A preliminary examination of the problem shows that it is of the $\frac{0}{0}$ type so that we can apply L'Hôpitals rule.
Solution

$$\begin{aligned}\lim_{x\to 0^+}\frac{\tan x}{x^2} &= \lim_{x\to 0^+}\frac{\sec^2 x}{2x}\\ &= \frac{1}{0^+}\\ &= \text{Does not exist}\\ &= \infty\end{aligned}$$

Note that in this problem, the limit does not exist, that is the limit does not approach a finite number. However, in this case,we can write down a complete description of the non- existence.

111. Exercises Evaluate the following limits.

(a) $\lim_{x \to 0^-} \frac{\tan x}{x^2}$

(b) $\lim_{x \to 0} \frac{\tan x}{x^4}$

(c) $\lim_{x \to 0} \frac{2\sqrt{x+1} - 2 - x}{x^2}$

(d) $\lim_{x \to 0} \frac{\sin x}{x + x^2}$

(e) $\lim_{x \to 0} \frac{1 - \cos x}{x^3 + x^2}$

(f) $\lim_{x \to 0} \frac{x \sin x}{1 + e^{x^2}}$

(g) $\lim_{x \to 0} \frac{e^x - x - 1}{x^4}$

112. Answers

(a) (111a) $-\infty$

(b) (111b) Does not exist, Cannot write down a description of the non-existence.

(c) (111c) $\frac{-1}{4}$

(d) (111d) 1

(e) (111e) $\frac{1}{2}$

(f) (111f) 1

(g) (111g) Does not exist, ∞

14.2 $\frac{\infty}{\infty}$ type.

113. L'Hôpitals Rule $\frac{\infty}{\infty}$ case. Let f and g be functions that are differentiable on an open interval (a, ∞) containing c , except possibly at c itself. Assume that

$g'(x) \neq 0$ for all x in (a, ∞) except possible at c itself. If $\lim\limits_{x \to c} f(x) = \infty$ and $\lim\limits_{x \to c} g(x) = \infty$ so that $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$ produces the indeterminate form $\frac{\infty}{\infty}$, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite)
Note: In the above rule, the following limit statements may be substituted for the limit statement $\lim\limits_{x \to c}$

(a) $\lim\limits_{x \to c^+}$

(b) $\lim\limits_{x \to c^-}$

(c) $\lim\limits_{x \to \infty}$

(d) $\lim\limits_{x \to -\infty}$

provided they lead to the indeterminate form $\frac{\infty}{\infty}$. Note, $\frac{-\infty}{\infty}$ or $\frac{\infty}{-\infty}$ are referred to as $\frac{\infty}{\infty}$ type indeterminate forms.

114. Example Evaluate the following limit.

$$\lim_{x \to \infty} \frac{3x^2 + 2x - 5}{5x^2 - 3x + 7}$$

A preliminary examination of the problem shows that it is of the $\frac{\infty}{\infty}$ type so that we can apply L'Hôpitals rule.Note that this problem could have been done using the methods of §1.3
Solution

$$\begin{aligned} \lim_{x \to \infty} \frac{3x^2 + 2x - 5}{5x^2 - 3x + 7} &= \lim_{x \to \infty} \frac{6x + 2}{10x - 3} \quad ; \frac{\infty}{\infty} \\ &= \lim_{x \to \infty} \frac{6}{10} \\ &= \frac{3}{5} \end{aligned}$$

115. Example Evaluate the following limit.

$$\lim_{x\to\infty} \frac{\ln(x^2+1)}{\ln(x+5)}$$

A preliminary examination of the problem shows that it is of the $\frac{\infty}{\infty}$ type so that we can apply L'Hôpitals rule.

Solution

$$\begin{aligned}
\lim_{x\to\infty} \frac{\ln(x^2+1)}{\ln(x+5)} &= \lim_{x\to\infty} \frac{\frac{2x}{x^2+1}}{\frac{1}{x+5}} \quad ;\text{Simplify} \\
&= \lim_{x\to\infty} \frac{2x^2+10x}{x^2+1} \quad ; \frac{\infty}{\infty} \\
&= \lim_{x\to\infty} \frac{4x+10}{2x} \quad ; \frac{\infty}{\infty} \\
&= \lim_{x\to\infty} \frac{4}{2} \\
&= 2
\end{aligned}$$

14.3 $0 \cdot \infty$ type.

