Math 2419 Student Notes #1A

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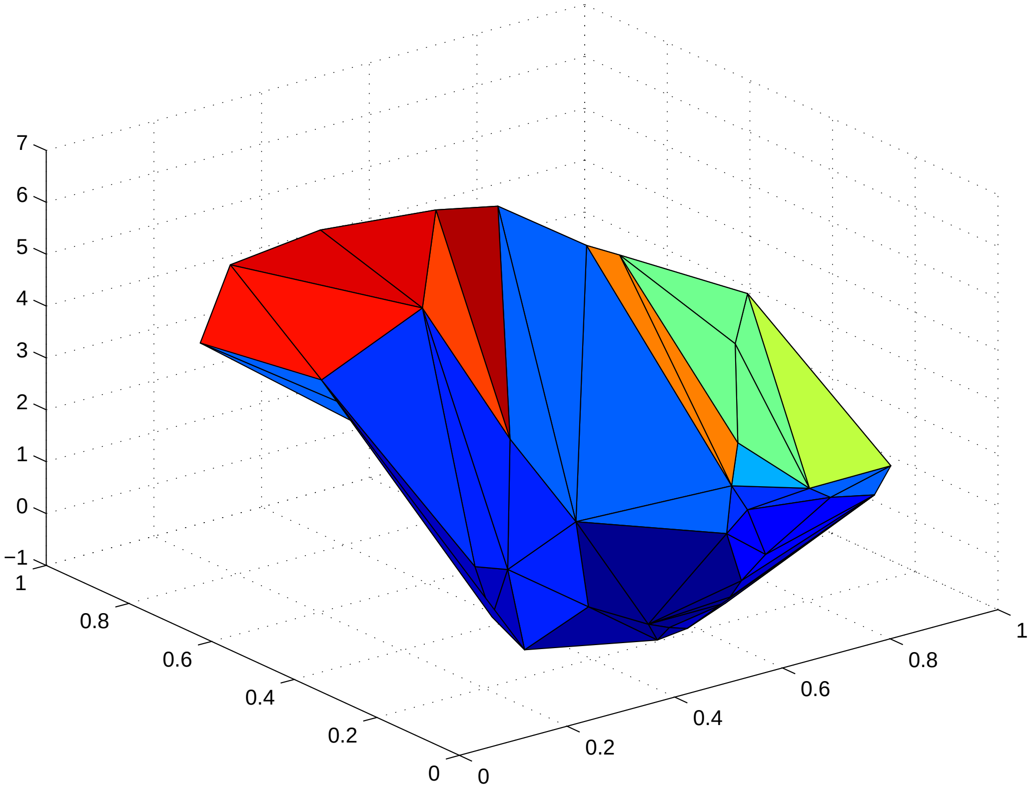

Figure 1: Cover:Triangular Surface Plot

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1 Introduction

. At this time, these notes are only a collection of odds and ends relating to Math 2419. However, this is a start and as time goes by, I will add to them and eventually present you with a coherent document.

2 Indeterminate Forms

1. In earlier sections, we met a few "indeterminate " forms, e.g. in §1.3, there is a worked example #6 on page 62

   $$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

   which is described as a $\frac{0}{0}$ problen. In general , if the problem is of the form $\lim_{x \to a} \frac{P(x)}{Q(x)}$ where P(x) and Q(x) are polynomials in x and $\lim_{x \to a} P(x) = 0$ and $\lim_{x \to a} Q(x) = 0$ then, the indeterminacy may be removed by cancelling the $(x - a)$ factor which is guaranteed to be present by the fundamental theorem of algebra,"If $P(\alpha) = 0, (x - \alpha)$ is a factor". In the same section on page 64, there is an example

   $$\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$$

   which requires rationalization of the numerator before the factor x can be cancelled. However, neither of these techniques would work for the following problem

   $$\lim_{x \to 1} \frac{\arctan x - (\pi/4)}{x - 1}$$

   Also in this section , we evaluated some trigonometric limits using the fundamental trigonometric limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$.
   In §3.5, we evaluated limits of the form

   $$\lim_{x \to \infty} \frac{2x^3 + 7x^2 - 5x + 83}{5x^3 - 25x^2 + 19x - 253}$$

   which is described as $\frac{\infty}{\infty}$ type problem. In this case, we factored out the highest power of x leading to reciprocals in x. We then took advantage of the known limits, $\lim_{x \to \infty} \frac{1}{x^n} = 0$ where n is a positive number. In this problem, the answer is 2/5. However, this technique will not work in the following case.

$$\lim_{x\to\infty} \frac{\ln(x^2+5)}{\ln(x+3)}$$

In future work, we have to guess intelligently what is likely to happen. In this case, try to guess what this limit will be. Remember, that this guess is NOT an answer to the problem. Clearly we need some additional tools to handle in a routine , mechanical manner, a wide variety of limit problems. First, let us summarize the various types of indeterminate forms

(a) $\frac{0}{0}$; $\frac{\infty}{\infty}$

(b) $0 \cdot \infty$

(c) $1^{\infty}, \infty^0, 0^0, [0^{\infty} = 0,\ 0^{-\infty} = \infty]$

(d) $\infty - \infty$

Remember that these statements are short-hand ways of writing limit statements.

2.1 $\frac{0}{0}$ type

2. L'Hôpitals Rule ,$\frac{0}{0}$ case. Let f and g be functions that are differentiable on an open interval (a, b) containing c , except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b) except possible at c itself. If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ so that $\lim_{x\to c} \frac{f(x)}{g(x)}$ produces the indeterminate form $\frac{0}{0}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite).
Note: In the above rule, the following limit statements may be substituted for the limit statement $\lim_{x\to c}$

(a) $\lim_{x\to c^+}$

(b) $\lim_{x\to c^-}$

(c) $\lim_{x\to\infty}$

(d) $\lim_{x \to -\infty}$

provided they lead to the indeterminate form $\frac{0}{0}$ In applying L'Hôpitals Rule, be aware that the right hand side is a quotient of derivatives and not the derivative of a quotient. Also be on the lookout for simplification through trigonometric identities or algebraic manipulation. You may apply L'Hôpitals Rule as long as you have the indeterminate form $\frac{0}{0}$ but dont apply the procedure if it is no longer an indeterminate form $\frac{0}{0}$. (Shortly ,we shall extend L'Hôpitals Rule, to cover the indeterminate form $\frac{\infty}{\infty}$)

3. First let us apply L'Hôpitals rule to some examples we could have worked from techniques given in §1.3
Example. Evaluate the following limits

(a) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

(b) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

Solution 3a

$$\begin{aligned} \lim_{x \to 1} \frac{x^3 - 1}{x - 1} &= \lim_{x \to 1} \frac{3x^2}{1} \\ &= 3 \end{aligned}$$

Solution 3b

$$\begin{aligned} \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} &= \lim_{x \to 4} \frac{\frac{1}{2\sqrt{x}}}{1} \\ &= \frac{1}{4} \end{aligned}$$

4. Example Evaluate the following limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}$$

Solution First, we see that the problem leads to an indeterminate form, $\frac{0}{0}$ and so we can apply L'Hôpitals rule.

$$\begin{aligned}\lim_{x\to 0}\frac{e^{2x}-1}{x} &= \lim_{x\to 0}\frac{2e^{2x}}{1}\\ &= 2\end{aligned}$$

5. Example Evaluate the following limit

$$\lim_{x\to 0}\frac{6e^x-3x^2+x^3-6x-6}{x^3}$$

First we determine that this limit is an indeterminate form, namely a $\frac{0}{0}$ type. Hence we can apply L'Hôpitals rule.

$$\begin{aligned}\lim_{x\to 0}\frac{6e^x-3x^2+x^3-6x-6}{x^3} &= \lim_{x\to 0}\frac{6e^x-6x+3x^2-6}{3x^2}\;;\;\frac{0}{0}\\ &= \lim_{x\to 0}\frac{6e^x-6+6x}{6x}\;;\;\frac{0}{0}\\ &= \lim_{x\to 0}\frac{6e^x+6}{6}\\ &= 2\end{aligned}$$

6. Example Evaluate the following limit.

$$\lim_{x\to 0^+}\frac{\tan x}{x^2}$$

A preliminary examination of the problem shows that it is of the $\frac{0}{0}$ type so that we can apply L'Hôpitals rule.

Solution

$$\begin{aligned}\lim_{x\to 0^+}\frac{\tan x}{x^2} &= \lim_{x\to 0^+}\frac{\sec^2 x}{2x}\\ &= \frac{1}{0^+}\\ &= \text{Does not exist}\\ &= \infty\end{aligned}$$

Note that in this problem, the limit does not exist, that is the limit does not approach a finite number. However, in this case,we can write down a complete description of the non-existence.

7. Exercises Evaluate the following limits.

   (a) $\lim_{x \to 0^-} \frac{\tan x}{x^2}$

   (b) $\lim_{x \to 0} \frac{\tan x}{x^4}$

   (c) $\lim_{x \to 0} \frac{2\sqrt{x+1} - 2 - x}{x^2}$

   (d) $\lim_{x \to 0} \frac{\sin x}{x + x^2}$

   (e) $\lim_{x \to 0} \frac{1 - \cos x}{x^3 + x^2}$

   (f) $\lim_{x \to 0} \frac{x \sin x}{1 + e^{x^2}}$

   (g) $\lim_{x \to 0} \frac{e^x - x - 1}{x^4}$

8. Answers

   (a) (7a) $-\infty$

   (b) (7b) Does not exist, Cannot write down a description of the non-existence.

   (c) (7c) $\frac{-1}{4}$

   (d) (7d) 1

   (e) (7e) $\frac{1}{2}$

   (f) (7f) 1

   (g) (7g) Does not exist, ∞

2.2 $\frac{\infty}{\infty}$ type.

9. L'Hôpitals Rule $\frac{\infty}{\infty}$ case. Let f and g be functions that are differentiable on an open interval (a, ∞) containing c , except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, ∞)

except possible at c itself. If $\lim\limits_{x\to c} f(x) = \infty$ and $\lim\limits_{x\to c} g(x) = \infty$ so that $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ produces the indeterminate form $\dfrac{\infty}{\infty}$, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is infinite)
Note: In the above rule, the following limit statements may be substituted for the limit statement $\lim\limits_{x\to c}$

(a) $\lim\limits_{x\to c^+}$
(b) $\lim\limits_{x\to c^-}$
(c) $\lim\limits_{x\to \infty}$
(d) $\lim\limits_{x\to -\infty}$

provided they lead to the indeterminate form $\dfrac{\infty}{\infty}$. Note, $\dfrac{-\infty}{\infty}$ or $\dfrac{\infty}{-\infty}$ are referred to as $\dfrac{\infty}{\infty}$ type indeterminate forms.

10. Example Evaluate the following limit.

$$\lim_{x\to\infty} \frac{3x^2+2x-5}{5x^2-3x+7}$$

A preliminary examination of the problem shows that it is of the $\dfrac{\infty}{\infty}$ type so that we can apply L'Hôpitals rule. Note that this problem could have been done using the methods of §1.3
Solution

$$\begin{aligned} \lim_{x\to\infty} \frac{3x^2+2x-5}{5x^2-3x+7} &= \lim_{x\to\infty} \frac{6x+2}{10x-3} \quad ; \frac{\infty}{\infty} \\ &= \lim_{x\to\infty} \frac{6}{10} \\ &= \frac{3}{5} \end{aligned}$$

11. Example Evaluate the following limit.

$$\lim_{x\to\infty}\frac{\ln(x^2+1)}{\ln(x+5)}$$

A preliminary examination of the problem shows that it is of the $\frac{\infty}{\infty}$ type so that we can apply L'Hôpitals rule.
Solution

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln(x^2+1)}{\ln(x+5)} &= \lim_{x\to\infty}\frac{\frac{2x}{x^2+1}}{\frac{1}{x+5}} \quad ;\text{Simplify}\\
&= \lim_{x\to\infty}\frac{2x^2+10x}{x^2+1} \quad ;\frac{\infty}{\infty}\\
&= \lim_{x\to\infty}\frac{4x+10}{2x} \quad ;\frac{\infty}{\infty}\\
&= \lim_{x\to\infty}\frac{4}{2}\\
&= 2
\end{aligned}$$

2.3 $0\cdot\infty$ type.

If $\lim_{x\to c} f(x)=0$ and $\lim_{x\to c} g(x)=\infty$, then $\lim_{x\to c} f(x)g(x)$ produces an indeterminate form of the type $0\cdot\infty$ .This can usually be solved by rewriting the limit as

$$\lim_{x\to c}\frac{f(x)}{\frac{1}{g(x)}}$$

which leads to a $\frac{0}{0}$ problem or rewriting in the form

$$\lim_{x\to c}\frac{g(x)}{\frac{1}{f(x)}}$$

which leads to a $\frac{\infty}{\infty}$ problem. In either case, it is then possible to use L'Hôpitals Rule. Whichever one leads to the easiest subsequent mathematics should be chosen.

12. Evaluate the following limit.

$$\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$$

. A preliminary investigation shows that this problem is of the $0\cdot\infty$ type. We shall rewrite this in the form

$$\lim_{x\to\infty}\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

. which now becomes a $\dfrac{0}{0}$ problem.

$$\begin{aligned}\lim_{x\to\infty}\frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{\frac{-1}{x^2}\sec^2\left(\frac{1}{x}\right)}{\frac{-1}{x^2}} \quad ;\text{Simplify}\\ &= \lim_{x\to\infty}\frac{\sec^2\left(\frac{1}{x}\right)}{1}\\ &= 1\end{aligned}$$

13. Example Evaluate the following limit.

$$\lim_{x\to\infty} x\left(e^{\frac{1}{x}}-1\right)$$

. A preliminary investigation shows that this problem is of the $0\cdot\infty$ type. We shall rewrite this in the form

$$\lim_{x\to\infty}\frac{\left(e^{\frac{1}{x}}-1\right)}{\frac{1}{x}}$$

. which now becomes a $\dfrac{0}{0}$ problem.

$$\begin{aligned}\lim_{x\to\infty}\frac{\left(e^{\frac{1}{x}}-1\right)}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{\frac{-1}{x^2}\left(e^{\frac{1}{x}}\right)}{\frac{-1}{x^2}} \quad ;\text{Simplify}\\ &= \lim_{x\to\infty} e^{\frac{1}{x}}\\ &= 1\end{aligned}$$

2.4 $1^\infty, \infty^0, 0^0, [\,0^\infty = 0,\ 0^{-\infty} = \infty]$ indeterminate forms.

14. Example Evaluate the following limit.

$$\lim_{x\to\infty} x^{1/x}$$

Clearly this limit can be described as a ∞^0 type indeterminate form.Let $y = x^{1/x}$.

$$\begin{aligned} y &= x^{1/x} \\ \ln y &= \ln\left(x^{1/x}\right) \\ \ln y &= \frac{\ln x}{x} \\ \lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{\ln x}{x} \ ; \ \frac{\infty}{\infty} \\ \lim_{x\to\infty} \ln y &= \lim_{x\to\infty} \frac{1}{x} \\ \ln\left(\lim_{x\to\infty} y\right) &= 0 \\ \lim_{x\to\infty} y &= e^{\,0} \\ \lim_{x\to\infty} x^{1/x} &= 1 \end{aligned}$$

Notice that in the above work, we have taken natural logarithms of both sides, used logarithm rules and applied L'Hôpitals Rule where appropriate.

15. Example Evaluate the following limit.

$$\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{3x}$$

Clearly this limit can be described as a 1^∞ type indeterminate form. Let us imitate the procedure used to solve the previous problem. Let $y = \left(1 - \dfrac{2}{x}\right)^{3x}$

$$\begin{aligned}
y &= \left(1 - \frac{2}{x}\right)^{3x} \\
\ln y &= 3x \ln\left(1 - \frac{2}{x}\right) \quad ; \\
\lim_{x\to\infty} \ln y &= \lim_{x\to\infty} 3x \ln\left(1 - \frac{2}{x}\right) \quad ; 0 \cdot \infty \\
\lim_{x\to\infty} \ln y &= 3 \lim_{x\to\infty} \frac{\ln\left(1 - \frac{2}{x}\right)}{1/x} \; ; \; \frac{0}{0} \\
\lim_{x\to\infty} \ln y &= 3 \lim_{x\to\infty} \frac{\frac{2/x^2}{(1-2/x)}}{-1/x^2} \\
\lim_{x\to\infty} \ln y &= 3 \lim_{x\to\infty} \frac{-2}{(1-2/x)} \\
\lim_{x\to\infty} \ln y &= -6 \\
\ln\left(\lim_{x\to\infty} y\right) &= -6 \\
\lim_{x\to\infty} y &= e^{-6} \\
\lim_{x\to\infty} \left(1 - \frac{2}{x}\right)^{3x} &= e^{-6}
\end{aligned}$$

16. Example Evaluate the following limit.

$$\lim_{x\to 0^+} x^{\,1/x}$$

Clearly this limit can be described as a 0^∞ type problem. While it is not indeterminate, let us imitate the procedure used to solve the previous example. Let $y = x^{\,1/x}$

$$\begin{aligned} y &= x^{\,1/x} \\ \ln y &= \ln\left(x^{\,1/x}\right) \quad ; \\ \ln y &= \frac{\ln x}{x} \quad ; \\ \lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} \frac{\ln x}{x} \quad ; -\infty \\ \ln\left(\lim_{x\to 0} y\right) &= -\infty \\ \lim_{x\to 0^+} y &= e^{-\infty} \\ \lim_{x\to 0^+} x^{1/x} &= 0 \end{aligned}$$

Note that for examples 14 and 15, an indeterminate form was obtained after we had taken logarithms of both sides and applied the given limit. However, in example 16, which is not indeterminate, when we took logarithms of both sides and applied the given limit, we did not obtain an indeterminate form. Rather we were able to evaluate the answer immediately.

2.5 $\infty - \infty$ indeterminate form

If $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = \infty$, them $\lim_{x\to c}[f(x) - g(x)]$ is an indeterminate form of the $\infty - \infty$ type. Note that under the same conditions, $\lim_{x\to c}[f(x) + g(x)]$ has ∞ for an answer.

17. Example Evaluate the following limit.

$$\lim_{x\to 2^+} \frac{8}{x^2 - 4} - \frac{x}{x-2}$$

A preliminary investigation shows that this problem is of the $\infty - \infty$ indeterminate form. We can solve this problem by combining on a common denominator and thereby changing the problem to a $\frac{0}{0}$ type. Hence, we will then be able to apply L'Hôpitals Rule.

$$\begin{aligned}
\lim_{x\to 2^+} \frac{8}{x^2-4} - \frac{x}{x-2} &= \lim_{x\to 2^+} \frac{8-x(x+2)}{x^2-4} \quad ; \text{Simplify} \\
&= \lim_{x\to 2^+} \frac{8-x^2-2x}{x^2-4} \quad ; \frac{0}{0} \\
&= \lim_{x\to 2^+} \frac{-2x-2}{2x} \quad ; \text{Simplify} \\
&= \frac{-3}{2}
\end{aligned}$$

18. Example Evaluate the following limit.

$$\lim_{x\to 0^+} (\cot x - \ln x)$$

A preliminary investigation shows that this problem is of the $\infty - (-\infty)$ or $\infty + \infty$ type and so is not indeterminate. Hence

$$\lim_{x\to 0^+} (\cot x - \ln x) = \infty$$

2.6 Additional Comments

19. Example Evaluate the following limit.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\sec 3x}$$

Solution A preliminary investigation shows that this problem is of the $\frac{\infty}{-\infty}$ or $\frac{\infty}{\infty}$ type and so that we can apply L'Hôpitals rule.

$$\lim_{x\to(\pi/2)^-} \frac{\sec x}{\sec 3x} = \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{3\sec 3x \tan 3x}$$

Clearly we are not making progress. However, if we rewrite the original problem, it is easily solved by application of L'Hôpitals rule.

$$\begin{aligned}
\lim_{x\to(\pi/2)^-}\frac{\sec x}{\sec 3x} &= \lim_{x\to(\pi/2)^-}\frac{\cos 3x}{\cos x} \quad ; \frac{0}{0}\\
&= \lim_{x\to(\pi/2)^-}\frac{-3\sin 3x}{-\sin x} \quad ;\\
&= \frac{-3}{1}\\
&= -3
\end{aligned}$$

20. Example Evaluate the following limit.

$$\lim_{x\to\infty}\frac{e^{-x}}{1+e^{2x}}$$

Solution A preliminary investigation shows that this problem is NOT an indeterminate form. Perhaps the easiest way to see this is to rewrite the problem in the following manner.

$$\begin{aligned}
\lim_{x\to\infty}\frac{e^{-x}}{1+e^{2x}} &= \lim_{x\to\infty}\frac{1}{e^{x}(1+e^{2x})}\\
&= \lim_{x\to\infty}\frac{1}{e^{x}+e^{3x}} \quad ; \frac{1}{\infty}\\
&= 0
\end{aligned}$$

21. Example Evaluate the following limit.

$$\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{5x-1}$$

Solution A preliminary investigation shows that this problem is an indeterminate form $\frac{\infty}{\infty}$ type. Let us see what happens if we apply L'Hôpitals rule.

