Differential length vectors
\[ d\mathbf{l} = (dx, dy, dz)_{\text{cart}} \]
\[ = (dr, r\,d\phi, dz)_{\text{cyl}} \]
\[ = (d\rho, \rho\,d\theta, \rho\sin(\phi)\,d\phi)_{\text{sph}} \]

Del Operator:
\[ \nabla \Phi = \left( \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \Phi = \left( \hat{r} \frac{\partial}{\partial r} + \hat{\phi} \frac{\partial}{\partial \phi} + \hat{\theta} \frac{\partial}{\partial \theta} \right) \Phi \]
\[ \nabla \cdot \mathbf{V} = \left( \frac{\partial}{\partial x} V_x + \frac{\partial}{\partial y} V_y + \frac{\partial}{\partial z} V_z \right) = \left( \frac{\partial}{\partial r} (r V_r) + \frac{\partial}{\partial \phi} \rho V_\phi + \frac{\partial}{\partial \theta} V_\theta \right) \]
\[ = \left( \rho^2 \frac{\partial}{\partial r} (\rho^2 V_r) + (\rho \sin(\theta))^2 \frac{\partial}{\partial \phi} (\sin(\theta) V_\phi) + (\rho \sin(\theta))^2 \frac{\partial}{\partial \theta} V_\theta \right) \]
\[ \nabla \wedge \mathbf{V} = \frac{1}{r} \begin{vmatrix} \hat{r} & r \hat{\phi} & \hat{\theta} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ V_r & \rho V_\phi & \rho \sin(\theta) V_\theta \end{vmatrix} \]
\[ \nabla \cdot \nabla \Phi = \nabla^2 \Phi = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \Phi = \left( \frac{\partial}{\partial r} (r \frac{\partial}{\partial r} + \frac{\partial}{\partial \phi} \rho + \frac{\partial}{\partial \theta} \rho \sin(\theta)) \right) \Phi \]
\[ = \left( \rho^2 \frac{\partial}{\partial r} (\rho^2 \frac{\partial}{\partial r} \Phi) + (\rho \sin(\theta))^2 \frac{\partial}{\partial \phi} (\sin(\theta) \frac{\partial}{\partial \phi} \Phi) + (\rho \sin(\theta))^2 \frac{\partial}{\partial \theta} \frac{\partial}{\partial \theta} \Phi \right) \]
\[ \nabla \wedge \nabla = 0 \]

Green’s Theorem
\[ \oint_D (Q(x, y) \, dy + P(x, y) \, dx) = \iint_{\text{enclosing curve}} \left( \frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P \right) \, dx \, dy \]

Divergence Theorem
\[ \iiint_{\text{Volume}} \nabla \cdot \mathbf{V} \, d\tau = \oiint_{\text{enclosing surface}} \mathbf{V} \cdot d\mathbf{S} \]

Stoke’s Theorem
\[ \oiint_{\text{surface}} (\nabla \wedge \mathbf{V}) \cdot d\mathbf{S} = \oint_{\text{enclosing curve}} \mathbf{V} \cdot d\mathbf{r} \]
Section 1. Basic concepts and basic mathematics

1.1. History

The knowledge of fluid mechanics is perhaps the oldest ‘scientific’ knowledge that humanity has. Humanity has interacted with fluids since antiquity. It is thought that our particular branch of humanity evolved along the shores of Africa. There we would have experienced and ‘understood’ wind and waves.

As our species grew in population and abilities, our link to fluids grew. We harnessed wind for transportation and water for our crops. This course is designed to help the student begin to understand fluids in a formal sense. Mathematically, the book will rely on vector calculus and some basic probability.

1.2. Concept, Nature and sources of vector fields

In Fluid Mechanics we deal with scalars vectors, scalar fields and vector fields. On top of this we have coordinate systems. Understanding these and how they relate will be critical to your understanding of fluids mechanics. In a nut shell:

Scalars: A number or magnitude (say a =1.25). A scalar does not have any direction associated with it. An example is the Temperature of the air in the room you are sitting in or the temperature at various locations in a hurricane.