If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \infty$, then $\lim_{x\to c} f(x)g(x)$ produces an indeterminate form of the type $0 \cdot \infty$ .This can usually be solved by rewriting the limit as

$$\lim_{x\to c} \frac{f(x)}{\frac{1}{g(x)}}$$

which leads to a $\frac{0}{0}$ problem or rewriting in the form

$$\lim_{x\to c} \frac{g(x)}{\frac{1}{f(x)}}$$

which leads to a $\frac{\infty}{\infty}$ problem. In either case, it is then possible to use L'Hôpitals Rule. Whichever one leads to the easiest subsequent mathematics should be chosen.

116. Evaluate the following limit.

$$\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$$

. A preliminary investigation shows that this problem is of the $0\cdot\infty$ type. We shall rewrite this in the form

$$\lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

. which now becomes a $\frac{0}{0}$ problem.

$$\begin{aligned}\lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{\frac{-1}{x^2}\sec^2\left(\frac{1}{x}\right)}{\frac{-1}{x^2}} \quad ;\text{Simplify} \\ &= \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{x}\right)}{1} \\ &= 1\end{aligned}$$

117. Example Evaluate the following limit.

$$\lim_{x\to\infty} x\left(e^{\frac{1}{x}} - 1\right)$$

. A preliminary investigation shows that this problem is of the $0\cdot\infty$ type. We shall rewrite this in the form

$$\lim_{x\to\infty} \frac{\left(e^{\frac{1}{x}} - 1\right)}{\frac{1}{x}}$$

. which now becomes a $\frac{0}{0}$ problem.

$$\begin{aligned}\lim_{x\to\infty} \frac{\left(e^{\frac{1}{x}} - 1\right)}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{\frac{-1}{x^2}\left(e^{\frac{1}{x}}\right)}{\frac{-1}{x^2}} \quad ;\text{Simplify} \\ &= \lim_{x\to\infty} e^{\frac{1}{x}} \\ &= 1\end{aligned}$$

14.4 $1^\infty, \infty^0, 0^0, [\,0^\infty = 0,\ 0^{-\infty} = \infty]$ indeterminate forms.

118. Example Evaluate the following limit.

$$\lim_{x\to\infty} x^{1/x}$$

Clearly this limit can be described as a ∞^0 type indeterminate form.Let $y = x^{1/x}$.

$$\begin{aligned}
y &= x^{1/x} \\
\ln y &= \ln\left(x^{1/x}\right) \\
\ln y &= \frac{\ln x}{x} \\
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{\ln x}{x}\ ;\ \frac{\infty}{\infty} \\
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{1}{x} \\
\ln\left(\lim_{x\to\infty} y\right) &= 0 \\
\lim_{x\to\infty} y &= e^{0} \\
\lim_{x\to\infty} x^{1/x} &= 1
\end{aligned}$$

Notice that in the above work, we have taken natural logarithms of both sides, used logarithm rules and applied L'Hôpitals Rule where appropriate.

119. Example Evaluate the following limit.

$$\lim_{x\to\infty} \left(1 - \frac{2}{x}\right)^{3x}$$

Clearly this limit can be described as a 1^∞ type indeterminate form. Let us imitate the procedure used to solve the previous problem. Let $y = \left(1 - \frac{2}{x}\right)^{3x}$

$$\begin{aligned}
y &= \left(1-\frac{2}{x}\right)^{3x} \\
\ln y &= 3x \ln\left(1-\frac{2}{x}\right) \quad ; \\
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} 3x \ln\left(1-\frac{2}{x}\right) \quad ; 0 \cdot \infty \\
\lim_{x\to\infty} \ln y &= 3 \lim_{x\to\infty} \frac{\ln\left(1-\frac{2}{x}\right)}{1/x} \; ; \; \frac{0}{0} \\
\lim_{x\to\infty} \ln y &= 3 \lim_{x\to\infty} \frac{\frac{2/x^2}{(1-2/x)}}{-1/x^2} \\
\lim_{x\to\infty} \ln y &= 3 \lim_{x\to\infty} \frac{-2}{(1-2/x)} \\
\lim_{x\to\infty} \ln y &= -6 \\
\ln\left(\lim_{x\to\infty} y\right) &= -6 \\
\lim_{x\to\infty} y &= e^{-6} \\
\lim_{x\to\infty} \left(1-\frac{2}{x}\right)^{3x} &= e^{-6}
\end{aligned}$$