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{5x-1} &= \lim_{x\to-\infty}\frac{1/2(2x^2+1)^{-1/2}4x}{5}\\
&= \lim_{x\to-\infty}\frac{2x}{5\sqrt{2x^2+1}} \quad \frac{\infty}{\infty}\\
&= \lim_{x\to-\infty}\frac{2}{5/2(2x^2+1)^{-1/2}}\\
&= \lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{5x} \quad \frac{\infty}{\infty}
\end{aligned}$$

It is obvious that nothing has been gained. Further applications of L'Hôpitals rule will only cause the problem to oscillate from one form with the radical in the numerator to another form in which the radical in the denominator. To solve this problem, we return to simpler methods.

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{5x-1} &= \lim_{x\to-\infty}\frac{\sqrt{x^2\left(2+\dfrac{1}{x}\right)}}{x\left(5+\dfrac{1}{x}\right)} \\
&= \lim_{x\to-\infty}\frac{\sqrt{x^2}\sqrt{\left(2+\dfrac{1}{x}\right)}}{x\left(5+\dfrac{1}{x}\right)} \\
&= \lim_{x\to-\infty}\frac{|x|\sqrt{\left(2+\dfrac{1}{x}\right)}}{x\left(5+\dfrac{1}{x}\right)} \\
&= \lim_{x\to-\infty}\frac{(-x)\sqrt{\left(2+\dfrac{1}{x}\right)}}{x\left(5+\dfrac{1}{x}\right)} \\
&= \lim_{x\to-\infty}\frac{(-)\sqrt{\left(2+\dfrac{1}{x}\right)}}{\left(5+\dfrac{1}{x}\right)} \\
&= \frac{-\sqrt{2}}{5}
\end{aligned}$$

In the above work, we have indicated that $\sqrt{x^2}=|x|$ and since we were interested in a limit where x was approaching $-\infty$, we could assume that $x<0$ and so $|x|=-x$

3 Infinite Series

3.1 Introduction

22. Basic Definition Many students have difficulty with Chapter 9 of our text, namely with Infinite Series. It is the purpose of these notes to help you overcome any difficulties and to approach the subject of Infinite Series with a great deal of confidence. First let us review the essential material. Consider the infinite series $\sum_{k=1}^{\infty} a_k$. A basic question is does this series have a sum and if so how does one find it and what meaning can we give to this word sum. Clearly we cannot use a calculator or a computer to sum up an infinite number of terms.

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots$$

Let us define the quantity S_n as the nth partial sum. That is , it is the sum of the first n terms.

$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$S_4 = a_1 + a_2 + a_3 + a_4$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$
$$\text{or } S_n = \sum_{k=1}^{n} a_k$$

We will define S the sum of the infinite series $\sum_{k=1}^{\infty} a_k$ to be

$$S = \lim_{n\to\infty} S_n$$

If this limit exists , that is, the limit is a finite number, we say that the series converges and has a sum equal to S. If the limit does not exist, for example, the limit does not approach a finite number or goes to infinity, we say that the series diverges. In this case the word "sum" has no meaning. This definition is fundamental to all work in series. The major difficulty is obtaining a suitable expression for the n^{th} partial sum S_n . Your text book has two cases where it is relatively easy to find an expression for S_n and hence be able to determine whether the given series converges or diverges.

3.2 Telescoping Series

23. Consider the following series.

$$\sum_{k=1}^{\infty} \frac{1}{k} - \frac{1}{k+1}$$

Lets write out a few partial sums.

$$S_1 = 1 - \frac{1}{2}$$

$$S_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$S_3 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4}$$

Write out as many terms as necessary in order to see a pattern develop. In this case, it is clear that $S_n = 1 - \frac{1}{n+1}$. From this expression we conclude that

$$S = \lim_{n \to \infty} S_n = 1$$

Hence the series converges (since the $\lim_{n \to \infty} S_n$exists). The sum of this convergent series is 1, the value of the previous limit. Try to develop 'recognition factors' which help you identify which test or theorem might be successfully applied to a given algebraic or trigonometric form used in the series. For telescoping series, look for two terms, algebraically similar, joined together by a subtraction sign. For a telescoping series, except for a finite number of terms at the beginning of the series and a finite number of terms at the end, all of the terms cancel leaving a relatively simple expression for the n^{th} partial sum

24. <u>Exercises</u>

    (a) $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+3} \right)$

    (b) $\sum_{k=2}^{\infty} \ln \left(\frac{k}{k+1} \right)$

    (c) $\sum_{k=1}^{\infty} \left(\frac{1}{2k+3} - \frac{1}{2k+7} \right)$

    (d) $\sum_{k=1}^{\infty} \frac{1}{4k^2 + 16k}$

Note , as illustrated in the last exercise, it is sometimes necessary to use the method of Partial Fractions in order to set up the algebraic form suitable for the identification of a telesoping series. Other tests will be available later to determine whether the given series converges or diverges. However, these tests will not give the sum directly.

3.3 Divergence Test

25. Divergence Test Let $\sum a_k$ be an infinite series.

    If $\lim_{k\to\infty} a_k \neq 0$, the series diverges.

    Note: This test gives no information on the convergence or divergence of the series if $\lim_{k\to\infty} a_k = 0$.

26. Example Use the divergence test on the series $\displaystyle\sum_{k=1}^{\infty} \frac{2k+7}{3k+5}$

    Solution $\displaystyle\lim_{k\to\infty} \frac{2k+7}{3k+5} = \frac{2}{3} \neq 0$

    Therefore, the series $\displaystyle\sum_{k=1}^{\infty} \frac{2k+7}{3k+5}$ diverges by the divergence test.

27. Example Use the divergence test on the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}\,(5k+3)}{2k-1}$

    Solution Since $\displaystyle\lim_{k\to\infty} \frac{(-1)^{k+1}\,(5k+3)}{2k-1}$ does not exist , that is $\neq 0$ the series diverges by the divergence test.

28. Exercises Use the divergence test on the following series.

    (a) $\displaystyle\sum_{k=1}^{\infty} \cos\frac{1}{k}$

    (b) $\displaystyle\sum_{k=1}^{\infty} \ln\frac{2k}{k+1}$

    (c) $\displaystyle\sum_{k=1}^{\infty} \arctan(k)$

(d) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{3k+5}{17k+2}$

(e) $\sum_{k=1}^{\infty} e^{1/k}$

(f) $\sum_{k=1}^{\infty} \frac{5k^3+3k+21}{7k^3+14k^2+93}$

(g) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{9k^5+2k^2+31}{17k^5-k+19}$

3.4 Geometric Series

29. <u>Geometric Series:</u> A series of the form $\sum_{n=0}^{\infty} a\, r^n$ is called a geometric series

$$\sum_{n=0}^{\infty} a\, r^n = a + ar + a\, r^2 + \ldots\ldots$$

a is the first term while r is the common ratio. Notice that the ratio of consecutive terms is constant, namely r. To prove that a given series is a geometric series, show that the ratio of the $(n+1)^{th}$ term to the n^{th} term is constant

30. <u>Example</u> Show that the series $\sum_{n=0}^{\infty} \frac{1}{5^n}$ is a geometric series.

<u>Solution</u> The n^{th} term is $\frac{1}{5^n}$; the $(n+1)^{th}$ term is $\frac{1}{5^{n+1}}$ (To find the $(n+1)^{th}$ term, replace n by $n+1$ in the algebraic formula for the series.)

The ratio of the two terms is given by.

$$\frac{\frac{1}{5^{n+1}}}{\frac{1}{5^n}} = \frac{1}{5^{n+1}} \frac{5^n}{1} = \frac{1}{5} = r$$

Since this ratio is a constant (independent of n), the series is a geometric series.

As noted above, it is important to develop "recognition factors". As a general guideline, if the series involves a $(\text{constant})^n$, it could be a geometric series. This statement should be interpreted liberally, eg $(\text{constant})^{cn+d}$ that is, a constant raised to a linear factor of n could be a geometric series.

31. Exercises: Show that the following series are geometric series

   (a) $\sum_{n=0}^{\infty} \frac{3^{n+2}}{5^{n+1}}$

   (b) $\sum_{n=0}^{\infty} \frac{3^{2n+1}}{4^{n+1}}$

   (c) $\sum_{n=0}^{\infty} \frac{4^{3n+1}}{5^{2n+3}}$

   (d) $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{5^{2k+1}}{3^{3k+1}}$

32. Exercise The following series do not fall under the general guidelines of a $(\text{constant})^{cn+d}$. Show that these series are not geometric series

   (a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ $(n)^{\text{constant}}$

   (b) $\sum_{k=0}^{\infty} \frac{1}{3^{k^2+1}}$ $(\text{constant})^{k^2}$

   (c) $\sum_{k=1}^{\infty} \frac{1}{\cos k}$

   (d) $\sum_{k=0}^{\infty} \frac{3k+1}{5k-2}$

33. Geometric Series Theorem For a geometric series, if $0 < |r| < 1$, the series converges. If $|r| \geq 1$,the series diverges. If the series converges, the sum S is given by $S = \frac{a}{1-r}$ where "a" is the first term and "r" is the common ratio.

34. Exercise Examine the following series for convergence or divergence. If the series converges, find the sum.

   (a) $\sum_{k=0}^{\infty} \frac{2^{3k+1}}{3^{2k+2}}$

   (b) $\sum_{k=2}^{\infty} \frac{2^{3k+1}}{3^{2k+2}}$

Solution (a) $\dfrac{(k+1)^{th} \text{ term}}{k^{th} \text{ term}} = \dfrac{2^{3(k+1)+1}}{3^{2(K+1)+2}} \dfrac{3^{2k+2}}{2^{3k+1}} = \dfrac{2^3}{3^2} = \dfrac{8}{9} = 2$

Since r is a constant, the series is a geometric series $|r| = |\frac{8}{9}| = \frac{8}{9} < 1,$ series converges

Therefore the sum S is given by $S = \dfrac{a}{1-r}$ where a is the first term. Substituting $k = 0$ into the algebraic formula we have $a = \frac{2}{9}$ and therefore $S = \dfrac{\frac{2}{9}}{1 - \frac{8}{9}} = 2$

Solution (b) $\dfrac{(k+1)^{th} \text{ term}}{k^{th}\text{term}} = \dfrac{2^{3(k+1)+1}}{3^{2(k+1)+2}} \dfrac{3^{2k+2}}{2^{3k+1}} = \dfrac{8}{9} = r$

Since r is a constant, (independent of k), the given series is a geometric series and we may apply the Geometric Series Theorem.

$|r| = |\frac{8}{9}| = \frac{8}{9} < 1,$ so the series converges. Therefore the sum S is given by $S = \dfrac{a}{1-r}$ where a is the first term of the series. Substituting $k = 2$ into the algebraic formula, we have $a = \dfrac{2^7}{3^6} = \dfrac{128}{729}$

$$S = \frac{\frac{128}{129}}{1 - \frac{8}{9}} = \frac{128}{729} \cdot \frac{9}{1} = \frac{128}{81}$$

You will notice that in the above example, both series converge. Only the sum differs. These series differ only at the beginning of the series. Let us generalize what we have discovered. Deleting a finite number of terms at the beginning of a series does not change the convergent or divergent behavior of the series. If the series is convergent, it will change the sum. Recall that this behaviour is similar to that of the infinite integral $\int_a^\infty f(x)dx$ where $f(x)$ is defined on $[a, \infty)$. Whether this integral converges or diverges depends on the behavior of $f(x)$ as x approaches infinity. The value of "a" only determines the value of the integral if it converges. When applying a theorem on infinite series in a later section, we may have to delete a finite number of terms at the beginning of the series in order to conform with conditions of the theorem.

3.5 Integral Test

35. Integral Test

Let $\sum\limits_{k=1}^{\infty} a_k$ be a positive term infinite series. Suppose that $f(x)$ is the function that results when k,an integer in the formula for a$_k$,is replaced by x (a continuous variable).
If $f(x)$ is continuous and decreasing for $x > 1$,
then if $\int\limits_1^{\infty} f(x)dx$ converges, the series $\sum\limits_{k=1}^{\infty} a_k$ converges.
If $\displaystyle\int_1^{\infty} f(x)dx$ diverges, then the series diverges.
Note that there are 3 conditions for this test

(a) Positive term series
(b) $f(x)$ continuous for $x \geqslant 1$
(c) $f(x)$ decreasing for $x \geqslant 1$

Recognition Factor: Use this test when the series satisfies the above 3 conditions and when the infinite integral is relatively easy to evaluate.

36. Example Use the integral test to determine whether the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{k^2+1}$converges or diverges.

Solution: Before proceeding with the solution, note that the previously mentioned tests do not apply, (Telescoping Series, Geometric Series, Divergence Test)

(a) Clearly this is a positive term series
(b) $f(x) = \dfrac{1}{x^2+1}$; $x \geqslant 1$
Since $x^2 + 1 \neq 0$, $f(x)$ is continuous everywhere and in particular for $x \geqslant 1$
(c) $f'(x) = \dfrac{-2x}{(x^2+1)^2}$; $f'(x) < 0$ for $x \geqslant 1$. Hence $f(x)$ is decreasing for $x \geqslant 1$.

All three conditions of the integral test have been satisfied.

Consider

$$\begin{aligned}
\int_1^\infty \frac{dx}{x^2+1} &= \lim_{t\to\infty} \int_1^t \frac{dx}{x^2+1} \\
&= \lim_{t\to\infty} [\arctan x]_{x=1}^{x=t} \\
&= \lim_{t\to\infty} \{\arctan t - \arctan 1\} \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}
\end{aligned}$$

Since the improper integral converges, the series $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ also converges. Always give the justification for your statement. In this case, say that the series converges by the Integral Test. Note that this test is not the only way to investigate the behavior of the series. In this particular case, the <u>limit comparison test</u> is probably the easiest to apply. Also note that the <u>lower</u> integration limit of 1 is quite arbitrary. In fact, if the series had been $\sum_{k=3}^{\infty} \frac{1}{k^2+1}$, the lower integration limit would have been 3.

37. <u>Example</u> Use the integral test to determine whether the series $\sum_{k=1}^{\infty} \frac{k}{k^2+100}$ converges or diverges

<u>Solution</u>

(a) Obviously this is a positive term series

(b) $f(x) = \frac{x}{x^2+100}; \quad x \geqslant 1$
Since $x^2 + 100 \neq 0$, $f(x)$ is continuous for all x and in particular, for $x \geqslant 1$

(c) $f'(x) = \frac{100 - x^2}{(x^2+100)^2}$
For $x < 10$, $f'(x) > 0$ and f is increasing
For $x > 10$ $f'(x) < 0$ and f is decreasing.

Consider

$$\begin{aligned}
\int_{11}^{\infty} f(x)dx &= \int_{11}^{\infty} \frac{x}{x^2+100}dx \\
&= \lim_{t\to\infty} \int_{11}^{t} \frac{x}{x^2+100}dx \\
&= \frac{1}{2}\lim_{t\to\infty} \left[\ln(x^2+100)\right]_{11}^{t} \\
&= \frac{1}{2}\lim_{t\to\infty} \left\{\ln(t^2+100) - \ln(221)\right\} \\
&= \infty
\end{aligned}$$

That is $\displaystyle\int_{11}^{\infty} \frac{x}{x^2+100}dx$ diverges and therefore the series $\displaystyle\sum_{k=11}^{\infty} \frac{k}{k^2+100}$ diverges. This series is <u>not</u> the given series but we can conclude that $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k^2+100}$ diverges since deleting a finite number of terms at the beginning of an infinite series does not affect the convergent or divergent behavior of the series. As with the previous example, the limit comparison test would probably be the preferred method for this example.

38. <u>Example</u> Examine the series $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k^2+1}}$ for convergence or divergence.

    <u>Solution</u> At first glance, this series seems to be a candidate for the integral test since it is a positive term series and relatively easy to integrate. In this case $f(x) = \dfrac{x}{\sqrt{x^2+1}}$; $x >, 1$ so that $f'(x) = \dfrac{1}{(x^2+1)^{3/2}}$ which is always positive. Hence f is increasing, not decreasing as required by the integral test. However, if we apply the Divergence Test, we see that the series diverges. $\displaystyle\lim_{k\to\infty} \frac{k}{\sqrt{k^2+1}} = 1 \neq 0$ Therefore the series diverges, by the Divergence Test.

4 Plane Curves

39. <u>Example:</u> Find all points (if any) of horizontal and vertical tangency to the curve defined by the parametric equations; $x = t^3 - 3t + 1$; $y = 2t^3 + 3t^2 - 12t$

    Also determine any points of inflection.

Solution

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{6t^2 + 6t - 12}{3t^2 - 3}$$

$$= \frac{6[t^2 + t - 2]}{3[t^2 - 1]}$$

$$= \frac{2(t+2)(t-1)}{(t+1)(t-1)}$$

Horizontal tangents occur when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

Vertical tangents occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

$\frac{dy}{dt} = 6t^2 + 6t - 12$

Set $\frac{dy}{dt} = 0$ and solve. $6t^2 + 6t - 12 = 0$

or $6(t^2 + t - 2) = 0$ or $6(t+2)(t-1) = 0$

Therefore $\frac{dy}{dt} = 0$ when $t = -2$ or $t = 1$

$\frac{dx}{dt} = 3t^2 - 3$

Set $\frac{dx}{dt} = 0$ and solve. $3t^2 - 3 = 0$

or $3(t+1)(t-1) = 0$

Therefore $\frac{dx}{dt} = 0$ when $t = 1$ or -1

Lets display this information in a table To determine the behavior at $t = 1$, we must investigate

$\lim_{t \to 1} \frac{6t^2 + 6t - 12}{3t^2 - 3}$; a $\frac{0}{0}$ problem

$= \lim_{t \to 1} \frac{12t + 6}{6t} = 3$

t	$\dfrac{dy}{dt}$	$\dfrac{dx}{dt}$	Conclusions	(x, y)
-2	0	$\neq 0$	Horizontal Tangent	$(-1, 20)$
-1	$\neq 0$	0	Vertical Tangent	$(3, 13)$
1	0	0	Singular Point	$(-1, -7)$

Table 1: Data for Example 39 and figure 2

The curve is not smooth at $t = 1$ since both $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ are simultaneously zero at $t = 1$ (p 714 of your textbook). At $t = 1$, we have a cusp. The graph of this set of parametric equations is shown in Figure 2. This figure was produced using Matlab. the instructions are given below, namely

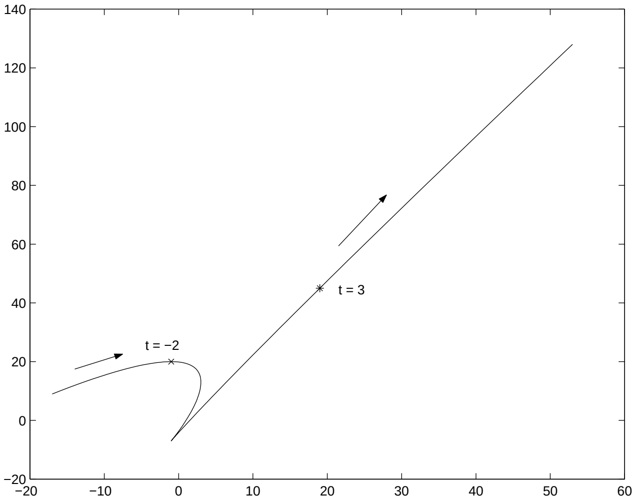

Figure 2: Diagram for example 39

$t = -3 : .05 : 4;$

$x = t.\hat{}3 - (3) * t.\hat{}1 + 1;$

Interval	$-\infty < t < -1$	$-1 < t < 1$	$1 < t < \infty$
Concavity	Down	Up	Down

Table 2: Concavity information for example 39

$y = (2) * t.\hat{}3 + (3) * t.\hat{}2 - (12) * t.\hat{}1;$

As $t \to \infty$, $x \to t^3$ and $y \to 2t^3$ so that as $t \to \infty$, $y = 2x$ is an asymptote. The same comment applies as $t \to -\infty$

To determine candidates for points of inflection, we must find $\dfrac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

For this exercise $\dfrac{d^2y}{dx^2} = \dfrac{-2}{3(t-1)(t+1)^3}$

Candidates occur at $t = 1$ and $t = -1$ (note, these points are defined on the graph). The results are summarized in table 2. Clearly there are points of inflection at $t = 1$, $(-1, -7)$ and $t = -1$, $(3, 13)$

40. Example: A curve is defined by the parametric equations

$$\begin{aligned} x &= t^2 + t && (1) \\ y &= t^2 - t && (2) \end{aligned}$$

(a) Draw the curve and show the direction of the curve

(b) Eliminate the parameter to find the equation in the rectangular system

(c) Locate any points of horizontal or vertical tangency.