Vectors: A number with an associated direction (say $A = 1.25$ North). An example is the wind in the room you are sitting in or the wind inside of a hurricane. The eye wall might have sustained winds of 200 MPH – rotating clockwise – but the air inside the eye is often very calm. The directionality of a vector requires that we describe what happens in each direction. For this we develop the concept of ‘unit vectors.’ Unit vectors are vectors of length one that point in a specific direction, say in the ‘north’ direction or the ‘x’ direction. The unit vectors that you use depend on the coordinate system that you are using. We will discuss the standard coordinate systems a little later. (In these notes scalars are notated as small letters while vectors will be notated with bold-faced capitol letters having a ‘bar’, $\bar{A}$, on top. The single exception to this rule is the ‘unit’ vector which is defined to have length 1 and in effect only contains directional
information. Unit vectors are symbolized with lower case bold-faced letters having a ‘carrot’, $\hat{\text{ }}$, on top.)

**Scalar Fields**: This is a scalar that depends on where and when it is measured (say $a = 3xt + 4y + z^2$). Scalar fields are more commonly called functions. An example is the Temperature inside a hurricane or in a our room.

Higher in the air $\rightarrow$ colder (usually)
Night $\rightarrow$ colder
North $\rightarrow$ colder
Texas in the summer $\rightarrow$ hot!

In our hurricane and room examples, the temperature changes, depending on the location. In a room it is usually hotter/colder near the air vent. In electromagnetism the scalars you deal with might be a functions of time, position and local velocity – 7 free parameters in all. An example of this would be:

$$t(x,y,z,t) = x^2 + yz\sqrt{t}$$

**Vector Fields**: This is a vector that depends on where and when it is measured (say $A = 3xt\hat{x} + 4y\hat{y} + z^2\hat{z}$). Vector fields are functions with directions associated with them. An example is the wind inside a hurricane. Note that each direction can have a different functional form, $A = f(x,y,z,v_x,v_y,v_z,t)\hat{x} + g(x,y,z,v_x,v_y,v_z,t)\hat{y} + h(x,y,z,v_x,v_y,v_z,t)\hat{z}$. **ONE OF THE MOST IMPORTANT THINGS THAT YOU MUST DO WHILE WORKING ON AN FLUID MECHANICS PROBLEM IS – KEEP TRACK OF ALL OF THE PARTS, e.g. ‘BOOKKEEPING’.** While this might be a little boring sounding, it is really not that hard. After all, if a person with an MBA can do a little bookkeeping, so can you.

This brings us to the concept of **coordinate systems**. In the proceeding paragraphs we have used two different coordinate systems, spherical (North-south-east-west-height is an example) and Cartesian (x, y and z). You should already be used to these through your earlier experiences.
While there are an infinite number of possible coordinate systems, we will only use three, Cartesian, spherical and cylindrical. (I will attempt to add a generalized discussion of coordinate systems as a later addendum.) Each coordinate system has a definite well-defined method for locating a point or position and a method for giving a direction based on the defined unit vectors. Each also relies on a well-defined origin. Our three main coordinate systems are shown graphically below. Note that the three unit vectors in each coordinate system are at right angles to each other and that one can get to any location in the system by simply adding various lengths of each unit vector. Further note that the cylindrical and spherical coordinate systems are defined in terms of the Cartesian (x, y, z) coordinate system.

Each of these coordinate systems have well defined methods for locating a position and giving a direction.

Almost all coordinate systems that we use are ‘right-hand-rule’. That is if we take our fingers and point them in the direction of the first unit vector and then bend them in the direction of the second unit vector our thumb will point in the direction of the third unit vector. (Note that most screws, bolts etc are right handed i.e. if you want them to move in a direction then point your right hand thumb in that direction and turn the bolt in the direction your fingers point.)