120. <u>Example</u> Evaluate the following limit.

$$\lim_{x\to 0^+} x^{\,1/x}$$

Clearly this limit can be described as a 0^∞ type problem. While it is not indeterminate, let us imitate the procedure used to solve the previous example.

Let $y = x^{\,1/x}$

$$\begin{aligned}
y &= x^{\,1/x} \\
\ln y &= \ln\left(x^{\,1/x}\right) \quad ; \\
\ln y &= \frac{\ln x}{x} \quad ; \\
\lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} \frac{\ln x}{x} \quad ; -\infty \\
\ln\left(\lim_{x\to 0} y\right) &= -\infty \\
\lim_{x\to 0^+} y &= e^{-\infty} \\
\lim_{x\to 0^+} x^{1/x} &= 0
\end{aligned}$$

Note that for examples 118 and 119, an indeterminate form was obtained after we had taken logarithms of both sides and applied the given limit. However, in example 120, which is not indeterminate, when we took logarithms of both sides and applied the given limit, we did not obtain an indeterminate form. Rather we were able to evaluate the answer immediately.

14.5 $\infty - \infty$ indeterminate form

If $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, them $\lim_{x\to c}[f(x) - g(x)]$ is an indeterminate form of the $\infty - \infty$ type. Note that under the same conditions, $\lim_{x\to c}[f(x) + g(x)]$ has ∞ for an answer.

121. Example Evaluate the following limit.

$$\lim_{x\to 2^+} \frac{8}{x^2 - 4} - \frac{x}{x - 2}$$

A preliminary investigation shows that this problem is of the $\infty - \infty$ indeterminate form. We can solve this problem by combining on a common denominator and thereby changing the problem to a $\frac{0}{0}$ type. Hence, we will then be able to apply L'Hôpitals Rule.

$$\begin{aligned}
\lim_{x\to 2^+} \frac{8}{x^2-4} - \frac{x}{x-2} &= \lim_{x\to 2^+} \frac{8-x(x+2)}{x^2-4} \quad ;\text{Simplify} \\
&= \lim_{x\to 2^+} \frac{8-x^2-2x}{x^2-4} \quad ;\frac{0}{0} \\
&= \lim_{x\to 2^+} \frac{-2x-2}{2x} \quad ;\text{Simplify} \\
&= \frac{-3}{2}
\end{aligned}$$

122. Example Evaluate the following limit.

$$\lim_{x\to 0^+} (\cot x - \ln x)$$

A preliminary investigation shows that this problem is of the $\infty - (-\infty)$ or $\infty + \infty$ type and so is not indeterminate. Hence

$$\lim_{x\to 0^+} (\cot x - \ln x) = \infty$$

14.6 Additional Comments

123. Example Evaluate the following limit.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\sec 3x}$$

Solution A preliminary investigation shows that this problem is of the $\frac{\infty}{-\infty}$ or $\frac{\infty}{\infty}$ type and so that we can apply L'Hôpitals rule.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\sec 3x} = \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{3\sec 3x \tan 3x}$$

Clearly we are not making progress. However, if we rewrite the original problem, it is easily solved by application of L'Hôpitals rule.

$$\begin{aligned}
\lim_{x\to(\pi/2)^-} \frac{\sec x}{\sec 3x} &= \lim_{x\to(\pi/2)^-} \frac{\cos 3x}{\cos x} \quad ;\frac{0}{0} \\
&= \lim_{x\to(\pi/2)^-} \frac{-3\sin 3x}{-\sin x} \quad ; \\
&= \frac{-3}{1} \\
&= -3
\end{aligned}$$