(d) Examine the curve for any points of inflection.

Solution: Let us begin by generating a table of values for t, x and y.(See Table 3)

The graph of this curve is shown in figure 3. The direction is given by the direction of increasing parameter and is indicated on the curve. By observation, the graph appears to be a parabola rotated through 45°. To eliminate the parameter, subtract equation 2 from equation 1

t	x	y	t	x	y
3	12	6	-1/2	-1/4	3/4
2	6	2	-1	0	2
1	2	0	-2	2	6
1/2	3/4	-1/4	-3	6	12
0	0	0			

Table 3: Data for Example 40

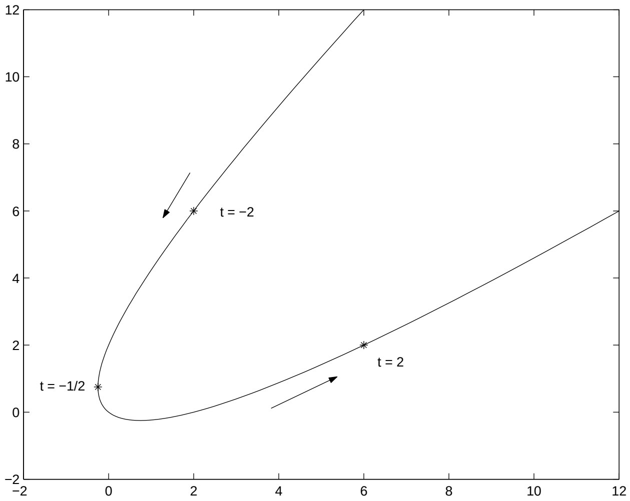

Figure 3: Diagram for example 40

Hence, $x - y = 2t$ or $t = \dfrac{x-y}{2}$ Substituting $t = \dfrac{x-y}{2}$ into equation 1 we have $x = \dfrac{(x-y)^2}{4} + \dfrac{x-4}{2}$

Simplifying, we have $4x = x^2 - 2xy + y^2 + 2x - 2y$

$$x^2 - 2xy + y^2 - 2x - 2y = 0$$

Clearly it is easier to draw this curve from the set of parametric equations rather than from the rectangular equation. The equation is that of a conic section. The presence of the "xy" term suggests that a rotation of axes will simplify the equation. In this particular case, the axes should be rotated clockwise 45°. When this is done it is found that this curve is indeed a parabola. The student is referred to an earlier edition of your text book (eg 5th edition; §9.4), or to a text on Analytic Geometry for a discussion of rotation of axes.

t	$\frac{dy}{dt}$	$\frac{dx}{dt}$	Conclusion	(x, y)
$1/2$	$= 0$	$\neq 0$	HorizontalTangent	$(3/4, -1/4)$
$-1/2$	$\neq 0$	$= 0$	VerticalTangent	$(-1/4, 3/4)$

Table 4: Horizontal and Vertical tangent information for example 40

$y = t^2 - t$ $\qquad$ $x = t^2 + t$

$\frac{dy}{dt} = 2t - 1$ $\qquad$ $\frac{dx}{dt} = 2t + 1$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-1}{2t+1}$$

$\frac{dy}{dx} = 0$ when $t = 1/2$; $\frac{dx}{dt} = 0$ when $t = -1/2$.

Horizontal tangents occur when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$

Vertical tangents occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$

To study the concavity, we must find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}} = \frac{\frac{4}{(2t+1)^2}}{2t+1} = \frac{4}{(2t+1)^3}$$

For $t < -1/2$, $\frac{d^2y}{dx^2} < 0$,graph is concave down

For $t > -1/2$ $\frac{d^2y}{dx^2} > 0$, graph is concave up

Since $t = -1/2$ is a point on the graph, $t = -1/2$ represents a point of inflection at $(-1/4, 3/4)$.

5 Summary of Vector Operations

41. Given the vectors $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$, and $\tilde{\mathbf{C}}$ where

$$\tilde{\mathbf{A}} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{\mathbf{k}}$$
$$\tilde{\mathbf{B}} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{\mathbf{k}}$$
$$\tilde{\mathbf{C}} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{\mathbf{k}}$$

   (a) $\vec{A} + \vec{B} = (a_1 + b_1)\hat{\imath} + (a_2 + b_2)\hat{\jmath} + (a_3 + b_3)\hat{k}$

   (b) $\|\vec{A}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

   (c) Unit vector, $\hat{A} = \dfrac{\vec{A}}{\|\vec{A}\|}$

   (d) $\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3$

   (e) $PROJ_{\vec{B}}\vec{A} = \dfrac{\vec{A} \cdot \vec{B}}{(\|\vec{B}\|)^2}\vec{B}$

   (f) $\|PROJ_{\vec{B}}\vec{A}\| = \dfrac{|\vec{A} \cdot \vec{B}|}{\|\vec{B}\|}$

   (g) $\vec{A} \cdot \vec{B} = \|\vec{A}\| \; \|\vec{B}\| \; \cos\theta; \quad 0 \le \theta \le \pi$

   (h) Direction cosines for $\vec{A}$; $\cos\alpha = \dfrac{a_1}{\|\vec{A}\|}$; $\cos\beta = \dfrac{a_2}{\|\vec{A}\|}$; $\cos\gamma = \dfrac{a_3}{\|\vec{A}\|}$; $\quad 0 \le \alpha, \beta, \gamma \; \le \pi$

   (i) $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\imath}(a_2b_3 - a_3b_2) - \hat{\jmath}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1)$

   (j) $\|\vec{A} \times \vec{B}\| = \|\vec{A}\|\|\vec{B}\| \sin\theta; \quad 0 \le \theta \le \pi$

   (k) $\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$

6 Vector Operations

The following questions will give you an opportunity to practice the various vector operations discussed in this course.

42. Given the vectors $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$, and $\tilde{\mathbf{C}}$ where

$$\tilde{\mathbf{A}} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

$$\tilde{\mathbf{B}} = 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

$$\tilde{\mathbf{C}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

find the following

(a) $\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}}$	ANSWER = -5
(b) $\tilde{\mathbf{A}} \cdot \tilde{\mathbf{C}}$	ANSWER = 22
(c) $\tilde{\mathbf{B}} \cdot \tilde{\mathbf{C}}$	ANSWER = -4
(d) $\tilde{\mathbf{A}} \times \tilde{\mathbf{B}}$	ANSWER = < 0, 18, 12>
(e) $\tilde{\mathbf{A}} \times \tilde{\mathbf{C}}$	ANSWER = < -2, 1, 2>
(f) $\tilde{\mathbf{B}} \times \tilde{\mathbf{C}}$	ANSWER = < 2,-25,-14>
(g) $\lVert\tilde{\mathbf{A}}\rVert$	ANSWER = $\sqrt{17}$
(h) $\lVert\tilde{\mathbf{B}}\rVert$	ANSWER = $\sqrt{29}$
(i) $\lVert\tilde{\mathbf{C}}\rVert$	ANSWER = $\sqrt{29}$
(j) $PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{B}}$	ANSWER = $\frac{-5}{17} < 2, -2, 3 >$
(k) $PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{A}}$	ANSWER = $\frac{-5}{29} < 4, 2, -3 >$
(l) $PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{A}}$	ANSWER = $\frac{22}{29} < 3, -2, 4 >$
(m) $PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{B}}$	ANSWER = $\frac{-4}{29} < 3, -2, 4 >$
(n) $PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{C}}$	ANSWER = $\frac{22}{17} < 2, -2, 3 >$
(o) $PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{C}}$	ANSWER = $\frac{-4}{29} < 4, 2, -3 >$
(p) $\lVert PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{A}}\rVert$	ANSWER = $\frac{5}{\sqrt{29}}$
(q) $\lVert PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{B}}\rVert$	ANSWER = $\frac{5}{\sqrt{17}}$
(r) $\lVert PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{A}}\rVert$	ANSWER = $\frac{22}{\sqrt{29}}$
(s) $\lVert PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{B}}\rVert$	ANSWER = $\frac{4}{\sqrt{29}}$
(t) $\lVert PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{C}}\rVert$	ANSWER = $\frac{22}{\sqrt{17}}$
(u) $\lVert PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{C}}\rVert$	ANSWER = $\frac{4}{\sqrt{29}}$
(v) $\tilde{\mathbf{A}} \cdot (\tilde{\mathbf{B}} \times \tilde{\mathbf{C}})$	ANSWER = 12
(w) $\tilde{\mathbf{B}} \cdot (\tilde{\mathbf{A}} \times \tilde{\mathbf{C}})$	ANSWER = -12
(x) $\tilde{\mathbf{C}} \cdot (\tilde{\mathbf{A}} \times \tilde{\mathbf{B}})$	ANSWER = 12
(y) θ, the angle between $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$	ANSWER : $\cos\theta = \frac{-5}{\sqrt{17}\sqrt{29}}$ = -0.2252 θ = 103.01
(z) θ, the angle between $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$	ANSWER : $\cos\theta = \frac{22}{\sqrt{17}\sqrt{29}}$ = 0.9908 θ = 7.77

(**aa**) θ, the angle between $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ ANSWER : $\cos\theta = \frac{-4}{\sqrt{29}\sqrt{29}}$ = -0.1379
$\theta = 97.93$

(**ab**) The direction cosines for $\tilde{\mathbf{A}}$ ANSWER : $\cos\alpha = \frac{2}{\sqrt{17}} = 0.4851$
$\cos\beta = \frac{-2}{\sqrt{17}}$ = -0.4851
$\cos\gamma = \frac{3}{\sqrt{17}} = 0.7276$

(**ac**) The direction cosines for $\tilde{\mathbf{B}}$ ANSWER : $\cos\alpha = \frac{4}{\sqrt{29}} = 0.7428$
$\cos\beta = \frac{2}{\sqrt{29}} = 0.3714$
$\cos\gamma = \frac{-3}{\sqrt{29}}$ = -0.5571

(**ad**) The direction cosines for $\tilde{\mathbf{C}}$ ANSWER : $\cos\alpha = \frac{3}{\sqrt{29}} = 0.5571$
$\cos\beta = \frac{-2}{\sqrt{29}}$ = -0.3714
$\cos\gamma = \frac{4}{\sqrt{29}} = 0.7428$

(**ae**) The direction angles for $\tilde{\mathbf{A}}$ ANSWER : $\alpha = 60.98$
$\beta = 119.02$
$\gamma = 43.31$

(**af**) The direction angles for $\tilde{\mathbf{B}}$ ANSWER : $\alpha = 42.03$
$\beta = 68.20$
$\gamma = 123.85$

(**ag**) The direction angles for $\tilde{\mathbf{C}}$ ANSWER : $\alpha = 56.15$
$\beta = 111.80$
$\gamma = 42.03$

(**ah**) $\tilde{\mathbf{A}} \times (\tilde{\mathbf{B}} \times \tilde{\mathbf{C}})$ ANSWER : $< 103, 34, -46 >$
(**ai**) $\tilde{\mathbf{B}} \times (\tilde{\mathbf{C}} \times \tilde{\mathbf{A}})$ ANSWER : $< -7, 2, -8 >$
(**aj**) $\tilde{\mathbf{C}} \times (\tilde{\mathbf{A}} \times \tilde{\mathbf{B}})$ ANSWER : $< -96, -36, 54 >$

43. Given the vectors $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$, and $\tilde{\mathbf{C}}$ where

$$\tilde{\mathbf{A}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\tilde{\mathbf{B}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

$$\tilde{\mathbf{C}} = 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

find the following

(a) $\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}}$	ANSWER = -3
(b) $\tilde{\mathbf{A}} \cdot \tilde{\mathbf{C}}$	ANSWER = 5
(c) $\tilde{\mathbf{B}} \cdot \tilde{\mathbf{C}}$	ANSWER = -6
(d) $\tilde{\mathbf{A}} \times \tilde{\mathbf{B}}$	ANSWER = < 7, -7, -7>
(e) $\tilde{\mathbf{A}} \times \tilde{\mathbf{C}}$	ANSWER = < 0, 1, 2>
(f) $\tilde{\mathbf{B}} \times \tilde{\mathbf{C}}$	ANSWER = < -7, 3, 6>
(g) $\|\tilde{\mathbf{A}}\|$	ANSWER = $\sqrt{6}$
(h) $\|\tilde{\mathbf{B}}\|$	ANSWER = $\sqrt{26}$
(i) $\|\tilde{\mathbf{C}}\|$	ANSWER = $\sqrt{5}$
(j) $PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{B}}$	ANSWER = $\frac{-3}{6} < 1, 2, -1 >$
(k) $PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{A}}$	ANSWER = $\frac{-3}{26} < 3, -1, 4 >$
(l) $PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{A}}$	ANSWER = $\frac{5}{5} < 0, 2, -1 >$
(m) $PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{B}}$	ANSWER = $\frac{-6}{5} < 0, 2, -1 >$
(n) $PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{C}}$	ANSWER = $\frac{5}{6} < 1, 2, -1 >$
(o) $PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{C}}$	ANSWER = $\frac{-6}{26} < 3, -1, 4 >$
(p) $\|PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{A}}\|$	ANSWER = $\frac{3}{\sqrt{26}}$
(q) $\|PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{B}}\|$	ANSWER = $\frac{3}{\sqrt{6}}$
(r) $\|PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{A}}\|$	ANSWER = $\frac{5}{\sqrt{5}}$
(s) $\|PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{B}}\|$	ANSWER = $\frac{6}{\sqrt{5}}$
(t) $\|PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{C}}\|$	ANSWER = $\frac{5}{\sqrt{6}}$
(u) $\|PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{C}}\|$	ANSWER = $\frac{6}{\sqrt{26}}$
(v) $\tilde{\mathbf{A}} \cdot (\tilde{\mathbf{B}} \times \tilde{\mathbf{C}})$	ANSWER = -7
(w) $\tilde{\mathbf{B}} \cdot (\tilde{\mathbf{A}} \times \tilde{\mathbf{C}})$	ANSWER = 7
(x) $\tilde{\mathbf{C}} \cdot (\tilde{\mathbf{A}} \times \tilde{\mathbf{B}})$	ANSWER = -7
(y) θ, the angle between $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$	ANSWER : $\cos\theta = \frac{-3}{\sqrt{6}\sqrt{26}}$ = -0.2402 θ = 103.90
(z) θ, the angle between $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$	ANSWER : $\cos\theta = \frac{5}{\sqrt{6}\sqrt{5}}$ = 0.9129 θ = 24.09

(**aa**) θ, the angle between $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ ANSWER : $\cos\theta = \frac{-6}{\sqrt{26}\sqrt{5}} = -0.5262$
$\theta = 121.75$

(**ab**) The direction cosines for $\tilde{\mathbf{A}}$ ANSWER : $\cos\alpha = \frac{1}{\sqrt{6}} = 0.4082$
$\cos\beta = \frac{2}{\sqrt{6}} = 0.8165$
$\cos\gamma = \frac{-1}{\sqrt{6}} = -0.4082$

(**ac**) The direction cosines for $\tilde{\mathbf{B}}$ ANSWER : $\cos\alpha = \frac{3}{\sqrt{26}} = 0.5883$
$\cos\beta = \frac{-1}{\sqrt{26}} = -0.1961$
$\cos\gamma = \frac{4}{\sqrt{26}} = 0.7845$

(**ad**) The direction cosines for $\tilde{\mathbf{C}}$ ANSWER : $\cos\alpha = \frac{0}{\sqrt{5}} = 0.0000$
$\cos\beta = \frac{2}{\sqrt{5}} = 0.8944$
$\cos\gamma = \frac{-1}{\sqrt{5}} = -0.4472$

(**ae**) The direction angles for $\tilde{\mathbf{A}}$ ANSWER : $\alpha = 65.91$
$\beta = 35.26$
$\gamma = 114.09$

(**af**) The direction angles for $\tilde{\mathbf{B}}$ ANSWER : $\alpha = 53.96$
$\beta = 101.31$
$\gamma = 38.33$

(**ag**) The direction angles for $\tilde{\mathbf{C}}$ ANSWER : $\alpha = 90.00$
$\beta = 26.57$
$\gamma = 116.57$

(**ah**) $\tilde{\mathbf{A}} \times (\tilde{\mathbf{B}} \times \tilde{\mathbf{C}})$ ANSWER : $< 15, 1, 17 >$
(**ai**) $\tilde{\mathbf{B}} \times (\tilde{\mathbf{C}} \times \tilde{\mathbf{A}})$ ANSWER : $< 6, 6, -3 >$
(**aj**) $\tilde{\mathbf{C}} \times (\tilde{\mathbf{A}} \times \tilde{\mathbf{B}})$ ANSWER : $< -21, -7, -14 >$

44. Given the vectors $\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$, and $\tilde{\mathbf{C}}$ where

$$\tilde{\mathbf{A}} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$$

$$\tilde{\mathbf{B}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$$\tilde{\mathbf{C}} = -\hat{\mathbf{i}} + 2\hat{\mathbf{k}}$$

find the following

(a) $\tilde{\mathbf{A}} \cdot \tilde{\mathbf{B}}$ ANSWER = -1

(b) $\tilde{\mathbf{A}} \cdot \tilde{\mathbf{C}}$ ANSWER = -1

(c) $\tilde{\mathbf{B}} \cdot \tilde{\mathbf{C}}$ ANSWER = -4

(d) $\tilde{\mathbf{A}} \times \tilde{\mathbf{B}}$ ANSWER = < 1, 1, 5>

(e) $\tilde{\mathbf{A}} \times \tilde{\mathbf{C}}$ ANSWER = < -2, -2, -1>

(f) $\tilde{\mathbf{B}} \times \tilde{\mathbf{C}}$ ANSWER = < 6, -3, 3>

(g) $\|\tilde{\mathbf{A}}\|$ ANSWER = $\sqrt{2}$

(h) $\|\tilde{\mathbf{B}}\|$ ANSWER = $\sqrt{14}$

(i) $\|\tilde{\mathbf{C}}\|$ ANSWER = $\sqrt{5}$

(j) $PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{B}}$ ANSWER = $\frac{-1}{2} < 1, -1, 0 >$

(k) $PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{A}}$ ANSWER = $\frac{-1}{14} < 2, 3, -1 >$

(l) $PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{A}}$ ANSWER = $\frac{-1}{5} < -1, 0, 2 >$

(m) $PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{B}}$ ANSWER = $\frac{-4}{5} < -1, 0, 2 >$

(n) $PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{C}}$ ANSWER = $\frac{-1}{2} < 1, -1, 0 >$

(o) $PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{C}}$ ANSWER = $\frac{-4}{14} < 2, 3, -1 >$

(p) $\|PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{A}}\|$ ANSWER = $\frac{1}{\sqrt{14}}$

(q) $\|PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{B}}\|$ ANSWER = $\frac{1}{\sqrt{2}}$

(r) $\|PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{A}}\|$ ANSWER = $\frac{1}{\sqrt{5}}$

(s) $\|PROJ_{\tilde{\mathbf{C}}}\tilde{\mathbf{B}}\|$ ANSWER = $\frac{4}{\sqrt{5}}$

(t) $\|PROJ_{\tilde{\mathbf{A}}}\tilde{\mathbf{C}}\|$ ANSWER = $\frac{1}{\sqrt{2}}$

(u) $\|PROJ_{\tilde{\mathbf{B}}}\tilde{\mathbf{C}}\|$ ANSWER = $\frac{4}{\sqrt{14}}$

(v) $\tilde{\mathbf{A}} \cdot (\tilde{\mathbf{B}} \times \tilde{\mathbf{C}})$ ANSWER = 9

(w) $\tilde{\mathbf{B}} \cdot (\tilde{\mathbf{A}} \times \tilde{\mathbf{C}})$ ANSWER = -9

(x) $\tilde{\mathbf{C}} \cdot (\tilde{\mathbf{A}} \times \tilde{\mathbf{B}})$ ANSWER = 9

(y) θ, the angle between $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ ANSWER : $\cos\theta = \frac{-1}{\sqrt{2}\sqrt{14}}$ = -0.1890
θ = 100.89

(z) θ, the angle between $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{C}}$ ANSWER : $\cos\theta = \frac{-1}{\sqrt{2}\sqrt{5}}$ = -0.3162
θ = 108.43

(**aa**) θ, the angle between $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$ ANSWER : $\cos\theta = \frac{-4}{\sqrt{14}\sqrt{5}}$ = -0.4781
$\theta = 118.56$

(**ab**) The direction cosines for $\tilde{\mathbf{A}}$ ANSWER : $\cos\alpha = \frac{1}{\sqrt{2}}$ = 0.7071
$\cos\beta = \frac{-1}{\sqrt{2}}$ = -0.7071
$\cos\gamma = \frac{0}{\sqrt{2}}$ = 0.0000

(**ac**) The direction cosines for $\tilde{\mathbf{B}}$ ANSWER : $\cos\alpha = \frac{2}{\sqrt{14}}$ = 0.5345
$\cos\beta = \frac{3}{\sqrt{14}}$ = 0.8018
$\cos\gamma = \frac{-1}{\sqrt{14}}$ = -0.2673

(**ad**) The direction cosines for $\tilde{\mathbf{C}}$ ANSWER : $\cos\alpha = \frac{-1}{\sqrt{5}}$ = -0.4472
$\cos\beta = \frac{0}{\sqrt{5}}$ = 0.0000
$\cos\gamma = \frac{2}{\sqrt{5}}$ = 0.8944

(**ae**) The direction angles for $\tilde{\mathbf{A}}$ ANSWER : $\alpha = 45.00$
$\beta = 135.00$
$\gamma = 90.00$

(**af**) The direction angles for $\tilde{\mathbf{B}}$ ANSWER : $\alpha = 57.69$
$\beta = 36.70$
$\gamma = 105.50$

(**ag**) The direction angles for $\tilde{\mathbf{C}}$ ANSWER : $\alpha = 116.57$
$\beta = 90.00$
$\gamma = 26.57$

(**ah**) $\tilde{\mathbf{A}} \times (\tilde{\mathbf{B}} \times \tilde{\mathbf{C}})$ ANSWER : $< -3, -3, 3 >$
(**ai**) $\tilde{\mathbf{B}} \times (\tilde{\mathbf{C}} \times \tilde{\mathbf{A}})$ ANSWER : $< 5, -4, -2 >$
(**aj**) $\tilde{\mathbf{C}} \times (\tilde{\mathbf{A}} \times \tilde{\mathbf{B}})$ ANSWER : $< -2, 7, -1 >$

7 Lines and Planes

In this section, I give some additional examples and exercises relating to lines and planes.