One can determine the transformation from one coordinate system to the next by simply drawing the vectors and examining each of the pieces. Let us take the transformation from cylindrical to Cartesian.
In the x-y plane this looks like

In the above picture we have divided a vector along r into its two components, one along the x-axis and one along the y-axis. Simple trigonometry tells us that these are of length

\[ x = r \cos \phi \]

and

\[ y = r \sin \phi \]

Thus

\[ \mathbf{r} = |\mathbf{r}| \cos(\phi) \mathbf{x} + |\mathbf{r}| \sin(\phi) \mathbf{y} \]

\[ \hat{\mathbf{r}} = \frac{\mathbf{r}}{|\mathbf{r}|} = \cos(\phi) \hat{\mathbf{x}} + \sin(\phi) \hat{\mathbf{y}} \]

The \( \hat{\phi} \) unit vector is at right angles to \( \hat{\mathbf{r}} \) and thus as can be seen

\[ \hat{\phi} = -\sin(\phi) \hat{\mathbf{x}} + \cos(\phi) \hat{\mathbf{y}} \]

Similar results are found for all of the other vectors/scalars leading to the results shown at the beginning of the book.
So now, we have scalar and vector fields and ways to describe specific locations in space. Hopefully we already know how to add and subtract, multiple and divide scalars (if not read the supplemental material at the end of this section.)

### 1.3. Proof of Divergence and Stokes theorems

#### 1.3.1. Divergence (Gauss’) Theorem

\[
\iiint_{\text{Volume}} \nabla \cdot \mathbf{V} \, d\tau = \iint_{\text{Volume surface}} \mathbf{V} \cdot \mathbf{d}\mathbf{s} \quad \leftarrow \text{Standard form}
\]

\[
\iiint_{\text{Volume}} f \, d\tau = \iint_{\text{Volume surface}} f \, d\mathbf{s}
\]

\[
\iiint_{\text{Volume}} \nabla \wedge \mathbf{V} \, d\tau = - \iint_{\text{Volume surface}} \mathbf{V} \wedge \mathbf{d}\mathbf{s}
\]

Let us consider the ‘flux’, \(d\Psi\) of a vector field, \(\mathbf{V}\), through an infinitesimal surface element, \(d\mathbf{s} = \hat{n} d\sigma\). (\(\hat{n}\) is normal to the surface and \(d\sigma\) is the area.) Now the only part of the vector that actually passes through the surface is \(\mathbf{V} \cdot \hat{n} = V_{\parallel}\). (The rest of the field, \(\mathbf{V} \wedge \hat{n} = V_{\perp}\), does not pass through.) Thus,

\[d\Psi = \mathbf{V} \cdot \hat{n} d\sigma\]

To find out how much of the field emanates from a volume, we must consider how much flux passes through a closed surface surrounding that volume. This is simply,

\[
\Psi = \iiint_{\sigma} d\Psi = \iint_{\sigma} \nabla \cdot \hat{n} d\sigma = \iint_{\sigma} \mathbf{V} \cdot d\mathbf{s}
\]
where \( \hat{n} \) is picked as the outward normal. \( \text{THIS FLUX IS A MEASURE OF THE SOURCE/SINK OF THE VECTOR FIELD IN THAT VOLUME. THIS IS A VERY IMPORTANT CONCEPT.} \) We can use the analogy of a bathtub (or sink). If the tap is on and the drain is plugged the bathtub will overflow and continue to overflow. If on the other hand the drain is open and the tap is off, we can continuously pour water in. This is in a nutshell what we find with the divergence theorem.

\[
\Psi = \iiint_{\sigma} \nabla \cdot \mathbf{d}s
\]

We can now look at this equation a bit more closely. Let us consider an infinitesimal volume shown in the figure. If we consider each of the six sides we find that

\[
\Psi = \iint_{\text{Surface 1}} \mathbf{\nabla} \cdot (+\hat{x})dydz + \iint_{\text{Surface 2}} \mathbf{\nabla} \cdot (-\hat{x})dydz + \iint_{\text{Surface 3}} \mathbf{\nabla} \cdot (+\hat{y})dxdz + \\
\iint_{\text{Surface 4}} \mathbf{\nabla} \cdot (-\hat{y})dxdz + \iint_{\text{Surface 5}} \mathbf{\nabla} \cdot (+\hat{z})dxdy + \iint_{\text{Surface 6}} \mathbf{\nabla} \cdot (-\hat{z})dxdy
\]
Accounting for the small change in the vector field across the volume, we find that