124. Example Evaluate the following limit.

$$\lim_{x\to\infty} \frac{e^{-x}}{1+e^{2x}}$$

Solution A preliminary investigation shows that this problem is NOT an indeterminate form. Perhaps the easiest way to see this is to rewrite the problem in the following manner.

$$\begin{aligned}\lim_{x\to\infty} \frac{e^{-x}}{1+e^{2x}} &= \lim_{x\to\infty} \frac{1}{e^{x}(1+e^{2x})} \\ &= \lim_{x\to\infty} \frac{1}{e^{x}+e^{3x}} \quad ; \frac{1}{\infty} \\ &= 0\end{aligned}$$

125. Example Evaluate the following limit.

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{5x-1}$$

Solution A preliminary investigation shows that this problem is an indeterminate form $\frac{\infty}{\infty}$ type. Let us see what happens if we apply L'Hôpitals rule.

$$\begin{aligned}\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{5x-1} &= \lim_{x\to-\infty} \frac{1/2(2x^2+1)^{-1/2}4x}{5} \\ &= \lim_{x\to-\infty} \frac{2x}{5\sqrt{2x^2+1}} \quad \frac{\infty}{\infty} \\ &= \lim_{x\to-\infty} \frac{2}{5/2(2x^2+1)^{-1/2}} \\ &= \lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{5x} \quad \frac{\infty}{\infty}\end{aligned}$$

It is obvious that nothing has been gained. Further applications of L'Hôpitals rule will only cause the problem to oscillate from one form with the radical in the numerator to another form in which the radical in the denominator. To solve this problem, we return to simpler methods.

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{5x-1} &= \lim_{x\to-\infty}\frac{\sqrt{x^2\left(2+\frac{1}{x}\right)}}{x\left(5+\frac{1}{x}\right)}\\
&= \lim_{x\to-\infty}\frac{\sqrt{x^2}\sqrt{\left(2+\frac{1}{x}\right)}}{x\left(5+\frac{1}{x}\right)}\\
&= \lim_{x\to-\infty}\frac{|x|\sqrt{\left(2+\frac{1}{x}\right)}}{x\left(5+\frac{1}{x}\right)}\\
&= \lim_{x\to-\infty}\frac{(-x)\sqrt{\left(2+\frac{1}{x}\right)}}{x\left(5+\frac{1}{x}\right)}\\
&= \lim_{x\to-\infty}\frac{(-)\sqrt{\left(2+\frac{1}{x}\right)}}{\left(5+\frac{1}{x}\right)}\\
&= \frac{-\sqrt{2}}{5}
\end{aligned}$$

In the above work, we have indicated that $\sqrt{x^2} = |x|$ and since we were interested in a limit where x was approaching $-\infty$, we could assume that $x < 0$ and so $|x| = -x$

15 Review Questions

126. Exercise: Answer the following questions

(a) Give the $\epsilon - \delta$ definition for the statement $\lim_{x\to c} f(x) = L$

(b) Find a suitable δ which proves the following statement $\lim_{x\to -3}(x^2+5x+1) = -5$

(c) State the Mean Value Theorem.

(d) $y = \ln \dfrac{x^3\sqrt{5x+2}}{(2x+1)^4}$; find $\dfrac{dy}{dx}$

(e) $y = \operatorname{arcsec}(e^x)$; find $\dfrac{dy}{dx}$

127. Exercise: Answer the following questions

(a) State Rolles Theorem

(b) $x^3y^2 + y^4 + x^3 = 10$; find $\dfrac{dy}{dx}$

(c) $y = \sqrt{e^{3x} + \sin^2(4x)}$; find $\dfrac{dy}{dx}$

(d) A mathematics book is to contain 48 square inches of print per page with margins of 1 1/2 inch along the sides and 2 inches along the top and bottom. Find the dimensions of the page that will require the minimum amount of paper.