7.1 Equation of a plane

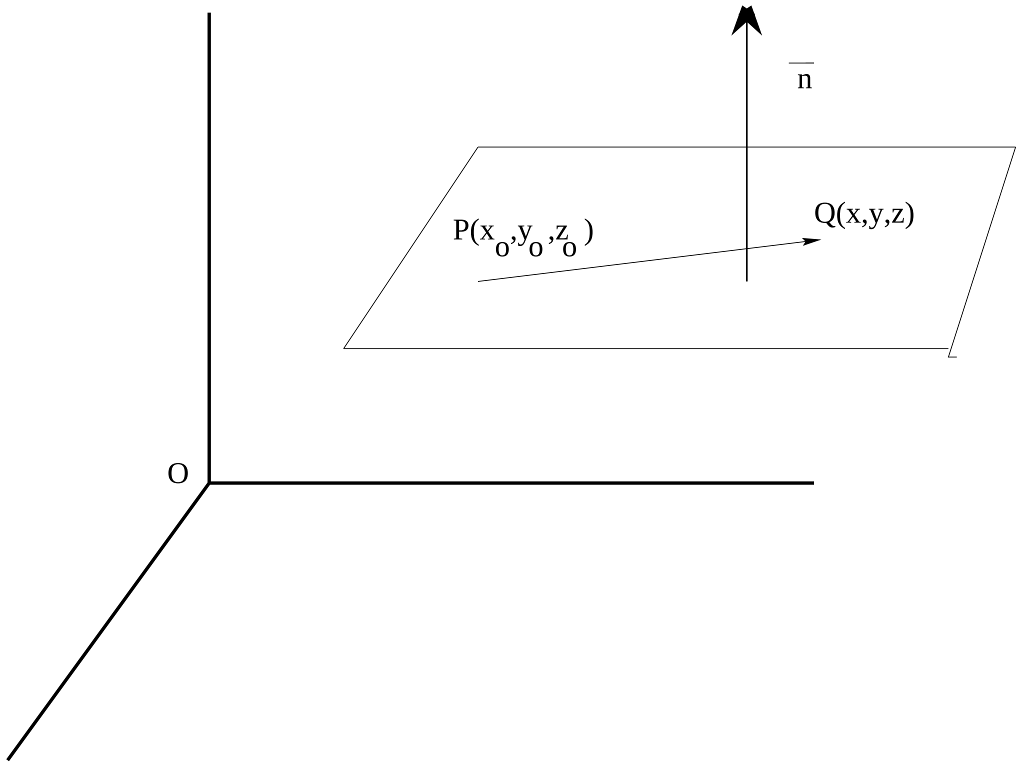

Figure 4: Equation of a plane passing through a point $P(x_0, y_0, z_0)$ with normal vector $\vec{n}$

Let $P(x_0, y_0, z_0)$ be a definite point in the plane and let $Q(x, y, z)$ be a general point in the plane. Let $\vec{n} =< a, b, c >$ be a vector normal to the plane. Since P and Q are in the plane, the vector$\overrightarrow{\mathbf{PQ}}$ is in the plane. Consequently a vector equation for the plane may be written as

$$\boxed{\text{Vector Equation of a Plane is} \quad \vec{n} \cdot \overrightarrow{\mathbf{PQ}} = 0}$$

If $\vec{X_0} =< x_0, y_0, z_0 >$ and $\vec{X} =< x, y, z >$ then $\vec{X} - \vec{X_0}$ is a vector in the plane and so we have

$$\boxed{\text{Vector Equation of a Plane is} \quad \vec{n} \cdot (\vec{X} - \vec{X_0}) = 0}$$

where $\vec{n}$ is a vector normal to the plane. Let $\vec{n} = a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}$. Then the vector equation of the plane can be written as

$$(a\,\hat{\imath} + b\,\hat{\jmath} + c\,\hat{k}) \cdot (\{x - x_0\}\,\hat{\imath} + \{y - y_0\}\,\hat{\jmath} + \{z - z_0\}\,\hat{k}) = 0$$

. Simplifying, we have

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

This is often called the Point-Normal form of the equation of a plane.Rewriting this equation, we have

$$\boxed{ax + by + cz - (ax_0 + by_0 + cz_0) = 0}$$

Since the normal vector is specified and the point is also given, the quantity in parentheses can be replaced by a single symbol, usually 'd' so that the equation becomes

$$\boxed{ax + by + cz + d = 0 \text{ where } d = -(ax_0 + by_0 + cz_0)}$$

This equation is often referred to as the standard or general equation of a plane. Note that it may also be written in the form

$$\boxed{ax + by + cz = d \text{ where } d = (ax_0 + by_0 + cz_0)}$$

$3x - 2y + 4z - 12 = 0$ would represent the equation of a plane. Immediately, you know that $\vec{n} = 3\,\hat{\imath} - 2\,\hat{\jmath} + 4\,\hat{k}$ is a vector normal to the plane. In many of the exercises that follow, we are required to find a point on the plane as part of a solution to a an exercise. The easiest way to do this is to set two of the variables to zero and hence find the value of the third variable. In the above equation, the point (0,0,3) is on the plane.

7.2 Equation of a plane through 3 points

45. Exercises

    (a) Find the equation of the plane which passes through the points P(2, 1, 1) , Q(0, 4, 1) and R(-2, 1, 4)
    Give your answer in the form ax + by + cz + d = 0
    Answer: $a = 9; b = 6; c = 12; d = -36$

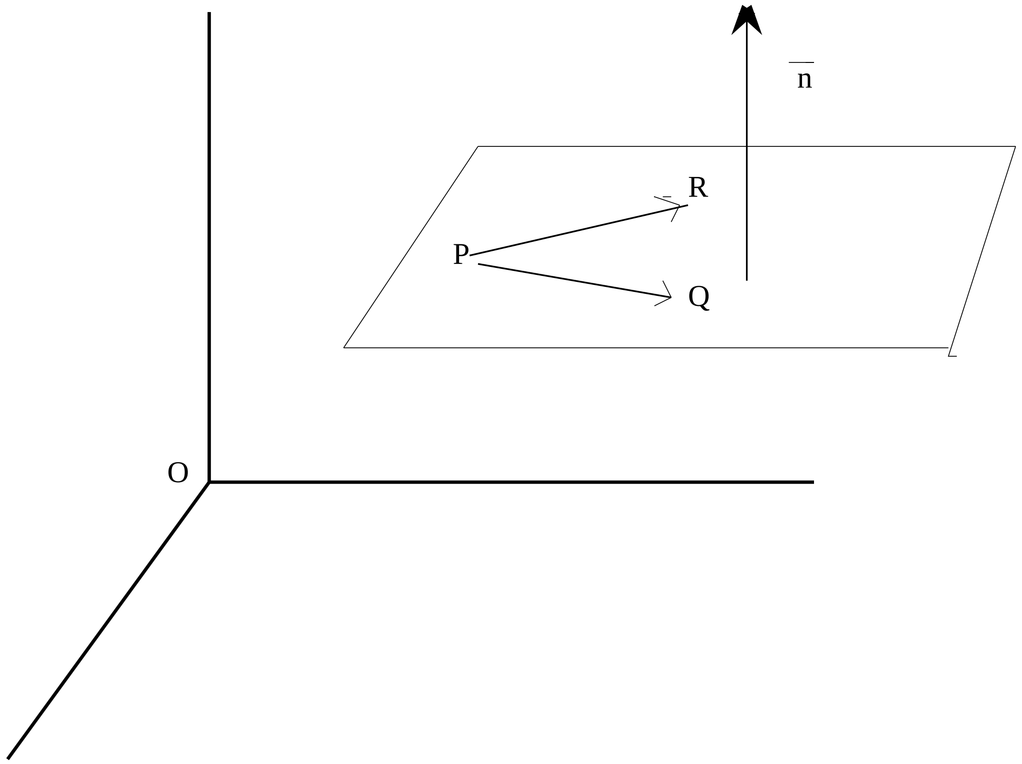

Figure 5: Equation of a plane passing through three points P, Q and R

$\overrightarrow{\mathbf{PQ}} =< -2, 3, 0 >$;$\overrightarrow{\mathbf{PR}} =< -4, 0, 3 >$;$\overrightarrow{\mathbf{QR}} =< -2, -3, 3 >$
Solution Since we are given 3 points on the plane ,the vectors $\overrightarrow{\mathbf{PQ}}$ and $\overrightarrow{\mathbf{PR}}$ will lie in the plane. Hence, the cross product $\overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}}$ will give a vector $\vec{n}$ normal to the plane.
$\overrightarrow{\mathbf{OP}} =< 2, 1, 1 >$; $\overrightarrow{\mathbf{OQ}} =< 0, 4, 1 >$; $\overrightarrow{\mathbf{OR}} =< -2, 1, 4 >$
$\overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{PQ}}$ so that $\overrightarrow{\mathbf{PQ}} = \overrightarrow{\mathbf{OQ}} - \overrightarrow{\mathbf{OP}} =< -2, 3, 0 >$
Similarly ,$\overrightarrow{\mathbf{PR}} = \overrightarrow{\mathbf{OR}} - \overrightarrow{\mathbf{OP}} =< -4, 0, 3 >$ Hence,

$$\vec{n} = \overrightarrow{\mathbf{PQ}} \times \overrightarrow{\mathbf{PR}} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix}$$

Hence $\vec{n} =< 9, 6, 12 >$. We can now use the point normal form for the equation of a plane. Select any point, lets say P so that we can write the equation of the plane as

$$9(x-2) + 6(y-1) + 12(z-1) = 0$$

After simplification, this becomes $3x + 2y + 4z - 12 = 0$

(b) Find the equation of the plane which passes through the points P(1, 0, -3) , Q(2, -5, -6) and R(6, 3, -4)
Give your answer in the form ax + by + cz + d = 0
<u>Answer</u>:$a = 14; b = -14; c = 28; d = 70$
$\overrightarrow{\mathbf{PQ}} =< 1, -5, -3 >$;$\overrightarrow{\mathbf{PR}} =< 5, 3, -1 >$;$\overrightarrow{\mathbf{QR}} =< 4, 8, 2 >$

(c) Find the equation of the plane which passes through the points P(0, 3, -1) , Q(2, 4, 2) and R(-1, 2, -3)
Give your answer in the form ax + by + cz + d = 0
<u>Answer</u>: $a = 1; b = 1; c = -1; d = -4$
$\overrightarrow{\mathbf{PQ}} =< 2, 1, 3 >$;$\overrightarrow{\mathbf{PR}} =< -1, -1, -2 >$;$\overrightarrow{\mathbf{QR}} =< -3, -2, -5 >$

(d) Find the equation of the plane which passes through the points P(1, 0, -1) , Q(2, 3, 1) and R(4, -3, 2)
Give your answer in the form ax + by + cz + d = 0
<u>Answer</u>: $a = 15; b = 3; c = -12; d = -27$
$\overrightarrow{\mathbf{PQ}} =< 1, 3, 2 >$;$\overrightarrow{\mathbf{PR}} =< 3, -3, 3 >$;$\overrightarrow{\mathbf{QR}} =< 2, -6, 1 >$

(e) Find the equation of the plane which passes through the points P(0, 0, 0) , Q(1, 1, 1) and R(1, 2, 3)
Give your answer in the form ax + by + cz + d = 0
<u>Answer</u>:$a = 1; b = -2; c = 1; d = 0$
$\overrightarrow{\mathbf{PQ}} =< 1, 1, 1 >$;$\overrightarrow{\mathbf{PR}} =< 1, 2, 3 >$;$\overrightarrow{\mathbf{QR}} =< 0, 1, 2 >$

(f) Find the equation of the plane which passes through the points P(0, 3, -1) , Q(2, 4, 2) and R(-2, 2, -4)
Give your answer in the form ax + by + cz + d = 0
<u>Answer</u>; NO UNIQUE PLANE; POINTS LIE ON A LINE

7.3 Equation of a line

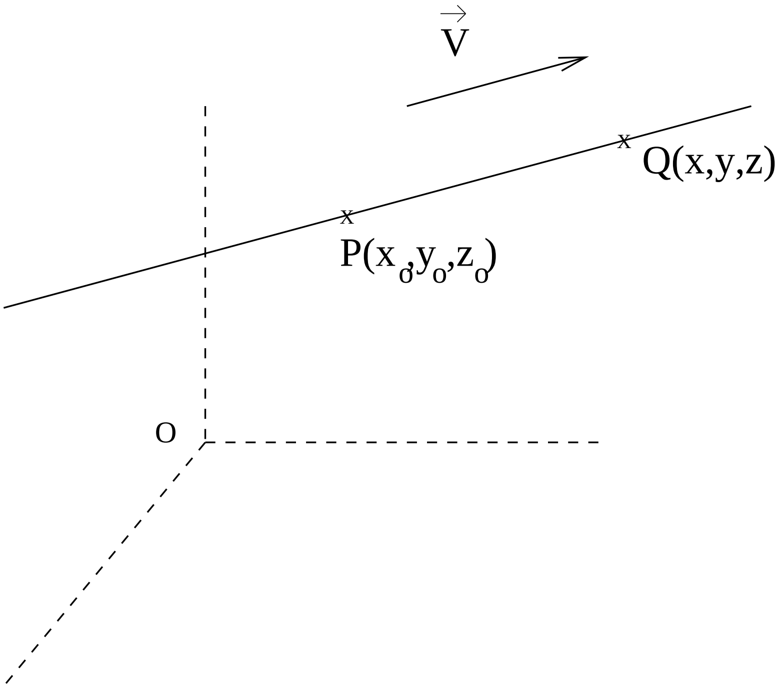

Figure 6: Equation of a line passing through a point $P(x_0, y_0, z_0)$, parallel to the vector $\vec{V}$. $Q(x, y, z)$ is a general point on the line.

46. Equation of a line passing through a point $P(x_0, y_0, z_0)$, and parallel to $\vec{V}$
Let $Q(x, y, z)$ be a general point on the line. Then $\overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{PQ}}$ or $\overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + t\vec{V}$
This is a vector equation for the line. If $\vec{V} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$, then this equation could be written

$$< x, y, z > = < x_0, y_0, z_0 > + t < a, b, c >$$

The parametric form can be obtained by equating the individual components, namely

$$x = x_0 + at; \quad y = y_0 + bt; \quad z = z_o + ct$$

Eliminating the parameter "t", we obtain the scalar symmetric form for the equation of a straight line.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

47. Example Find the equation of the straight line through the point $P(1, -3, -2)$ and parallel to the vector$\vec{V} =< 4, -5, 3 >$ Give your answer in parametric form.

Solution Let $Q(x, y, z)$ be a general point on the line. $\overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + t\vec{V}$. For this exercise , we have

$$< x, y, z >=< 1, -3, -2 > +t < 4, -5, 3 >$$

Writing the answer in parametric form, we have

$$x = 1 + 4t \ ; \ y = -3 - 5t \ ; \ z = -2 + 3t$$

48. Exercises

    (a) Find the equation of the straight line through the point $P(5, 2, -3)$ and parallel to the vector$\vec{V} =< 2, -4, 1 >$ Give your answer in parametric form.

    (b) Find the equation of the straight line through the point $P(7, -3, -5)$ and parallel to the vector$\vec{V} =< 1, -3, -2 >$ Give your answer in symmetric scalar form.

    (c) Find the equation of the straight line through the point $P(-1, 7, 0)$ and parallel to the vector$\vec{V} =< 4, -5, -2 >$ Give your answer in parametric form.

    (d) Find the equation of the straight line through the point $P(6, -9, 7)$ and parallel to the vector$\vec{V} =< -3, 2, 2 >$ Give your answer in symmetric scalar form.

49. Equation of a line passing through the points $P(x_0, y_0, z_0)$, and $R(x_1, y_1, z_1)$

    Let $Q(x, y, z)$ be a general point on the line. Then $\overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{PQ}}$ or $\overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + t\overrightarrow{\mathbf{PR}}$ This is a vector equation for the line. Since $\overrightarrow{\mathbf{OR}} = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{PR}}$, we can write $\overrightarrow{\mathbf{PR}} = \overrightarrow{\mathbf{OR}} - \overrightarrow{\mathbf{OP}}$, or $\overrightarrow{\mathbf{PR}} =< (x_1 - x_0), (y_1 - y_0), (z_1 - z_0) >$ Hence, we can write the vector equation in the following form.

$$< x, y, z >=< x_0, y_0, z_0 > +t < (x_1 - x_0), (y_1 - y_0), (z_1 - z_0) >$$

The parametric form can be obtained by equating the individual components, namely

$$x = x_0 + t(x_1 - x_0); \quad y = y_0 + t(y_1 - y_0); \quad z = z_o + t(z_1 - z_0)$$

Eliminating the parameter "t", we obtain the scalar symmetric form for the equation of a straight line.