\begin{align*}
\text{Surface 1} &- \vec{\nabla}(x,y,z,t) \Rightarrow \vec{\nabla}
\left( x_0 + \frac{\Delta x}{2}, y, z, t \right) \\
\text{Surface 2} &- \vec{\nabla}(x,y,z,t) \Rightarrow \vec{\nabla}
\left( x_0 - \frac{\Delta x}{2}, y, z, t \right) \\
\text{Surface 3} &- \vec{\nabla}(x,y,z,t) \Rightarrow \vec{\nabla}
\left( x, y_0 + \frac{\Delta y}{2}, z, t \right) \\
\text{Surface 4} &- \vec{\nabla}(x,y,z,t) \Rightarrow \vec{\nabla}
\left( x, y_0 - \frac{\Delta y}{2}, z, t \right) \\
\text{Surface 5} &- \vec{\nabla}(x,y,z,t) \Rightarrow \vec{\nabla}
\left( x, y, z_0 + \frac{\Delta z}{2}, t \right) \\
\text{Surface 6} &- \vec{\nabla}(x,y,z,t) \Rightarrow \vec{\nabla}
\left( x, y, z_0 - \frac{\Delta z}{2}, t \right)
\end{align*}

Where \((x_0, y_0, z_0, t)\) is the center of the volume at time \(t\). We can now do a series expansion of each term of the individual \(\vec{\nabla}\), so that

\begin{align*}
\text{Surface 1} &- \vec{\nabla}
\left( x_0 + \frac{\Delta x}{2}, y, z, t \right) \Rightarrow \vec{\nabla}(x_0, y, z, t) + \frac{\Delta x}{2} \frac{\partial}{\partial x} \vec{\nabla} \bigg|_{x=0} + \text{very small terms} \\
\text{Surface 2} &- \vec{\nabla}
\left( x_0 - \frac{\Delta x}{2}, y, z, t \right) \Rightarrow \vec{\nabla}(x_0, y, z, t) - \frac{\Delta x}{2} \frac{\partial}{\partial x} \vec{\nabla} \bigg|_{x=0} + \text{very small terms} \\
\text{Surface 3} &- \vec{\nabla}
\left( x, y_0 + \frac{\Delta y}{2}, z, t \right) \Rightarrow \vec{\nabla}(x, y_0, z, t) + \frac{\Delta y}{2} \frac{\partial}{\partial y} \vec{\nabla} \bigg|_{y=0} + \text{very small terms} \\
\text{Surface 4} &- \vec{\nabla}
\left( x, y_0 - \frac{\Delta y}{2}, z, t \right) \Rightarrow \vec{\nabla}(x, y_0, z, t) - \frac{\Delta y}{2} \frac{\partial}{\partial y} \vec{\nabla} \bigg|_{y=0} + \text{very small terms} \\
\text{Surface 5} &- \vec{\nabla}
\left( x, y, z_0 + \frac{\Delta z}{2}, t \right) \Rightarrow \vec{\nabla}(x, y, z_0, t) + \frac{\Delta z}{2} \frac{\partial}{\partial z} \vec{\nabla} \bigg|_{z=0} + \text{very small terms} \\
\text{Surface 6} &- \vec{\nabla}
\left( x, y, z_0 - \frac{\Delta z}{2}, t \right) \Rightarrow \vec{\nabla}(x, y, z_0, t) - \frac{\Delta z}{2} \frac{\partial}{\partial z} \vec{\nabla} \bigg|_{z=0} + \text{very small terms}
\end{align*}