(e) Use the second fundamental theorem of calculus to find $F'(x)$ where $F(x) = \displaystyle\int_{\sin x}^{\tan x} \frac{dt}{\sqrt{1+t^3}}$;

128. Exercise: Answer the following questions

(a) $y = \dfrac{x^7(5x+3)^4}{\sqrt[3]{x^5+2}(2x+3)^9}$; find $\dfrac{dy}{dx}$

(b) $\displaystyle\int_{-2}^{-1} \frac{(x-5)}{x^4}dx$

(c) $y = \arctan(e^{2x})$; find $\dfrac{dy}{dx}$

(d) Evaluate $\displaystyle\lim_{x\to\infty} \cos\left(\frac{\pi x}{9+3x}\right)$

(e) Evaluate $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{2x^2-5x+9}}{5x+3}$

129. Exercise: Answer the following questions

(a) Evaluate $\lim_{x\to\pi/3} \dfrac{\sin 3x}{1-\cos 3x}$

(b) Examine the following function for vertical and horizontal asymptotes $f(x) = \dfrac{\sqrt{x^2-2x-15}}{x^2-9x+18}$

(c) $f(x) = (x-6)\sqrt{x}$ Find

i. The open intervals on which the function is increasing, decreasing

ii. Any relative extrema

iii. Absolute extrema on the interval [0, 16]

(d) $f(x) = 3x^5 + 5x^4 - 60x^3 + x + 1$
Determine the open intervals on which the graph is concave upward, the intervals on which the graph is concave downward, and find the point(s) of inflection (if any).

(e) Using the definition $f'(x) = \lim_{\Delta x\to 0} \dfrac{f(x+\Delta x)-f(x)}{\Delta x}$
find $f'(x)$ where $f(x) = \dfrac{1}{3+5x}$

130. Exercise: Perform the following integrations

(a) $\int (4-x^2)^{3/2}dx$

(b) $\int \dfrac{dx}{x\ln x}$

(c) $\int e^{2x}(1+e^x)^{1/2}dx$

(d) $\int \dfrac{5x+15}{\sqrt{x^2+6x+10}}$

(e) $\int x^3 \ln x \ dx$

(f) $\int \dfrac{26\ dx}{x^2+11x-12}$

131. Exercise: Perform the following integrations

(a) $\displaystyle\int \frac{2x+1}{x^2+x-12}dx$

(b) $\displaystyle\int \arcsin x\ dx$

(c) $\displaystyle\int \frac{x}{(x+5)^7}dx$

(d) $\displaystyle\int (x^2+4)^3 dx$

(e) $\displaystyle\int \sin^5 x\ \cos^2 x\ dx$

(f) $\displaystyle\int \frac{dx}{x[5+3\ln x]^2}$

132. <u>Exercise:</u> Perform the following intergrations.

(a) $\displaystyle\int \frac{\mathrm{cosec}^3 x}{\cot^4 x}dx$

(b) $\displaystyle\int x^2 \arctan x\ dx$

(c) $\displaystyle\int e^{7x}(2+e^{3x})^2 dx$

(d) $\displaystyle\int \frac{x+2}{x^2+8x+20}dx$

(e) $\displaystyle\int x \sec x \tan x\ dx$

(f) $\displaystyle\int \frac{3x+5}{x^2+x-12}dx$

133. <u>Exercise:</u> Perform the following integrations

(a) $\displaystyle\int \frac{\sqrt{9+4x^2}}{x^4}dx$

(b) $\displaystyle\int \tan^5 x \sec^3 x\ dx$

(c) $\displaystyle\int \ln(2x+39)dx$

(d) $\displaystyle\int \frac{34}{(x^2+25)(x+3)}dx$

(e) $\displaystyle\int \frac{x^3}{(2+x^2)^5}dx$

(f) $\displaystyle\int \frac{(x^2-16)^{3/2}}{x}dx$

134. Exercise: Perform the following integrations.

(a) $\displaystyle\int \frac{dx}{x^2+4x+8}$

(b) $\displaystyle\int \frac{dx}{\sqrt{-x^2-6x}}$

(c) $\displaystyle\int \frac{dx}{\sqrt{1+e^x}}$

(d) $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$

(e) $\displaystyle\int e^{2x}\operatorname{arcsec}(e^x)dx$

(f) $\displaystyle\int \frac{108}{x^4-81}dx$

1125
March 16, 2005
The time is 18h 45min.