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

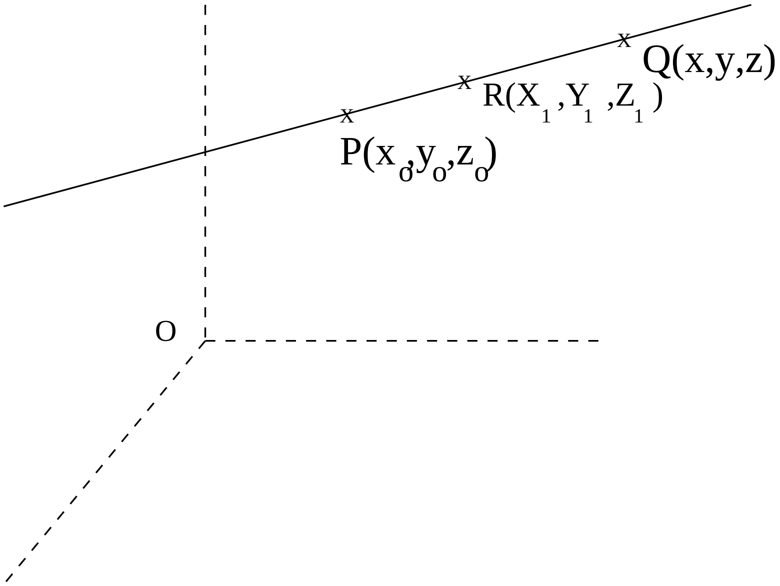

Figure 7: Equation of a line passing through the points $P(x_0, y_0, z_0)$, and $R(x_1, y_1, z_1)$. $Q(x, y, z)$ is a general point on the line.

50. Exercises

(a) Find an equation of the line which passes through the points $P(3, 4, 5)$ and $R(2, 1, 6)$ Give your answer in symmetric scalar form.

Answer : $\dfrac{x-3}{-1} = \dfrac{y-4}{-3} = \dfrac{z-5}{1}$

$$\frac{x-2}{-1} = \frac{y-1}{-3} = \frac{z-6}{1}$$

(b) Find an equation of the line which passes through the points $P(2, 5, 3)$ and $R(4, 6, 5)$ Give your answer in symmetric scalar form.

Answer : $\dfrac{x-2}{2} = \dfrac{y-5}{1} = \dfrac{z-3}{2}$

$$\frac{x-4}{2} = \frac{y-6}{1} = \frac{z-5}{2}$$

(c) Find an equation of the line which passes through the points $P(-4, 3, 6)$ and $R(5, 9, 2)$ Give your answer in symmetric scalar form.

Answer : $\dfrac{x+4}{9} = \dfrac{y-3}{6} = \dfrac{z-6}{-4}$

$$\frac{x-5}{9} = \frac{y-9}{6} = \frac{z-2}{-4}$$

(d) Find an equation of the line which passes through the points $P(0,3,7)$ and $R(4,2,11)$ Give your answer in symmetric scalar form.

Answer : $\frac{x}{4} = \frac{y-3}{-1} = \frac{z-7}{4}$

$$\frac{x-4}{4} = \frac{y-2}{-1} = \frac{z-11}{4}$$

(e) Find an equation of the line which passes through the points $P(4,0,3)$ and $R(2,9,1)$ Give your answer in symmetric scalar form.

Answer : $\frac{x-4}{-2} = \frac{y}{9} = \frac{z-3}{-2}$

$$\frac{x-2}{-2} = \frac{y-9}{9} = \frac{z-1}{-2}$$

(f) Find an equation of the line which passes through the points $P(3,7,0)$ and $R(5,2,12)$ Give your answer in symmetric scalar form.

Answer : $\frac{x-3}{2} = \frac{y-7}{-5} = \frac{z}{12}$

$$\frac{x-5}{2} = \frac{y-2}{-5} = \frac{z-12}{12}$$

(g) Find an equation of the line which passes through the points $P(0,-3,8)$ and $R(1,-4,13)$ Give your answer in symmetric scalar form.

Answer : $\frac{x}{1} = \frac{y+3}{-1} = \frac{z-8}{5}$

$$\frac{x-1}{1} = \frac{y+4}{-1} = \frac{z-13}{5}$$

(h) Find an equation of the line which passes through the points $P(2,4,8)$ and $R(-6,-5,-9)$ Give your answer in symmetric scalar form.

Answer : $\frac{x-2}{-8} = \frac{y-4}{-9} = \frac{z-8}{-17}$

$$\frac{x+6}{-8} = \frac{y+5}{-9} = \frac{z+9}{-17}$$

(i) Find an equation of the line which passes through the points $P(3,5,1)$ and $R(2,8,1)$ Give your answer in symmetric scalar form.

Answer : $\frac{x-3}{-1} = \frac{y-5}{3} \ : \ z = 1$

(j) Find an equation of the line which passes through the points $P(8,3,7)$ and $R(4,3,2)$ Give your answer in symmetric scalar form.
Answer : $\dfrac{x-8}{-4} = \dfrac{z-7}{-5}$: $y = 3$

7.4 Distance of a point from a plane

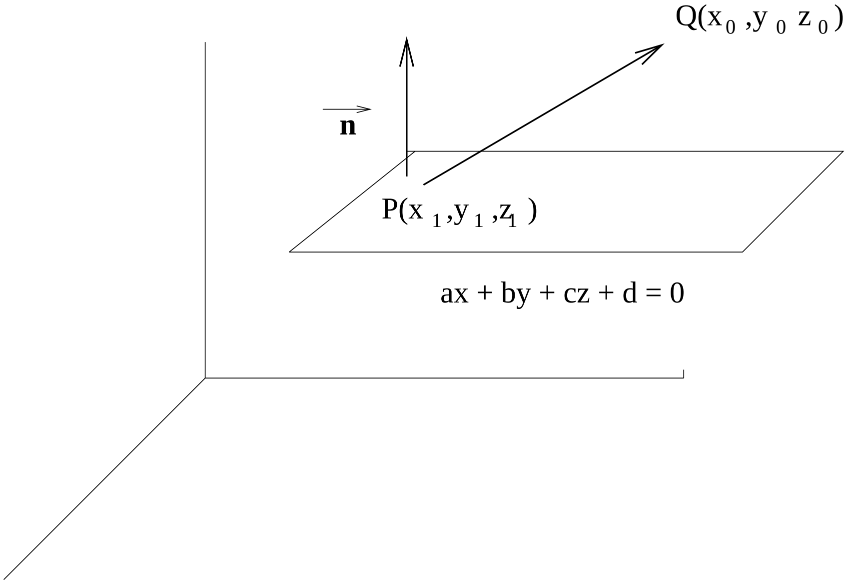

Figure 8: Distance of a point $Q(x_0, y_0, z_0)$,from a plane $ax + by + cz + d = 0$. $P(x_1, y_1, z_1)$ is a point on the plane.

The required distance is given by $\|PROJ_{\tilde{\mathbf{n}}}\overrightarrow{\mathbf{PQ}}\|$ where $\tilde{\mathbf{n}}$ is a vector normal to the plane.
Recall from our work on vectors that $\|PROJ_{\vec{B}}\vec{A}\| = \dfrac{|\vec{A}\cdot\vec{B}|}{\|\vec{B}\|}$.
Hence,$\|PROJ_{\tilde{\mathbf{n}}}\overrightarrow{\mathbf{PQ}}\| = \dfrac{|\overrightarrow{\mathbf{PQ}}\cdot\tilde{\mathbf{n}}|}{\|\tilde{\mathbf{n}}\|}$
Since we are given the equation of the plane, the normal vector $\tilde{\mathbf{n}}$ can be determined easily. $\tilde{\mathbf{n}} =< \mathbf{a}, \mathbf{b}, \mathbf{c} >$ and hence $\|\tilde{\mathbf{n}}\| = \sqrt{\mathbf{a^2} + \mathbf{b^2} + \mathbf{c^2}}$ Also since we are given the equation of the plane, we can always find a point on the plane. Let that point be $P(x_1, y_1, z_1)$. We can

also write $\overrightarrow{\mathbf{OQ}} = \overrightarrow{\mathbf{OP}} + \overrightarrow{\mathbf{PQ}}$ where $\overrightarrow{\mathbf{OQ}} =< \mathbf{x_0}, \mathbf{y_0}, \mathbf{z_0} >$ and $\overrightarrow{\mathbf{OP}} =< x_1, y_1, z_1 >$. Hence

$$\begin{aligned}\overrightarrow{\mathbf{PQ}} &= \overrightarrow{\mathbf{OQ}} - \overrightarrow{\mathbf{OP}} \\ \overrightarrow{\mathbf{PQ}} &= < x_0 - x_1, y_0 - y_1, z_0 - z_1 >\end{aligned}$$

$$\begin{aligned}|\overrightarrow{\mathbf{PQ}} \cdot \tilde{\mathbf{n}}| &= |a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)| \\ &= |ax_0 + by_0 + cz_0 - ax_1 - by_1 - cz_1| \\ &= |ax_0 + by_0 + cz_0 + d|\end{aligned}$$

Since $x_1, y_1, z_1)$ is a point on the plane, we may write $ax_1 + by_1 + cz_1 + d = 0$ and hence $d = -ax_1 - by_1 - cz_1$
Hence the distance of a point $Q(x_0, y_0, z_0)$,from a plane $ax + by + cz + d = 0$. is given by the formula

$$\boxed{\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Note , to use this formula, the equation of the plane must be written in the form
$ax + by + cz + d = 0$

51. Distance of a point from a plane

(a) Exercises Find the distance between the point Q(1, 5, -4)
and the plane $3x - y + 2z - 6 = 0$
Answer: distance $= \frac{16}{\sqrt{14}}$
Solution

$$\text{Distance} = \frac{|(3)(1) + (-1)(5) + (2)(-4) + (-6)|}{\sqrt{9 + 1 + 4}} = \frac{|3 - 5 - 8 - 6|}{\sqrt{14}} = \frac{16}{\sqrt{14}}$$

(b) Find the distance between the point Q(0, 0, 0)
and the plane $2x + 3y + z - 12 = 0$
Answer: distance $= \frac{12}{\sqrt{14}}$

(c) Find the distance between the point Q(1, 2, 3)
and the plane $2x - y + z - 4 = 0$
Answer: distance $= \frac{1}{\sqrt{6}}$

(d) Find the distance between the point Q(1, 1, 3)
and the plane $2x - y + z - 4 = 0$
Answer = 0 ; Point is on the plane

(e) Find the distance between the point Q(1, 0, 2)
and the plane $2x - 3y + 6z - 6 = 0$
Answer: distance = $\frac{8}{7}$

(f) Find the distance between the point Q(0, 0, 0)
and the plane $4x - 7y + z - 2 = 0$
Answer: distance = $\frac{2}{\sqrt{66}}$

(g) Find the distance between the point Q(-1, 1, 2)
and the plane $3x - 2y + z - 1 = 0$
Answer: distance = $\frac{4}{\sqrt{14}}$

(h) Find the distance between the point Q(3, -1, 4)
and the plane $2x - y + z - 5 = 0$
Answer: distance = $\frac{6}{\sqrt{6}}$

(i) Find the distance between the point Q(2, 0, -4)
and the plane $x + 2y + 4z - 3 = 0$
Answer: distance = $\frac{17}{\sqrt{21}}$

7.5 Distance between two planes

This question is only sensible if the planes are parallel. If not, they will intersect and give zero for the distance. It is possible to develop a formula for the distance between two parallel planes. Given two parallel planes

$$\begin{aligned} ax + by + cz + d_1 &= 0 \\ ax + by + cz + d_2 &= 0 \end{aligned}$$

the distance between these two planes can be shown to be

$$\frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$$

However rather than memorize yet another formula, I suggest that you use the previous formula for the distance between a point and a plane.Find a point on one plane and then find the distance between that point and the other plane.

52. Exercises

(a) Find the distance between the parallel planes

$$\begin{aligned} x - 3y + 4z - 10 = 0 \qquad &\text{Plane 1} \\ x - 3y + 4z - 6 = 0 \qquad &\text{Plane 2} \end{aligned}$$

Answer : distance $= \frac{4}{\sqrt{26}}$

Solution For plane 1, set $y = 0$ and $z = 0$ so that $x = 10$. Hence $(10, 0, 0)$ is a point on plane 1. Using this as the point in the formula for the distance of a point from a plane and plane 2 as the plane, we have

$$\text{Distance } = \frac{|10 + 0 + 0 - 6|}{\sqrt{1 + 9 + 16}} = \frac{4}{\sqrt{26}}$$

(b) Find the distance between the parallel planes

$3x - y + 2z - 6 = 0$ and $6x - 2y + 4z + 4 = 0$

Answer : distance $= \frac{48}{\sqrt{504}}$

(c) Find the distance between the parallel planes

$2x - 4z - 4 = 0$ and $2x - 4z + 10 = 0$

Answer : distance $= \frac{28}{\sqrt{80}}$

(d) Find the distance between the parallel planes

$x - y + z - 2 = 0$ and $3x - 3y + 3z - 1 = 0$

Answer : distance $= \frac{5}{\sqrt{27}}$

(e) Find the distance between the parallel planes

$y - 2z - 4 = 0$ and $2y + 4z - 6 = 0$

Answer : distance $= \frac{14}{\sqrt{20}}$

(f) Find the distance between the parallel planes

$2x - 3y + 4z - 5 = 0$ and $4x - 6y + 8z + 1 = 0$

Answer : distance $= \frac{22}{\sqrt{464}}$

(g) Show that the distance between two parallel planes

$$\begin{aligned} ax + by + cz + d_1 &= 0 \\ ax + by + cz + d_2 &= 0 \end{aligned}$$

is given by the formula

$$\frac{|d_2 - d_1|}{\sqrt{a^2+b^2+c^2}}$$

7.6 Distance of a point from a line

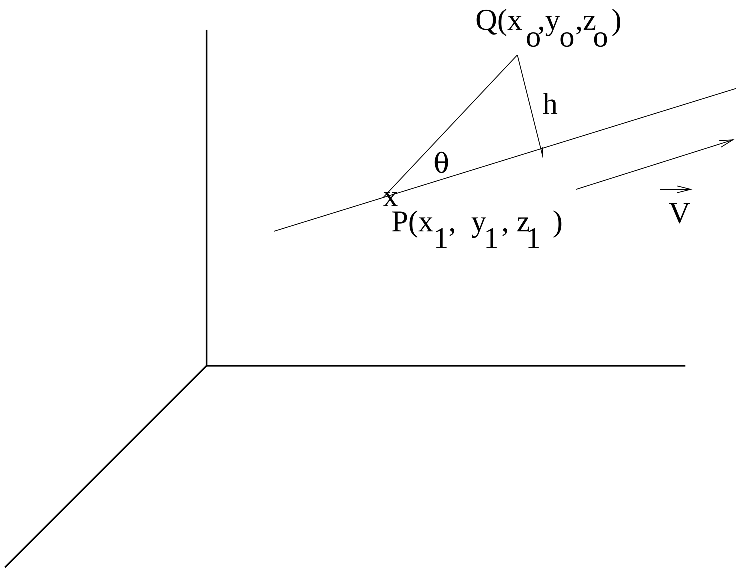

Figure 9: Distance(h) of a point $Q(x_0, y_0, z_0)$,from a line with direction vector $\vec{V}$. $P(x_1, y_1, z_1)$ is a point on the line.

$$\begin{aligned}
\sin\theta &= \frac{h}{\|\overrightarrow{\mathbf{PQ}}\|} \\
h &= \|\overrightarrow{\mathbf{PQ}}\| \sin\theta \\
h\,\|\vec{V}\| &= \|\overrightarrow{\mathbf{PQ}}\|\;\|\vec{V}\| \sin\theta \\
h\,\|\vec{V}\| &= \|\overrightarrow{\mathbf{PQ}} \times \vec{V}\| \\
\text{Distance} &= \frac{\|\overrightarrow{\mathbf{PQ}} \times \vec{V}\|}{\|\vec{V}\|}
\end{aligned}$$

53. Example. Find the distance of the point $Q(1,2,3)$ from the line

$$\frac{x-1}{1}=\frac{y}{-2}=\frac{z-2}{3}$$

Rewriting the equation of the line in parametric form, we have $x=1+t$; $y=-2t$; $z=2+3t$ The direction vector for the line is given by $V=<1,-2,3>$. Setting $t=0$,we find that $P(1,0,2)$ is a point on the line. $\overrightarrow{\mathbf{OP}}+\overrightarrow{\mathbf{PQ}}=\overrightarrow{\mathbf{OQ}}$ or $\overrightarrow{\mathbf{PQ}}=\overrightarrow{\mathbf{OQ}}-\overrightarrow{\mathbf{OP}}$ where $\overrightarrow{\mathbf{OQ}}=<1,2,3>$ and $\overrightarrow{\mathbf{OP}}=<1.0,2>$ Hence $\overrightarrow{\mathbf{PQ}}=<0,2,1>$

$$\overrightarrow{\mathbf{PQ}}\times\vec{V}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\0&2&1\\1&-2&3\end{vmatrix}$$

$$\overrightarrow{\mathbf{PQ}}\times\vec{V}=8\hat{\imath}+\hat{\jmath}-2\hat{k}$$

$$\text{Distance}=\frac{\|\overrightarrow{\mathbf{PQ}}\times\vec{V}\|}{\|\vec{V}\|}=\frac{\sqrt{69}}{\sqrt{14}}$$

54. Exercises

(a) Find the distance between the point $Q(3,-8,1)$ and the line $\dfrac{x-3}{3}=\dfrac{y+7}{-1}=\dfrac{z+2}{5}$

Answer : $\dfrac{\sqrt{94}}{\sqrt{35}}$

(b) Find the distance between the point $Q(2,3,4)$ and the line $\dfrac{x-5}{7}=\dfrac{y+3}{2}=\dfrac{z-3}{-3}$

Answer : $\dfrac{\sqrt{2708}}{\sqrt{62}}$

(c) Find the distance between the point $Q(1,0,-1)$ and the line $\dfrac{x-2}{3}=\dfrac{y+1}{1}=\dfrac{z-1}{2}$

Answer : $\dfrac{\sqrt{48}}{\sqrt{14}}$

(d) Find the distance between the point $Q(1,-2,2)$ and the line $\dfrac{x}{2}=\dfrac{y}{-1}=\dfrac{z}{2}$

Answer : $\dfrac{\sqrt{17}}{\sqrt{9}}$

(e) Find the distance between the point $Q(1,-2,2)$ and the line $\dfrac{x}{2}=\dfrac{y}{1}=\dfrac{z}{2}$

Answer : $\dfrac{\sqrt{65}}{\sqrt{9}}$

7.7 Volume of a parallelepiped

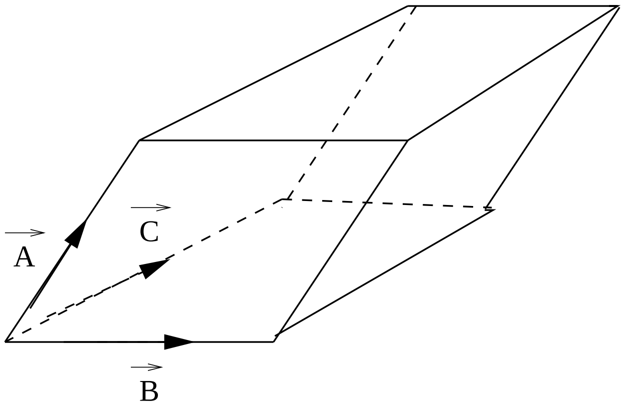

Figure 10: Parallelepiped having the vectors $\vec{A}, \vec{B}$ and $\vec{C}$ as adjacent edges.

55. The volume of a parallelepiped,V, is obtained by the formula

$$V = (\text{Base area})\ (\text{ Height}).$$

The base is a parallelogram formed by the vectors $\vec{B}$ and $\vec{C}$ and from our previous work has an area of $\|\vec{B} \times \vec{C}\|$. The height can be obtained by finding the magnitude of the projection of the vector $\vec{A}$ onto a vector normal to the plane of the base. $\vec{B} \times \vec{C}$ is such a vector. Recall from our previous work that

$$\|PROJ_{\vec{B}}\vec{A}\| = \frac{|\vec{A} \cdot \vec{B}|}{\|\vec{B}\|}$$

Hence, the height of the parallelepiped is given by

$$h = \|PROJ_{\vec{B} \times \vec{C}}\vec{A}\| = \frac{|\vec{A} \cdot (\vec{B} \times \vec{C})|}{\|\vec{B} \times \vec{C}\|}$$

Hence

$$V = (\|\vec{B} \times \vec{C}\|)\left(\frac{|\vec{A} \cdot (\vec{B} \times \vec{C})|}{\|\vec{B} \times \vec{C}\|}\right)$$

or

$$\boxed{V = |\vec{A} \cdot (\vec{B} \times \vec{C})|}$$

56. Example Find the volume of the parallelepiped having

$$\tilde{\mathbf{A}} = 3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + \hat{\mathbf{k}}, \quad \tilde{\mathbf{B}} = 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}} \quad \text{and} \quad \tilde{\mathbf{C}} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

as adjacent edges.
Solution The volume is given by the absolute value of the scalar triple product $\vec{A} \cdot (\vec{B} \times \vec{C})$. From our previous work , the scalar triple product can be written as a third order determinant.