Plugging this into our equation for the outward flux we find

\[
\Psi = \iiint_{\text{Surface 1}} \frac{\Delta x}{2} \frac{\partial}{\partial x} V \bigg|_{x=0} \, dydz + \iiint_{\text{Surface 2}} \frac{\Delta x}{2} \frac{\partial}{\partial x} V \bigg|_{x=0} \, dydz + \iiint_{\text{Surface 3}} \frac{\Delta y}{2} \frac{\partial}{\partial y} V \bigg|_{y=0} \, dxdz + \\
\iiint_{\text{Surface 4}} \frac{\Delta y}{2} \frac{\partial}{\partial y} V \bigg|_{y=0} \, dxdz + \iiint_{\text{Surface 5}} \frac{\Delta z}{2} \frac{\partial}{\partial z} V \bigg|_{z=0} \, dx dy + \iiint_{\text{Surface 6}} \frac{\Delta z}{2} \frac{\partial}{\partial z} V \bigg|_{z=0} \, dx dy
\]
Letting our infinitesimal volume go to zero so that \( \Delta x \to dx \), \( \Delta y \to dy \), \( \Delta z \to dz \) we find that by putting the pieces together we get,

\[
\Psi = \iiint_{\text{Volume}} \left( \partial_x V_x + \partial_y V_y + \partial_z V_z \right) dx dy dz
\]

or

\[
\iiint_{\text{Volume}} \nabla \cdot \mathbf{V} dx dy dz
\]

This is the Divergence Theorem. It is extremely important to the understanding of electromagnetism. Further it is important for solving problems in electromagnetism. **Physically, the theorem states the total outward flux of a vector field is directly related to the source of the vector field.** (Think of a sink that can be filled forever – or for those of you with small children - floods forever.)

The latter two forms shown at the beginning of the section can be proved from the standard form by letting

\[
\mathbf{V} = f(x,y,z) \mathbf{a}
\]

\[
\mathbf{V}' = \mathbf{a} \wedge \mathbf{V}(x,y,z)
\]

where \( \mathbf{a} \) is a constant vector. The exact derivations of these proofs are left as problems.

### 1.3.2. Stokes’ Theorem

\[
\oint_{\text{closed loop}} \mathbf{V} \cdot d\mathbf{l} = \iint_{\text{enclosed surface}} \left[ \nabla \wedge \mathbf{V} \right] \cdot d\mathbf{s} \quad \leftarrow \text{Standard form}
\]

\[
\oint_{\text{closed loop}} f d\mathbf{l} = - \iint_{\text{enclosed surface}} \left[ \nabla f \right] \wedge d\mathbf{s}
\]

\[
- \oint_{\text{closed loop}} \mathbf{V} \wedge d\mathbf{l} = \iiint_{\text{enclosed surface}} \left( d\mathbf{s} \wedge \nabla \right) \wedge \mathbf{V}
\]
Stokes’ Theorem deals with the twist, or shear, of a vector field. Stokes’ Theorem can be proved in much the same way that the Divergence theorem is proved. To examine shear, one must look at how the field changes as one moves around a loop, as in the figure below.

Hence we look at

\[ \oint \mathbf{V} \cdot d\mathbf{r} \]

Since this loop is entirely in two dimensions we can pick the coordinate system so that we are \( \parallel \) to the x-y plane. Thus,

\[
\oint \mathbf{V} \cdot d\mathbf{r} = \int_{\text{side 1}} (V_x)_{x_0, y_0, z} (\Delta x, y, z) \, dx - \int_{\text{side 2}} (V_y)_{x_0, y_0, z} (\Delta x, y, z) \, dy
\]
\[
+ \int_{\text{side 3}} (V_x)_{x_0, y_0, z} (\Delta x, y, z) \, dx + \int_{\text{side 4}} (V_y)_{x_0, y_0, z} (\Delta x, y, z) \, dy
\]

Expanding, as before, we find

\[
\oint \mathbf{V} \cdot d\mathbf{r} = \int_{\text{side 1}} V_x(x, y_0, z) + \frac{\Delta y}{2} \frac{\partial V_y}{\partial x} \bigg|_{\Delta y=0} \, dx - \int_{\text{side 2}} V_y(x, y_0, z) - \frac{\Delta x}{2} \frac{\partial V_x}{\partial y} \bigg|_{\Delta x=0} \, dy
\]
\[
+ \int_{\text{side 3}} V_x(x_0, y, z) - \frac{\Delta y}{2} \frac{\partial V_y}{\partial x} \bigg|_{\Delta y=0} \, dx + \int_{\text{side 4}} V_y(x_0, y, z) + \frac{\Delta x}{2} \frac{\partial V_x}{\partial y} \bigg|_{\Delta x=0} \, dy
\]
Now flipping integration order and letting $\Delta x, \Delta y \to 0$, we find

$$\oint_{\text{closed loop}} \mathbf{V} \cdot d\mathbf{r} = \iint_{\text{enclosed surface}} \left[ \partial_x V_y - \partial_y V_x \right] dxdy$$

$$= \iint_{\text{enclosed surface}} \left[ \nabla \wedge \mathbf{V} \right] \cdot \mathbf{\hat{z}} dxdy$$

$$= \iint_{\text{enclosed surface}} \left[ \nabla \wedge \mathbf{V} \right] \cdot d\mathbf{s}$$

The left-hand side of this equation shows us that we are measuring the twist of the vector field. The right-hand side of the equation tells us that the twist is related to the curl of the field. Thus, the curl is a measure of the twist of the vector field.

As with the alternate forms of the divergence theorem, the alternate forms of Stokes’ Theorem, shown at the beginning of the section, can be proved from the standard form by letting

$$\mathbf{V} = f(x,y,z) \mathbf{a}$$

$$\mathbf{V}' = \mathbf{a} \wedge \mathbf{V}(x,y,z)$$

where $\mathbf{a}$ is a constant vector. The exact derivations of these proofs are left as problems.

### 1.4. Problems

### 1.5. Supplemental material

### 1.6. Vector Mathematics

Vectors also have an algebra associated with them.

#### 1.6.1.1. Vector Addition and Subtraction

Vector addition:

$$\mathbf{A} + \mathbf{B} = \mathbf{C}$$
If we pick the correct coordinate system – these are very easy problems. What we need is to use an orthogonal coordinate system with unit vectors to describe the direction. Fortunately, in Cartesian coordinates works well. One must be careful using addition in Cylindrical and spherical coordinate as the unit vectors vary from point to point.

\[ A = a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \]
\[ B = b_x \hat{x} + b_y \hat{y} + b_z \hat{z} \]

\[ C = (a_x + b_x) \hat{x} + (a_y + b_y) \hat{y} + (a_z + b_z) \hat{z} \]
\[ = c_x \hat{x} + c_y \hat{y} + c_z \hat{z} \]

Vector Subtraction works in much the same way except one replaces the ‘b’ with ‘-b’

1.6.1.2. More Advanced Vector Algebra and Calculus

Now we need to begin using our vector algebra and calculus. First the algebra. We have two forms of multiplication, \( A \cdot B \) and \( A \wedge B \) for vectors. Can we mix these operations? Yes but what happens? E.g. what happens if \( A \cdot (B \wedge C) \) or \( A \wedge (B \cdot C) \)? The second one is obvious. Noting that \((B \cdot C)\) is a scalar and \(^\wedge\) is a vector operation then the second one is zero. The first complex term is more difficult. We however can solve it following our few simple rules. First,
\[
\mathbf{B} \wedge \mathbf{C} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\mathbf{b}_x & \mathbf{b}_y & \mathbf{b}_z \\
\mathbf{c}_x & \mathbf{c}_y & \mathbf{c}_z \\
\end{vmatrix}
= (\mathbf{b}_y \mathbf{c}_z - \mathbf{b}_z \mathbf{c}_y) \hat{x} + (\mathbf{b}_z \mathbf{c}_x - \mathbf{b}_x \mathbf{c}_z) \hat{y} + (\mathbf{b}_x \mathbf{c}_y - \mathbf{b}_y \mathbf{c}_x) \hat{z}
= (\mathbf{B} \wedge \mathbf{C})_x \hat{x} + (\mathbf{B} \wedge \mathbf{C})_y \hat{y} + (\mathbf{B} \wedge \mathbf{C})_z \hat{z}
\]