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix}$$

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = 3(2+2) - (-5)(0+6) + 1(0-6) = 36$$

Hence the volume $V = |36| = 36$

57. Exercises

    (a) Find the volume of the parallelepiped having
    $\tilde{\mathbf{A}} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}, \tilde{\mathbf{B}} = 5\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ and $\tilde{\mathbf{C}} = 4\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$
    as adjacent edges.
    Answer: Volume = 60

    (b) Find the volume of the parallelepiped having
    $\tilde{\mathbf{A}} = \hat{\mathbf{i}} + \hat{\mathbf{j}}, \tilde{\mathbf{B}} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\tilde{\mathbf{C}} = \hat{\mathbf{i}} + \hat{\mathbf{k}}$
    as adjacent edges.
    Answer: Volume = 2

    (c) Find the volume of the parallelepiped having
    $\tilde{\mathbf{A}} = -\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}, \tilde{\mathbf{B}} = -3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\tilde{\mathbf{C}} = -4\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$
    as adjacent edges.
    Answer: Volume = 7

    (d) Find the volume of the parallelepiped having
    $\tilde{\mathbf{A}} = \hat{\mathbf{i}} + 6\hat{\mathbf{k}}, \tilde{\mathbf{B}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 8\hat{\mathbf{k}}$ and $\tilde{\mathbf{C}} = 8\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$
    as adjacent edges.
    Answer: Volume = 226

    (e) Find the volume of the parallelepiped having
    $\tilde{\mathbf{A}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}, \tilde{\mathbf{B}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\tilde{\mathbf{C}} = 7\hat{\mathbf{i}} + 5\hat{\mathbf{k}}$
    as adjacent edges.
    Scalar Triple Product is zero: Vectors are coplanar

7.8 Equation of a plane through two points and perpendicular to another plane

58. Equation of a plane through two points and perpendicular to another plane

(a) Find the equation of the plane which passes through the points P(3, 2, 1) , Q(3, 1, -5)) and is perpendicular to the plane $6x + 7y + 2z = 10$
Give your answer in the form ax + by + cz + d = 0
Answer: $a = 40$; b = -36 ; c = 6 ; d = -54
$\overrightarrow{\mathbf{PQ}} =< 0, -1, -6 >$

(b) Find the equation of the plane which passes through the points P(2, 2, 1) , Q(-1, 1, -1)) and is perpendicular to the plane $2x - 3y + z = 3$
Give your answer in the form ax + by + cz + d = 0
Answer: $a = 7$; b = 1 ; c = -11 ; d = -5
$\overrightarrow{\mathbf{PQ}} =< -3, -1, -2 >$

(c) Find the equation of the plane which passes through the points P(4, -2, 1) , Q(3, 1, 2)) and is perpendicular to the plane $3x + 2y - 4z = 5$
Give your answer in the form ax + by + cz + d = 0
Answer: $a = 14$; b = 1 ; c = 11 ; d = -65
$\overrightarrow{\mathbf{PQ}} =< -1, 3, 1 >$

(d) Find the equation of the plane which passes through the points P(-2, 1, 4) , Q(1, 0, 3)) and is perpendicular to the plane $4x - y + 3z = 2$
Give your answer in the form ax + by + cz + d = 0
Answer: $a = 4$; b = 13 ; c = -1 ; d = -1
$\overrightarrow{\mathbf{PQ}} =< 3, -1, -1 >$

(e) Find the equation of the plane which passes through the points P(-3, 5, 2) , Q(4, 1, 2)) and is perpendicular to the plane $3x + 2y - 2z = 5$
Give your answer in the form ax + by + cz + d = 0
Answer: $a = 8$; b = 14 ; c = 26 ; d = -98
$\overrightarrow{\mathbf{PQ}} =< 7, -4, 0 >$

7.9 Intersection of a line and a plane

59. Example Find (if any) the intersection point of the line $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z}{4}$ and the plane $3x - 2y + z + 5 = 0$
Solution Rewriting the equation of the line in parametric form, we have $x = 1 + t$: $y = -2 + 2t$; and $z = 4t$. Substituting these parametric equations into the equation of the plane, we have

$$3(1 + t) - 2(-2 + 2t) + (4t) + 5 = 0$$

Simplifying, we have $3+3t+4-4t+4t+5=0$ or $3t=-12$, giving $t=-4$
Substituting this value of t into the parametric equations, we have $x=-3, y=-10$ and $z=-16$. Hence , the intersection point is (-3,-10,-16) Note that $\vec{n}$ the normal vector to the plane is $< 3,-2,1 >$ while the direction vector $\vec{V}$ for the line is $< 1,2,4 >$. The dot product $\vec{n}\cdot\vec{V}$ is 3 or $\neq 0$ so that $\vec{n}$ and $\vec{V}$ are not orthogonal. In the following exercises , let us examine what happens if $\vec{n}$ and $\vec{V}$ are orthogonal.

60. Example Find the intersection point (if any) of the line $\dfrac{x-2}{2}=\dfrac{y+3}{-2}=\dfrac{Z-1}{1}$ and the plane $3x+y-4z+21=0$
Solution Rewriting the equation of the line in parametric form, we have $x=2+2t;\ y=-3-2t;\ \ z=1+t$. Substituting these parametric equations into the equation of the plane, we have

$$3(2+2t)+(-3-2t)-4(1+t)+21=0$$

Simplifying, we have

$$6+6t-3-2t-4-4t+21=0 \ \text{ or } 0t+20=0$$

From this obviously ridiculous equation, we conclude that there is no intersection between the line and the given plane. Note that if $\vec{V}$ is the direction vector for the line and $\vec{n}$ is the normal vector to the plane. $\vec{V}=< 2,-2,1 >$ and $\vec{n}=< 3,1,-4 >$ so that $\vec{n}\cdot\vec{V}=6-2-4=0$. That is, $\vec{n}$ and $\vec{V}$ are orthogonal. To summarize the situation, the line is in a plane parallel to the given plane. As an additional exercise, select any two points on the line and show that the distance between each point and the plane is the same, that is, independent of point selection and is equal to $\frac{20}{\sqrt{26}}$ You might use the point $(2,-3,1)$ or perhaps $(6,-7,3)$. Also, find the equation of the plane which contains the given line.

61. Example Find the intersection point (if any) of the line $\dfrac{x+7}{8}=\dfrac{y-1}{-12}=\dfrac{z-1}{-2}$ and the plane $5x+3y+2z+30=0$
Solution Rewriting the equation of the line in parametric form, we have $x=-7+8t;\ \ y=1-12t;\ \ z=1-2t$ Substituting these parametric equations into the equation of the plane, we have

$$5(-7+8t)+3(1-12t)+2(1-2t)+30=0$$

Simplifying, we have

$$-35+40t+3-36t+2-4t+30=0 \quad \text{or } 0t+0=0$$

Clearly this is a valid statement, good for all values of t. If $\vec{V}$ is the direction vector for the line and $\vec{n}$ is the normal vector for the plane $\vec{v} = < 8, -12, -2 >$; $\vec{n} = 25, 3, 2 >$ so that $\vec{n} \cdot \vec{V} = 40 - 36 - 4 = 0$. Hence $\vec{n}$ and $\vec{V}$ are othogonal. We conclude that the line is actually in the given plane $(-7, 1, 1)$ is a point on the line. Show that the distance between this point and the plane is 0

7.10 Equation of a plane determined by two non parallel lines

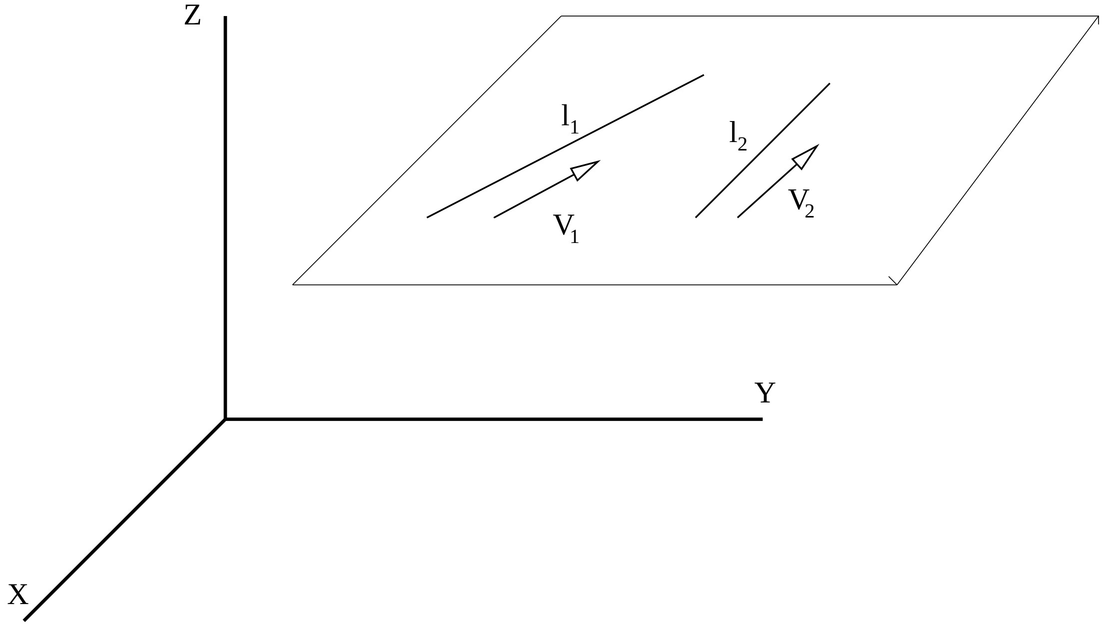

Figure 11: Equation of a plane containing two lines, l_1, l_2 with direction vectors $\vec{V_1}, \vec{V_2}$

Let $\vec{V_1}$ be the direction vector for the first line and let $\vec{V_2}$ be the direction vector for the second line. It is assumed that the lines are not parallel. If we form the cross product, $\vec{V_1} \times \vec{V_2}$,we will obtain a vector normal to the plane containing these two lines. Since we are given the equations of both lines, it is easy to obtain a point on one of the lines and so a point in the plane. Hence, we can the use the point-normal form for the equation of a plane.

62. Example Find the equation of the plane containing the lines

$$\frac{x-1}{-2} = \frac{y-4}{1} = \frac{z}{1} \text{ and } \frac{x-2}{-3} = \frac{y-1}{4} = \frac{z-2}{-1}$$

Give the answer in the form $ax + by + cz + d = 0$

Solution A direction vector for line 1 is $\vec{V_1} =< -2, 1, 1 >$ and for line 2, a direction vector is $\vec{V_2} =< -3, 4, -1 >$. We find a vector normal to the plane by forming the cross product $\vec{n} = \vec{V_1} \times \vec{V_2}$

$$\vec{n} = \vec{V_1} \times \vec{V_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix}$$

Hence, $\vec{n} = -5\hat{\imath} - 5\hat{\jmath} - 5\hat{k}$. Since all we need is the simplest possible normal vector, we will take $\vec{n} = \hat{\imath} + \hat{\jmath} + \hat{k}$. From line 1, we see that (1,4,0) is on the line and so is a point on the plane. Hence we can now use the point-normal form for the equation of a plane,namely

$$1(x-1) + 1(y-4) + 1(z-0) = 0 \text{ or } x + y + z - 5 = 0$$

7.11 Equation of a plane determined by two distinct parallel lines

63. If the plane contains two parallel but distinct lines, it is possible to find the equation of the plane.

Example Find the equation of the plane containing the two parallel lines

$$\frac{x+2}{5} = \frac{y-1}{-2} = \frac{z+4}{1} \quad l_1$$

$$\frac{x-3}{-5} = \frac{y+4}{2} = \frac{z-3}{-1} \quad l_2$$

Method Since the two lines are parallel, we cannot use the cross product of the direction vectors to obtain a vector normal to the plane as we did in the previous example. However, since we are given the line equations, we can find a point,A, on line one l_1 and a point ,B, on line two l_2. Hence, we can find the vector $\vec{AB}$. A normal vector to the plane can then be obtained by calculating the cross product of the vector $\vec{V_1}$ with $\vec{AB}$. Then we can use the point-normal form for the equation of a plane.

Solution First lets show that the lines are parallel but distinct.The direction vector for line 1,$\vec{V_1}$ is $< 5, -2, 1 >$ while the direction vector for line 2 $\vec{V_2}$ is $< -5, 2, -1 >$ .Since $\vec{V_1} = -\vec{V_2}$, the lines are parallel. The point(-2,1,-4) is on line 1 but is not on line 2 as you may verify by substitution. Hence, we have the case of two parallel but distinct lines. Taking A(-2,1,-4) on line 1 and B(3,-4,3) on line 2, we can find the vector $\vec{AB} =< 5, -5, 7 >$.

$$\tilde{\mathbf{n}} = \overrightarrow{\mathbf{AB}} \times \vec{V_1} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & -5 & 7 \\ 5 & -2 & 1 \end{vmatrix}$$

$$\tilde{\mathbf{n}} = \overrightarrow{\mathbf{AB}} \times \vec{V_1} = 9\hat{\imath} + 30\hat{\jmath} + 15\hat{k}$$

Using the point $B(3, -4, 3)$ and the calculated normal vector $\vec{n} = 9\hat{\imath} + 30\hat{\jmath} + 15\hat{k}$ in the point normal form for the equation of a plane, we have

$$9(x-3) + 30(y + 4 + 15(z-3) = 0 \text{ or } 3x + 10y + 5z + 16 = 0$$

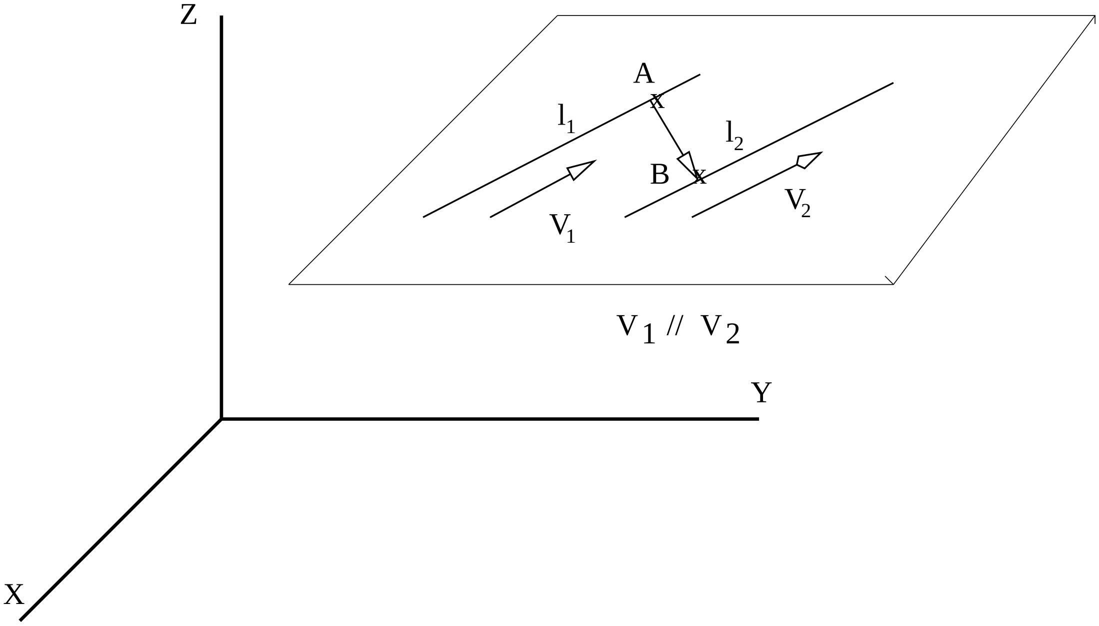

Figure 12: Equation of a plane containing two distinct lines, l_1, l_2 with direction vectors $\vec{V_1}, \vec{V_2}$, with $\vec{V_1}$ parallel to $\vec{V_2}$

8 Partial Derivatives

64. Definition: Assume that $z = f(x, y)$.

    The partial derivative of f with respect to x, denoted by $f_x(x, y)$, is defined as follows:

$$f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

The partial derivative of f with respect to y, denoted by $f_y(x, y)$, is defined by

$$f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

65. Example Using the above definitions, find $f_x(x, y)$ and $f_y(x, y)$ where

$f(x, y) = x^2 + 2xy + 3y^2$

$$\begin{aligned}
f_x(x, y) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta y} \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + 2(x + \Delta x)y + 3y^2 - (x^2 + 2xy + 3y^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 2xy + 2y\Delta x + 3y^2 - x^2 - 2xy - 3y^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2y\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x + 2y) \\
&= 2x + 2y
\end{aligned}$$

$$\begin{aligned}
f_y(x, y) &= \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{x^2 + 2x(y + \Delta y) + 3(y + \Delta y)^2 - (x^2 + 2xy + 3y^2)}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{x^2 + 2xy + 2x\Delta y + 3y^2 + 6y\Delta y 3(\Delta y)^2 - x^2 - 2xy - 3y^2}{\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{2x\Delta y + 6y\Delta y + 3(\Delta y)^2}{\Delta y} \\
&= \lim_{\Delta y \to 0} 2x + 6y + 3\Delta y \\
&= 2x + 6y
\end{aligned}$$

66. Exercises: Using the basic definitions, find $f_x(x, y)$ and $f_y(x, y)$ for the following functions

(a) $f(x, y) = 5x^2 + xy^2 + y$

(b) $f(x, y) = x^3 + xy^2 + 2y^3$

(c) $f(x,y) = x^2y^2 - 2x^2 - 4y$

(d) $f(x,y) = \sin(2x + 3y)$

67. An examination of the basic definitions and the above examples clearly indcate the procedure for partial differentiation. If you are seeking the partial derivative of f with respect to x, everywhere you see an x, differentiate it using the usual rules developed in Math 2417; treat all other variables as constants for this operation. To find the partial derivative of f with respect to y, everywhere you see a y, differentiate it using the rules given in Math 2417, for example, if $f(x,y)$ contains a "y^3 ", its derivative is $3y^2$. All other variables are to be regarded as constants.

68. Example: $f(x,y) = x^2 \arctan(3y)$

Find (a) $f_x(x,y)$; $(b) f_y(x,y)$

Answer $f_x(x,y) = 2x \arctan(3y)$; $f_y(x,y) = \dfrac{3x^2}{1+9y^2}$

69. Example $f(x,y) = x^2 e^{\frac{x}{y}}$

$$f_x(x,y) = 2xe^{\frac{x}{y}} + \frac{x^2}{y}e^{\frac{x}{y}}$$

$$f_x(x,y) = e^{\frac{x}{y}}\left(2x + \frac{x^2}{y}\right)$$

$$f_y(x,y) = x^2\left(\frac{-x}{y^2}\right)e^{\frac{x}{y}}$$

$$f_y(x,y) = x^2\left(\frac{-x}{y^2}\right)e^{\frac{x}{y}}$$

$$f_y(x,y) = \frac{-x^3}{y^2}e^{\frac{x}{y}}$$

70. Example $f(x,y) = \arctan(x^2y^3)$

$$f_x(x,y) = \frac{2xy^3}{1+x^4y^6} \qquad f_y(x,y) = \frac{3xy^2}{1+x^4y^6}$$

71. Example $w = x^2y^3 + y^2z^2 + x^3z$

$$\frac{\partial w}{\partial x} = 2xy^3 + 3x^2z; \quad \frac{\partial w}{\partial y} = 3x^2y^2 + 2yz^2; \quad \frac{\partial w}{\partial z} = 2y^2z + x^3$$

72. Exercises: Find $f_x(x,y)$ and $f_y(x,y)$ for the following functions

(a) $f(x,y) = x^2 \arcsin(xy^3)$

(b) $f(x,y) = e^{x^3+y^2}$
(c) $f(x,y) = \sec(3x^2 + 2y^3)$
(d) $f(x,y) = \arctan(\sqrt{xy})$

73. Another notation for the partial derivative of f with respect to x is $\dfrac{\partial f}{\partial x}$. Similarly, the partial derivative of f with respect to y is represented by $\dfrac{\partial f}{\partial y}$

74. Exercises: Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ for the following functions

    (a) $f(x,y) = \sec^3(x^2y^3)$

    (b) $f(x,y) = x^3y^2e^{\frac{y}{x}}$

    (c) $f(x,y) = \ln\sqrt[3]{x^2 + y^5}$

    (d) $f(x,y) = x^3\tan^2(3x + 2y^2)$

75. Higher order derivatives

    It is sometimes necessary to find higher order derivatives. The possibilities and notation for $z = f(x,y)$ are given in the accompanying diagrams,Figure 13and Figure 14 For the case of $z = f(x,y)$, we see that these are 4 possible second order derivatives, namely $f_{xx}(x,y)$, $f_{xy}(x,y)$, $f_{yx}(x,y)$, $f_{yy}(x,y)$. These derivatives are sometimes written as a matrix, called a Hessian matrix.

$$H_f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

76. Example $f(x,y) = x^3y^2 + \sin(xy)$

    Find all second order partial derivatives. The starting point is to find both first order partial derivatives.

    $f_x(x,y) = 3x^2y^2 + y\cos(xy)$; $f_y(xy) = 2x^3y + x\cos(xy)$

    To find $f_{xx}(x,y)$, we start with $f_x(x,y)$ and differentiate that answer with respect to x

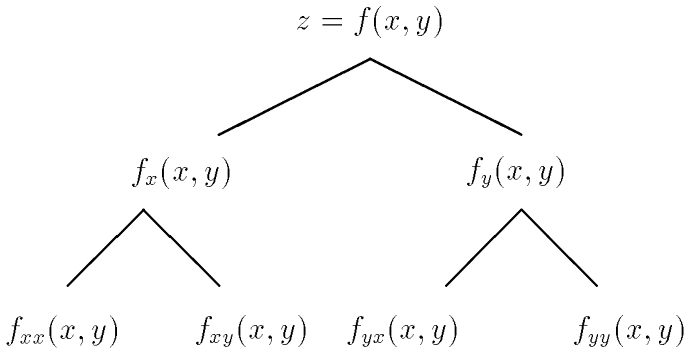

Figure 13: First and second order partial derivatives ; subscript notation

To find $f_{xy}(x,y)$, we start with $f_x(x,y)$ and differentiate that answer with respect to y.

Therefore

$f_{xx}(x,y) = 6xy^2 - y^2\sin(xy)$

$f_{xy}(x,y) = 6x^2y + \cos(xy) - xy\sin(xy)$

To find $f_{yx}(x,y)$, we start with $f_y(x,y)$ and differentiate that answer with respect to x.

To find $f_{yy}(x,y)$, we start with $f_y(x,y)$ and differentiate that answer with respect to y.

Therefore

$f_{yx}(x,y) = 6x^2y + \cos xy - xy\sin(xy)$

$f_{yy}(x,y) = 2x^3 - x^2\sin(xy)$

Note that in this notation, the order of differentiations from left to right.

This notation is easily extended to higher order differentiation and additional variables

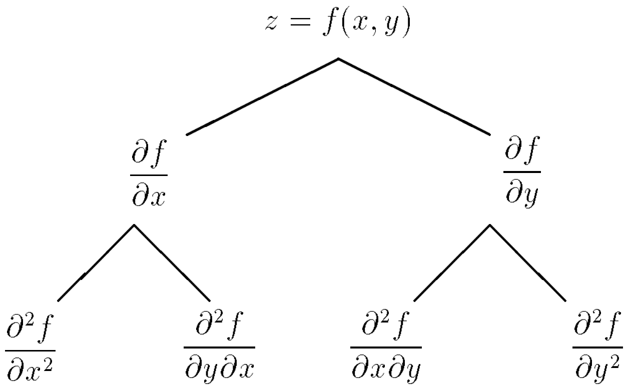

Figure 14: First and second order partial derivatives ; alternate notation

77. <u>Example</u>: $f(x,y,z,t) = x^2y^3tz + xz^4t^2y + x^3z^3y^2$

    Find $f_{xytz}(x,y,z,t)$.

    $f_x(x,y,z,t) = 2xy^3tz + z^4t^2y + 3x^2z^3y^2$

    $f_{xy}(x,y,z,t) = 6xy^2tz + 6x^2z^3y$

    $f_{xyt}(x,y,z,t) = 6xy^2z$

    $f_{xytz}(x,y,z,t) = 6xy^2$

78. <u>Alternate notation</u> Figure 14 shows the second order partial derivatives in the alternate notation.The Hessian matrix is then written as

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

79. To find $\frac{\partial^2 f}{\partial x^2}$, we start with $\frac{\partial f}{\partial x}$ and differentiate this function with respect to x.

To find $\frac{\partial^2 f}{\partial x \partial y}$, start with $\frac{\partial f}{\partial y}$ and differentiate it with respect to x To find $\frac{\partial^2 f}{\partial y^2}$, we start with $\frac{\partial f}{\partial y}$ and differentiate this function with respect to y.

To find $\frac{\partial^2 f}{\partial y \partial x}$, start with $\frac{\partial f}{\partial x}$ and differentiate it with respect to y Note in this symbolism, the order of differentiation is from right to left.

80. Example $f(x,y) = \ln(x^2 - y^2)$

Find all first and second order partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 - y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2 - y^2)\,2 - 2x(2x)}{(x^2 - y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-2x^2 - 2y^2}{(x^2 - y^2)^2}$$

$$\frac{\partial f}{\partial y \partial x} = \frac{(x^2 - y^2)(0) - (2x)(-2y)}{(x^2 - y^2)^2}$$

$$\frac{\partial f}{\partial y \partial x} = \frac{4xy}{(x^2 - y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{-2y}{x^2 - y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2 - y^2)\,(-2) - (-2y)(-2y)}{(x^2 - y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2x^2 - 2y^2}{(x^2 - y^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^2 - y^2)(0) - (-2y)(2x)}{(x^2 - y^2)^2}$$

$$\frac{\partial f}{\partial x \partial y} = \frac{4xy}{(x^2 - y^2)^2}$$

In this example, it is a coincidence that $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ give the same result. This is NOT true in general as may be seen from the example in item 76 above. However, it is true in general (but not always) that the order of differentiation is unimportant, so that, $\frac{\partial^2 f}{\partial x \partial y}$ should give the same result as $\frac{\partial^2 f}{\partial y \partial x}$. Situations in which this is not true are given below (see items 82g and 82h)

81. Exercises Find all first and second order partial derivatives, for the following functions.

(a) $f(x,y) = 2x^3y^2 + \cos(xy) + e^{x^2y}$

(b) $f(x,y) = \sin^2(2x+3y)$

(c) $f(x,y) = \arctan(xy)$

(d) $f(x,y) = \ln\sqrt{x^2+y^2}$

82. Example

$$\begin{aligned} f(x,y) &= \frac{xy(x^2-y^2)}{x^2+y^2}; (x,y) \neq (0,0) \\ f(0,0) &= 0 \end{aligned}$$

(a) Find $f_x(x,y)$; $(x,y) \neq (0,0)$

Answer

$$\begin{aligned} f(x,y) &= \frac{x^3y - xy^3}{x^2+y^2} \\ f_x(x,y) &= \frac{(x^2+y^2)(3x^2-y^3) - (2x)(x^3y - xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \end{aligned}$$

(b) Find $f_y(x,y)$; $(x,y) \neq (0,0)$

Answer

$$\begin{aligned} f_y(x,y) &= \frac{(x^2+y^2)(x^3-3xy^2) - (2y)(x^3y - xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} \end{aligned}$$

(c) Find $f_x(0,0)$

Answer We must use the fundamental definition in this case.

$$\begin{aligned} f_x(0,0) &= \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{0}{(\Delta x)^2} - 0}{\Delta x} \\ &= 0 \end{aligned}$$

(d) Find $f_y(0,0)$

Answer Again, using the definition, we have

$$\begin{aligned} f_y(0,0) &= \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{\frac{0}{(\Delta y)^2} - 0}{\Delta y} \\ &= 0 \end{aligned}$$

(e) Find $f_{xy}(x,y)$ for $(x,y) \neq (0,0)$

Answer Using the quotient rule and after simplification

$$f_{xy}(x,y) = \frac{x^6 - y^6 + 9x^4y^2 - 9x^2y^4}{(x^2+y^2)^3}; \ (x,y) \neq (0,0)$$

(f) Find $f_{yx}(x,y)$ for $(x,y) \neq (0,0)$

Answer Using the quotient rule and after simplification

$$f_{yx}(x,y) = \frac{x^6 - y^6 + 9x^4y^2 - 9x^2y^4}{(x^2+y^2)^3}; \ (x,y) \neq (0,0)$$

(g) Find $f_{xy}(0,0)$

Answer In this case, we have to use the basic definition, namely

$$\begin{aligned}
f_{xy}(0,0) &= \lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}\\
&= \lim_{\Delta x\to 0}\frac{\dfrac{-\Delta y^5}{\Delta y^4}-0}{\Delta y}\\
&= -1
\end{aligned}$$

(h) Find $f_{yx}(0,0)$.
Answer As in the previous item, we must use the basic definition to find this derivative at (0,0)

$$\begin{aligned}
f_{yx}(0,0) &= \lim_{\Delta x\to 0}\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\dfrac{\Delta x^5}{\Delta x^4}-0}{\Delta x}\\
&= 1
\end{aligned}$$

83. Notice that the $\lim\limits_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist and the $\lim\limits_{(x,y)\to(0,0)} f_{yx}(x,y)$ does not exist.(Hint, Try approaching $(0,0)$ along a family of lines $y=mx$). The value of the limit depends on "m". Hence f_{xy} and f_{yx} are not continuous at (0,0). See Theorem 13.3,page 911 of your text book. We do not expect f_{xy} and f_{yx} to be equal at the point $(0,0)$

84. Figures 15 and 16 show the first and second order partial derivatives in both notations for the case of $w=f(x,y,z)$

85. The Hessian matrix for $w=f(x,y,z)$is shown in the subscript and alternate notation.

$$H_f=\begin{pmatrix} f_{xx} & f_{xy} & f_{xz}\\ f_{yx} & f_{yy} & f_{yz}\\ f_{zx} & f_{zy} & f_{zz}\end{pmatrix}$$

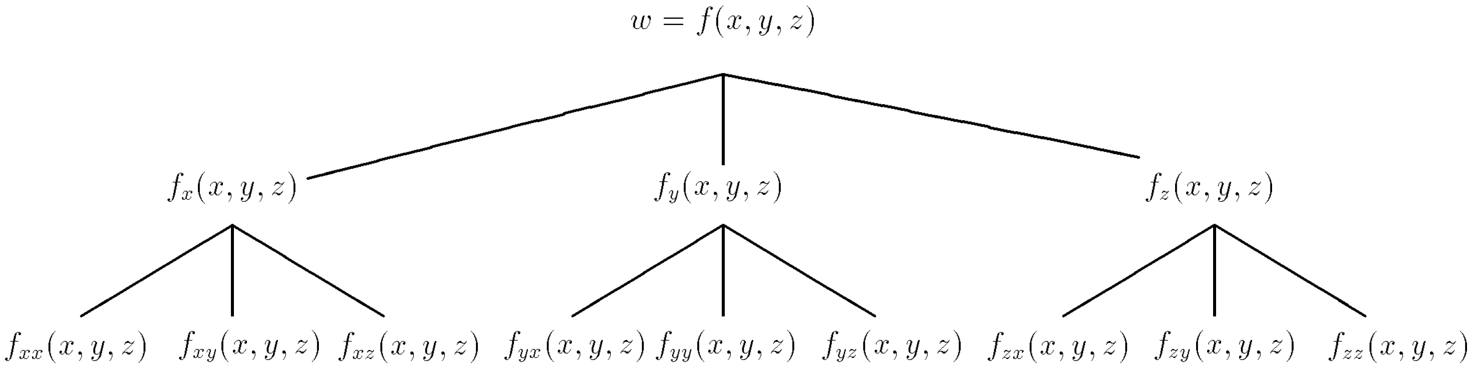

Figure 15: First and second order partial derivatives;subscript notation. $w = f(x,y,z)$

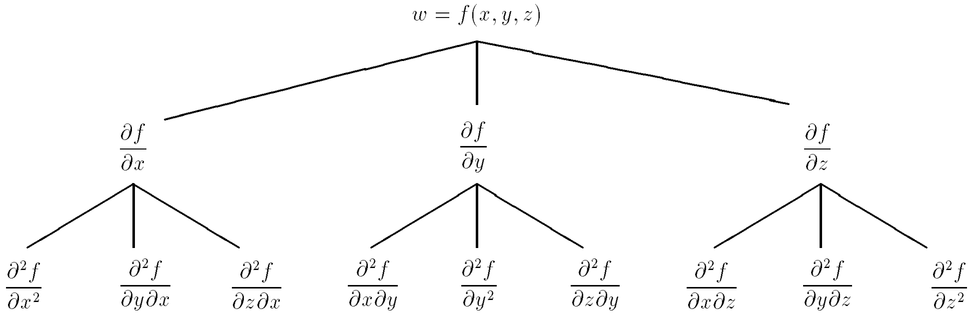

Figure 16: First and second order partial derivatives;alternate notation .$w = f(x,y,z)$

$$H_f = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial z \partial x} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} & \dfrac{\partial^2 f}{\partial z \partial y} \\ \dfrac{\partial^2 f}{\partial x \partial z} & \dfrac{\partial^2 f}{\partial y \partial z} & \dfrac{\partial^2 f}{\partial z^2} \end{pmatrix}$$

9 Relative Extrema of $z = f(x, y)$

86. In this section, we find the relative extrema and saddle points for a function of two variables, namely $z = f(x, y)$. The procedure for finding relative extrema of a function of 2 variables is analogous to the procedure for finding relative extrema of a function of one variable.

    (a) Step 1 We must find candidates, that is critical points as discussed in the section on directional derivatives, the requirement is that $\|\nabla f(x, y)\| = 0$. Since $\nabla f(x, y) = f_x(x, y)\hat{\imath} + f_y(x, y)\hat{\jmath}$, this statement requires that both partial derivatives be zero simultaneously. Let $f_x(x, y) = 0$ and $f_y(x, y) = 0$ Solve these equations and obtain all solutions. Note, although we are only considering a function of 2 variables, this step is valid for functions of many variables e.g. if $w = F(x, y, z, \lambda_1, \lambda_2)$, then critical points would be obtained by solving simultaneously the 5 equations

    $$F_x = 0;\ F_y = 0;\ F_z = 0;\ F_{\lambda_1} = 0;\ F_{\lambda_2} = 0$$

    Also note that other critical points may be obtained if f_x, f_y are undefined but $f(x, y)$ exists. We shall not deal with these critical points in this course. Recall the Hessian Matrix mentioned in the earlier section on partial derivatives

    $$H_f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

    (b) Step 2 We must test the critical points and determine whether they are relative maxima, relative minima or saddle points. On the assumption that $f_{xy} = f_{yx}$, we have the determinant of the Hessian Matrix

    $$\det H_f = D = f_{xx}f_{yy} - (f_{xy})^2$$

$D = f_{xx}f_{yy} - (f_{xy})^2$	f_{xx}	Result
$D > 0$	$f_{xx} > 0$	Relative Minimum
$D > 0$	$f_{xx} < 0$	Relative Maximum
$D < 0$		Saddle Point
$D = 0$		No information

Table 5: Second Derivative Test for a Function of Two Variables

Let us assume that (a, b) is critical point If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then there is a relative minimum at (a, b) The value of the minimum is $f(a, b)$. If $D(a, b) < 0$, then there is a saddle point at (a, b). If $D(a, b) = 0$, the test yields no information. This information is summarized in Table 5

87. <u>Example</u>

Find all relative extrema and saddle points for the following function.

$$f(x, y) = 6x^3 + 4y^3 + 51x^2 - 9y^2 + 60x - 12y + 1$$

A good procedure is to find all of the first and second order partial derivatives at the beginning of the problem and locate them on one place on your paper.

$f_x(x, y) = 18x^2 + 102x + 60$ $\qquad$ $f_y(x, y) = 12y^2 - 18y - 12$

$f_{xx}(x, y) = 36x + 102$ $\qquad$ $f_{yy}(x, y) = 24y - 18$

$f_{xy}(x, y) = 0$ $\qquad$ $f_{yx}(x, y) = 0$

Let $f_x(x, y) = 0$ and $f_y(x, y) = 0$ and solve.

In this case, since $f_x(x, y)$ contains only x's and $f_y(x, y)$ contains only y's, these equations can be solved independently.

$$
\begin{aligned}
f_x(x, y) = 0 \qquad \text{gives} \qquad & 18x^2 + 102x + 60 = 0 \\
& 6(3x^2 + 1yx + 10) = 0 \\
& 6(3x + 2)(x + 5) = 0 \\
\text{or} \qquad & x = -\tfrac{2}{3}, \quad x = -5 \\
\mathrm{f}_y(x, y) = 0 \qquad \text{gives} \qquad & 12y^2 - 18y - 12 = 0
\end{aligned}
$$

CP	$f_{xx} = 36x + 102$	$f_{yy} = 24y - 18$	$f_{xy} = 0$	$D = f_{xx}f_{yy} - (f_{xy})^2$	Conclusion
$\left(-\frac{2}{3},\ -\frac{1}{2}\right)$	> 0	< 0	0	< 0	Saddle Point
$\left(-\frac{2}{3},\ 2\right)$	> 0	> 0	0	$> 0,\ f_{xx} > 0$	Relative Min
$\left(-5,\ -\frac{1}{2}\right)$	< 0	< 0	0	$> 0,\ f_{xx} < 0$	Relative Max
$(-5,\ 2)$	< 0	> 0	0	< 0	Saddle Point

Table 6: Test results for Example 87

$$6(2y^2 - 3y - 2) = 0$$

$$6(2y + 1)(y - 2) = 0$$

$$\text{or} \qquad y = -\tfrac{1}{2}, \quad y = 2$$

From these results, we may conclude that there are 4 critical points

$\left(-\frac{2}{3},\ -\frac{1}{2}\right)$, $\left(-\frac{2}{3},\ 2\right)$, $\left(-5,\ -\frac{1}{2}\right)$, $(-5,\ 2)$

Each critical number makes both f_x and f_y simultaneously zero. When there are several critical points, it is probably best to do the second derivative test in tabular form.

88. Example

Find all relative extrema and saddle points for the following function.

$$f(x, y) = 4xy - x^4 - y^4$$

Find all of the first and second order partial derivatives at the beginning of the problem .

$f_x(x, y) = 4y - 4x^3$ $\qquad$ $f_y(x, y) = 4x - 4y^3$

$f_{xx}(x, y) = -12x^2$ $\qquad$ $f_{yy} = -12y^2$

$f_{xy}(x, y) = 4$ $\qquad$ $f_{yx}(x, y) = 4$

To find the critical numbers, set $f_x(x, y) = 0$ and $f_y(x, y) = 0$ and solve these simultaneous equations.

$$f_x(x, y) = 0 \quad \text{gives} \quad 4y - 4x^3 = 0 \text{ or } y = x^3$$
$$f_y(x, y) = 0 \quad \text{gives} \quad 4x - 4y^3 = 0 \text{ or } x = y^3$$

CP	$f_{xx} = -12x^2$	$f_{yy} = -12y^2$	$f_{xy} = 4$	$D = f_{xx}f_{yy} - (f_{xy})^2$	Conclusion
$(0,0)$	0	0	4	$-16 \quad < 0$	Saddle Point
$(1,1)$	-12	-12	4	$128; > 0,\ f_{xx} < 0$	Relative Max
$(-1,-1)$	-12	-12	4	$128;\ > 0,\ f_{xx} < 0$	Relative Max

Table 7: Test results for Example 88

Substituting $y = x^3$ into $x = y^3$, we have $x = (x^3)^3$ or $x^9 - x = 0$. Factoring, we have $x(x^4+1)(x^2+1)(x+1)(x-1) = 0$ Hence there are three real solutions,namely$x = 0, x = 1$, and $x = -1$. The corresponding y values are $y = 0$,$y = 1$ and $y = -1$. Hence there are three critical points,$(0,0), (1,1)$ and $(-1,-1)$. The second derivative test is shown in Table 7. There is a saddle point at (0,0,0) and relative maxima at (1,1,2) and (-1,-1,2). Note that unlike the case for a continuous function of one variable, it is possible for a continuous function of two variables to have two relative maxima without having a relative minimum

10 Method of Lagrange Multipliers

89. Example: Find the critical points of $2x + y$ subject to the constraint $xy = 32$
Solution: First form the function $F = f - \lambda g$ or in our case

$$F(x, y, \lambda) = 2x + y - \lambda(xy - 32)$$

Now lets obtain the partial derivatives

$$F_x(x, y, \lambda) = 2 - \lambda y$$

$$F_y(x, y, \lambda) = 1 - \lambda x$$

$$F_\lambda(x, y, \lambda) = -(xy - 32)$$

Set these partial derivatives equal to zero and solve.

$$\begin{aligned} 2 - \lambda y &= 0 \quad (3)\\ 1 - \lambda x &= 0 \quad (4)\\ xy - 32 &= 0 \quad (5) \end{aligned}$$

Eliminate λ from equations 3 and 4
$\lambda = \frac{2}{y}$; $\lambda = \frac{1}{x}$. Hence $\frac{2}{y} = \frac{1}{x}$ or $y = 2x$. Substituting $y = 2x$ into equation 5,

det $\bar{H}_f(x,y,\lambda)$	Result
If det $\bar{H}_f(x,y,\lambda) < 0$	Constrained Minimum
If det $\bar{H}_f(x,y,\lambda) > 0$	Constrained Maximum
If det $\bar{H}_f(x,y,\lambda) = 0$	No Information

Table 8: Test for Constrained Extrema

we have $2x^2 = 32$ or $x = \pm 4$. So the critical points are $(4,8)$ and $(-4,-8)$.$f(4,8) = 16$; $f(-4,-8) = -16$. It would appear that the critical point $(4,8)$ leads to a maximum of 16 while the critical point $(-4,-8)$ gives a minimum of -16. As we shall see,this is not true.