Now
\[
\mathbf{A} \bullet (\mathbf{B} \wedge \mathbf{C}) = a_x (\mathbf{B} \wedge \mathbf{C})_x + a_y (\mathbf{B} \wedge \mathbf{C})_y + a_z (\mathbf{B} \wedge \mathbf{C})_z
= a_x (\mathbf{b}_y \mathbf{c}_z - \mathbf{b}_z \mathbf{c}_y) + a_y (\mathbf{b}_z \mathbf{c}_x - \mathbf{b}_x \mathbf{c}_z) + a_z (\mathbf{b}_x \mathbf{c}_y - \mathbf{b}_y \mathbf{c}_x)
\]
\[
= \begin{vmatrix}
a_x & a_y & a_z \\
\mathbf{b}_x & \mathbf{b}_y & \mathbf{b}_z \\
\mathbf{c}_x & \mathbf{c}_y & \mathbf{c}_z \\
\end{vmatrix}
= (\mathbf{A} \wedge \mathbf{B}) \bullet \mathbf{C}
= -(\mathbf{B} \wedge \mathbf{A}) \bullet \mathbf{C}
= -\mathbf{B} \bullet (\mathbf{A} \wedge \mathbf{C})
\]

Finally, we have the similar terms \(\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C})\) or \(\mathbf{A} \bullet (\mathbf{B} \wedge \mathbf{C})\)? Again the second type is simple \((= 0)\) – for the same reasons as above. The first form however is more difficult. Can we tell anything \textit{a priori}? Yes, we know that the final vector has to be at perpendicular to \(\mathbf{A}\). This means that final vector can have components along \(\mathbf{B}\) and \(\mathbf{C}\) but not along \(\mathbf{A}\). Hence \(\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = ? \mathbf{B} + ? \mathbf{C}\). We can figure these terms out, again, by simple algebra.

\[
\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
a_x & a_y & a_z \\
(b_y \mathbf{c}_z - b_z \mathbf{c}_y) & (b_z \mathbf{c}_x - b_x \mathbf{c}_z) & (b_x \mathbf{c}_y - b_y \mathbf{c}_x) \\
\end{vmatrix}
\]
\[
\hat{x}(a_y(b_z \mathbf{c}_x - b_x \mathbf{c}_z) - a_z(b_x \mathbf{c}_y - b_y \mathbf{c}_x))
- \hat{y}(a_x(b_z \mathbf{c}_x - b_x \mathbf{c}_z) - a_z(b_x \mathbf{c}_y - b_y \mathbf{c}_x))
+ \hat{z}(a_x(b_z \mathbf{c}_x - b_x \mathbf{c}_z) - a_y(b_x \mathbf{c}_y - b_y \mathbf{c}_x))
\]
This is somewhat of a mess but we notice that each term has two ‘x’ or two ‘y’ or two ‘z’ terms – so we gather them together and find that

\[
A \wedge (B \wedge C) = \hat{x}(b_x(a_y c_z + a_z c_y) - c_x(a_y b_z + a_z b_y)) \\
+ \hat{y}(a_x b_c c_x - a_x b_y c_x) \\
+ \hat{z}(b_y(a_x c_z + a_z c_y) + c_y(a_x b_z + a_z b_x)) \\
- \hat{y}(a_x b_c c_y - a_x b_y c_y) \\
+ \hat{z}(b_y(a_x c_z + a_z c_y) - c_y(a_x b_z + a_z b_x)) \\
- \hat{y}(a_x b_c c_z - a_x b_z c_z) \\
= (a_x c_z + a_y c_y + a_z c_z)B - (a_x b_z + a_y b_y + a_z b_z)C \\
= B(A \cdot C) - C(A \cdot B)
\]

This is the ‘famous’ BAC-CAB formula that you should have learned in previous classes.