90. <u>Test for constrained extrema</u> Using the notation $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ we form the bordered Hessian matrix .

$$\bar{H}_f(x,y,\lambda) = \begin{pmatrix} 0 & -g_x & -g_y \\ -g_x & F_{xx} & F_{xy} \\ -g_y & F_{yx} & F_{yy} \end{pmatrix}$$

If det $\bar{H}_f = 0$, the test is inconclusive.If det $\bar{H}_f > 0$, there is a constrained maximum.If det $\bar{H}_f < 0$, there is a constraint minimum. This information is summarized in Table 8

In our case ,$F_{xx}(x,y,\lambda) = 0$, $F_{yy}(x,y,\lambda) = 0$, $F_{xy}(x,y,\lambda) = F_{yx}(x,y,\lambda) = -\lambda$, $g_x(x,y) = y$; $g_y(x,y) = x$
Therefore in our case

$$\bar{H}_f(x,y,\lambda) = \begin{pmatrix} 0 & -y & -x \\ -y & 0 & -\lambda \\ -x & -\lambda & 0 \end{pmatrix}$$

or

$$\bar{H}_f\left(4,8,\frac{1}{4}\right) = \begin{pmatrix} 0 & -8 & -4 \\ -8 & 0 & -\frac{1}{4} \\ -4 & -\frac{1}{4} & 0 \end{pmatrix}$$

$$\det \bar{H}_f\left(4,8,\frac{1}{4}\right) = 8(-1) - 4(2) = -8 - 8 = -16 < 0$$

According to this test, the point $(4,8)$ is a constrained minimum.

$$\bar{H}_f\left(-4,-8,-\frac{1}{4}\right) = \begin{pmatrix} 0 & 8 & 4 \\ 8 & 0 & \frac{1}{4} \\ 4 & \frac{1}{4} & 0 \end{pmatrix}$$

$$\det \bar{H}_f\left(-4,-8,-\frac{1}{4}\right)=(-8)(-1)+4(2)=16>0$$

According to this test, the point $(-4,-8)$ is a constrained maximum. Note that the value of $f(x,y)$ in each case is obtained from $f(x,y)=2x+y$.

This exercise is shown graphically in figure 17. The constraint equation $xy=32$ is shown

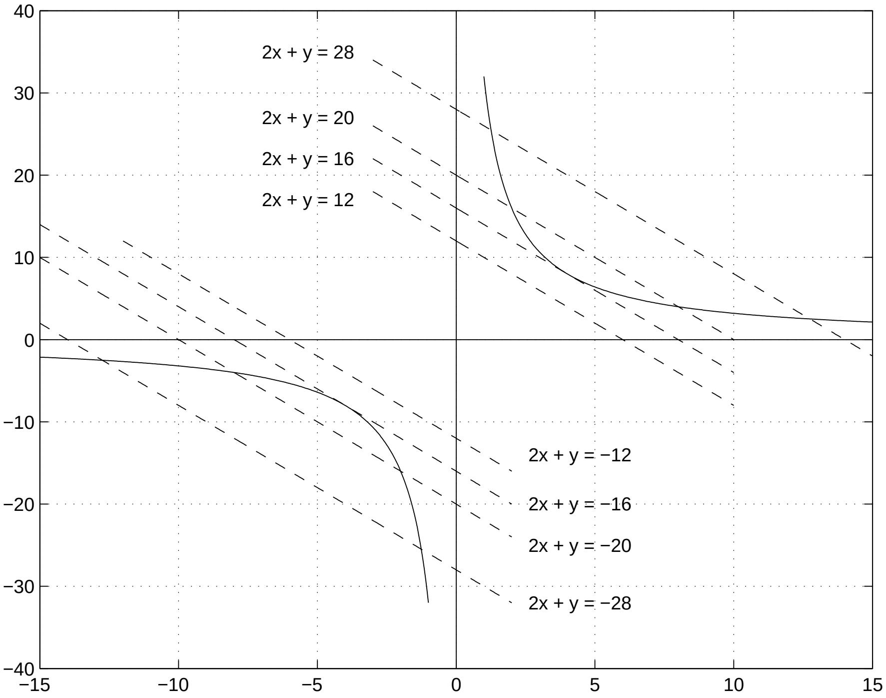

Figure 17: Figure for example 89

for both positive and negative values of x,y. Also shown is a family of lines, $2x+y=k$, for various values of k. For positive values of k, such as $28, 20$, the lines intersect the constraint curves at two places. For $k=16$, the line $2x+y=16$ just touches the constraint curve. For a smaller value of k, the line $2x+y=k$ no longer intersects the upper branch of the constraint curve. Looking at the family of lines intersecting the lower branch of the constraint curve, we see that as k increases from -28 to -16, the line $2x+y=k$ intersects the constraint curve. For $k=-16$, the line touches the constraint curve, that is for $k=-16$, $2x+y=-16$ is the equation of the tangent line to $xy=32$ at $(-4,-8)$. For values of k greater than -16, the line $2x+y=k$ no longer cuts or touches the lower branch of the constraint curve. Note ,for this course, you are <u>NOT</u> required to use this test for constrained extrema. Rather, we shall ask you to find critical points subject to a constraint.

11 Double Integrals

91. Evaluate $\int_{2}^{4} \int_{1}^{3} xy^2 dy\ dx$

As a first step, I suggest that you rewrite the problem. In this case the inner integral is with respect to the variable "y" so that 1 and 3 are y limits. The outer integral is with respect to x and so 2 and 4 are x limits.

$$\int_{x=2}^{x=4} \int_{y=1}^{y=3} xy^2 dy \ \ dx$$

In this case, to evaluate the inner integral, we integrate with respect to y, holding x constant,

$$\begin{aligned}
\int_{x=2}^{x=4} \int_{y=1}^{y=3} xy^2 dy \ \ dx &= \int_{x=2}^{x=4} \left[\frac{x\ \ y^3}{3}\right]_{y=1}^{y=3} dx \\
&= \int_{x=2}^{x=4} 9x - \frac{x}{3}\ dx \\
&= \int_{x=2}^{x=4} \frac{26x}{3}\ dx \\
&= \left[\frac{13\ x^2}{3}\right]_{x=2}^{x=4} \\
&= \frac{13}{3}(16-4) \\
&= 52
\end{aligned}$$

Notice that after the inner integral is evaluated,the remaining integral contains only one variable and may be evaluated using techniques from Math 2417.

92. Example Set up $\displaystyle\iint\limits_{\mathrm{R}} f(x,y)dA$

where R is the region bounded by $y=\sqrt{x},\quad x=4,\quad y=0.$

Solution If this were an exam question, we would specify the order in which the iterated integral was to be set up. However, as it is an illustrative example, we will set it up both ways.

Case 1 $\displaystyle\int\int f(x,y)dy\,dx.$

Begin by writing the limits using the appropriate variables. Since the integration in the inner integral is with respect to y, the integration limits must be y values. So we write

$$\int_{x=}^{x=}\int_{y=}^{y=} f(x,y)dy\,dx$$

A vitally important step is to draw the region R. (see figure 18). For the inner integral, we are integrating with respect to y, holding x constant. Therefore we draw a line of x=constant in the region. Label the end points of the line in R as P and Q, with P as the low value. Then ask the question, "Does it matter where you draw the line". In this case, the answer is NO. Why? Because P is always on $y=0$ and Q is always on $y=\sqrt{x}$. These are the integration limits for the inner integral. Then allow this line to sweep out the region.

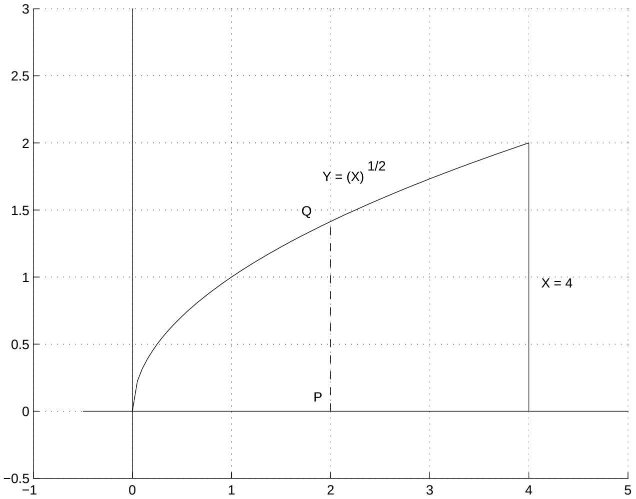

Figure 18: Diagram for example 92, Case(I)

It will go from x=0 to x=4 so that the iterated integral becomes

$$\int_{x=0}^{x=4}\int_{y=0}^{y=\sqrt{x}} f(x,y)dy\ dx$$

Case II In this solution, we want the inner integral to be with respect to x, that is

$$\int\int f(x,y)dx\ dy$$

Because the inner integration is with respect to x, the integration limits must be x values, so we write

$$\int_{y=}^{y=}\int_{x=}^{x=} f(x,y)dx\ dy$$

Once again, examine the inner integral. In this case, we are integrating with respect to x, and holding y constant. Therefore, draw a line of $y =$ constant in R. Label the end points of the line as P and Q, with P being before Q as you move along the line in the direction of increasing x.(see figure 19) Then ask the question, "Does it matter where you draw the line". Again, in this case, the answer is NO. Why? Because P is always on y=$\sqrt{x}$ and Q is always on $x = 4$. Remember that we want the limits in the form $x =$ so we must solve $y = \sqrt{x}$ to become $x = y^2$. Then allow the line to sweep out the entire region, going from $y = 0$ to $y = 2$. Hence the iterated integral becomes

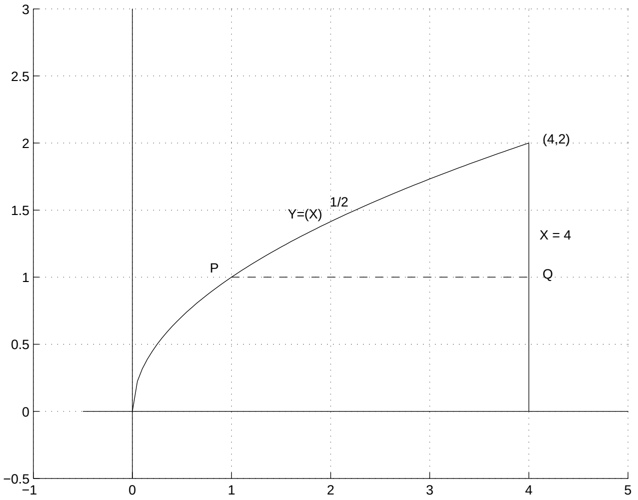

Figure 19: Diagram for example 92, Case(II)

$$\int_{y=0}^{y=2} \int_{x=y^2}^{x=4} f(x,y)dx\ dy.$$

93. <u>Example</u> Set up $\displaystyle\iint\limits_{\text{R}} f(x,y)dA$

where R is the region bounded by $y = x$, $y = 2x$ and $x = 4$. As a first step in solving this problem, <u>draw</u> the region. (Figure 20)

<u>Case I</u> Set up the interated integral in the form

$$\int_{x=}^{x=} \int_{y=}^{y=} f(x,y)dy\ dx$$

For the inner integral, we are integrating with respect to y, holding x constant. Therefore draw a line of x=constant in the region. Label the end points of the line P, Q with P being the low point. Then ask the question, "Does it make any difference where I draw the line" Again, the answer is <u>NO</u> because P is always on $y = x$ and Q is always on $y = 2x$. These boundaries are the integration limits for the inner integral. Then allow the line to sweep out the region. It will go from $x = 0$ to $x = 4$. Hence the iterated integral becomes

$$\int_{x=0}^{x=4} \int_{y=x}^{y=2x} f(x,y)dy\ dx$$

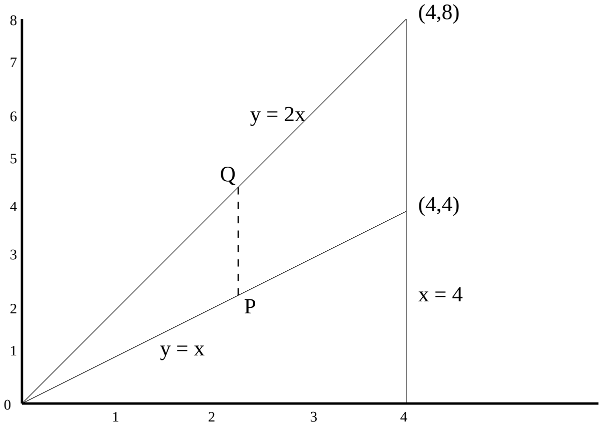

Figure 20: Diagram for example 93,Case(I)

Case (II) Set up the iterated integral in the form

$$\int_{y=}^{y=}\int_{x=}^{x=} f(x,y)dxdy$$

Since the inner integral involves an integration with respect to x, holding y constant, we draw a line of y=constant in the region and label the end points P, and Q with P being before Q as you move along the line in the direction of increasing x. Again we ask the question "Does it matter where I draw the line". This time, the answer is yes. While P is on the same curve throughout the region namely, $y = 2x$, Q is on $y = x$ in the lower part of the region and on $x = 4$ in the upper part of R. When this happens, we subdivide the region into a number of contiguous subregions such that in each subregion when we draw a line of y =constant, it is independent of where it is drawn in that subregion. In the present case, we divide R into two subregions R_1 and R_2 by a line AB (Figure 21).

The iterated integrals now become

$$\underset{R_1}{\int\int} f(x,y)dx\ dy + \underset{R_2}{\int\int} f(x,y)dx\ dy$$

or

$$\int_{y=}^{y=}\int_{x=}^{x=} f(x,y)dx\ dy + \int_{y=}^{y=}\int_{x=}^{x=} f(x,y)dx\ dy$$

For R_1, P is always on $y = 2x$ and Q is always on $y = x$. Hence the limits are $x = \frac{y}{2}$ and $x = y$. We then allow the line of y = constant to sweep out R_1 namely from $y = 0$ to $y = 4$. For R_2, P is always on $y = 2x$ and Q is always on $x = 4$. Allowing the line of y = constant to sweep out the region R_2,we have y going from $y = 4$ to $y = 8$. Hence the answer is

$$\int_{y=0}^{y=4}\int_{x=\frac{y}{2}}^{x=y} f(x,y)dxdy + \int_{y=4}^{y=8}\int_{x=\frac{y}{2}}^{x=4} f(x,y)dxdy$$

94. Example Set up $\underset{R}{\int\int} e^{-x^2}dydx$

where R is the region bounded by $y = x$, $y = 0$ and $x = 4$. Draw the region R (Figure 22)

Case(I)$\int_{x=}^{x=}\int_{y=}^{y=} e^{-x^2}dydx$. Integrating with respect to y, holding x constant, we draw a line of x = constant and ask the question "Does it matter where I draw the line "?" No!

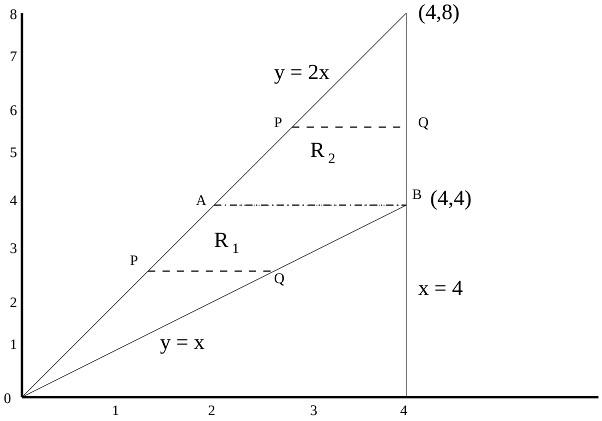

Figure 21: Diagram for example 93,Case(II)

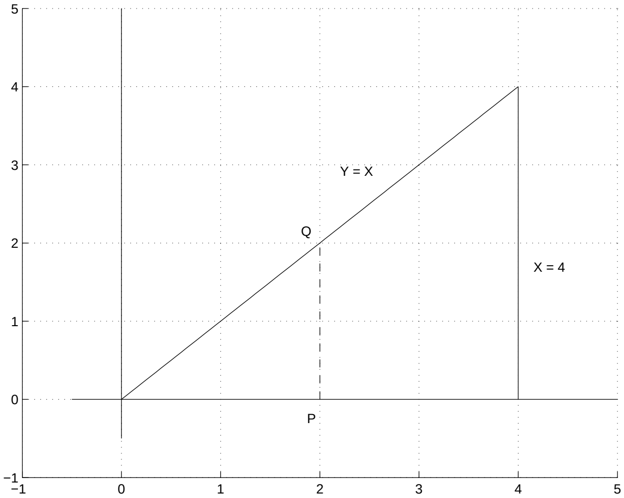

Figure 22: Diagram for example 94, Case(I)

why? P is always on $y = 0$ and Q is always on $y = x$. Allow this line to sweep out the region. Hence we can write $\int_{x=0}^{x=4} \int_{y=0}^{y=x} e^{-x^2} dy\, dx$

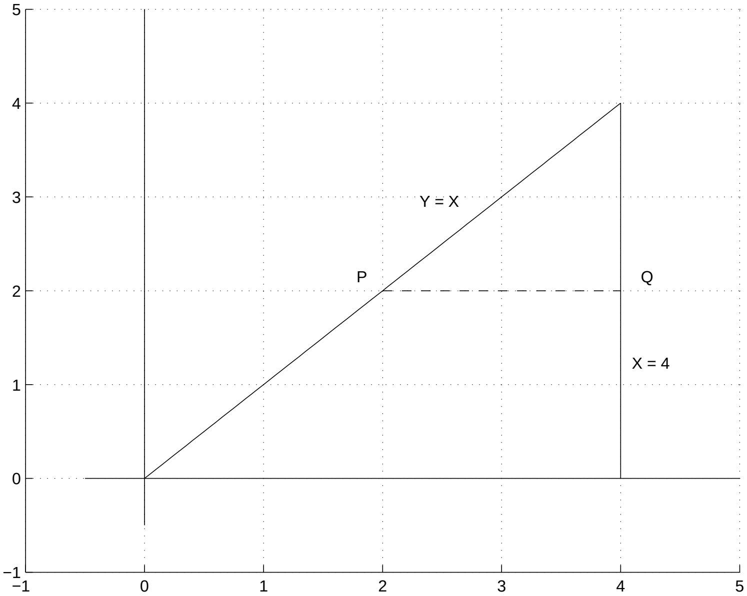

Figure 23: Diagram for example 94, Case(II)

Case (II) $\int_{y=}^{y=} \int_{x=}^{x=} e^{-x^2} dxdy$. Integrating with respect to x, holding y constant, we draw a line of y=constant (see Figure 23) and ask the question "Does it matter where I draw the line?" No! Why? Because P is always on the line y=x and Q is always on the line x=4. Allowing the line to sweep out the region, we have $\int_{y=0}^{y=4} \int_{x=y}^{x=4} e^{-x^2} dxdy$ In this exercise, as far as the region is concerned, the order of integration is unimportant. It is just as easy to set up one way as it is the other way. However, in this exercise, the integration determines which is the preferred order. Clearly case (I) is recommended for this exercise.

1245
March 31, 2008
The time is 20h 45min.