Now we need to go back and look at the differentiation operators and consider the means of the \(\nabla = \hat{x} \partial_x + \hat{y} \partial_y + \hat{z} \partial_z\) operator. First let us consider a scalar field \(j\), which might be the height on the side of a hill or mountain range. Then let us consider a path \(s\). How does \(j\) change as we move along the path \(s\)? For a very small change along \(s\) we find

\[
\frac{\Delta j}{\Delta s \lim_{s \to 0}} = \frac{dj}{ds} = \frac{\partial j}{\partial x} \frac{dx}{ds} + \frac{\partial j}{\partial y} \frac{dy}{ds} + \frac{\partial j}{\partial z} \frac{dz}{ds} \\
= \frac{\partial j}{\partial x} a + \frac{\partial j}{\partial y} b + \frac{\partial j}{\partial z} c; \quad U = a\hat{x} + b\hat{y} + c\hat{z} \\
= \nabla j \cdot U
\]

After a moments consideration one realizes that \(U\) is the direction along which the path \(s\) follows. Hence \(U\) is the instantaneous tangent to the path \(s\).

**Example 1.1**
Let $T = x^2 - y^2 + xyz + 273 \, ^\circ K$ be the temperature variation in a room. In what direction is the temperature varying most rapidly? Where $\frac{dT}{ds}$ is a maximum? From above we know that

$$\frac{dT}{ds} = \mathbf{U} \cdot \nabla T \quad and$$

$$\nabla T = (2x - yz)\hat{x} + (-2y + xz)\hat{y} + (xy)\hat{z}$$

to find the maximum, we simply need to maximize the inner product – hence the angle between $\mathbf{U}$ and $\nabla T$ must be zero. This implies that the direction of maximum temperature change must be along $\nabla T$. Therefore $\nabla T = (2x - yz)\hat{x} + (-2y + xz)\hat{y} + (xy)\hat{z}$ is the direction that we seek.

We can look at this is a more general form. Assume that we have a path along a surface in three dimensions. Can we find a path on the surface in which our scalar field is a constant $= f$. Hence we seek a path in which $\frac{df}{ds} = 0$. Thus, $\frac{df}{ds} = 0 = \mathbf{U} \cdot \nabla f$, where as before $\mathbf{U}$ is the tangent to a path. Using our knowledge of the inner product we find that $\frac{df}{ds} = 0$ if and only if $\mathbf{U} \perp \nabla f$. Therefore $\nabla f$ is a perpendicular or ‘normal’ to the direction of the constant curve. $\nabla f$ is sometimes called the ‘normal’ derivative and $|\nabla f|$ is often written $\frac{df}{dn}$.

Previously we defined the following derivatives

The gradient $\nabla f$

The Divergence $\nabla \cdot \mathbf{J}$

and the Curl $\nabla \times \mathbf{J}$.

Is there anything else that we can do? Why of course.
Putting all of the parts together, we find only four that are useful here and two of those are zero. Of the two that remain the first is easiest to obtain but neither is hard. First,

\[ \nabla \cdot (\nabla \Phi) = \nabla \cdot (\hat{x} \partial_x + \hat{y} \partial_y + \hat{z} \partial_z) \Phi \]

\[ = \left( \hat{x} \partial_x + \hat{y} \partial_y + \hat{z} \partial_z \right) \cdot \left( \hat{x} \partial_x + \hat{y} \partial_y + \hat{z} \partial_z \right) \Phi \]

\[ = \left( \partial_x^2 + \partial_y^2 + \partial_z^2 \right) \Phi \]

\[ = \nabla^2 \Phi \]

is simply the **Laplacian**. The second, and harder one, we have already solved for. We know from our BAC-CAB rule that \( A \wedge (B \wedge C) = B(A \cdot C) - C(A \cdot B) \). Therefore replacing \( A = \nabla \) and \( B = \nabla \) we find, (keeping the order of differentiation correct)

\[ \nabla \wedge (\nabla \wedge J) = \nabla(\nabla \cdot J) - (\nabla \cdot \nabla)J \]

\[ = \nabla(\nabla \cdot J) - (\nabla^2)J \]

Now we need to change course and look at integration.

### 1.6.1.3. Vector Multiplication

Vector Multiplication is not so easy. There are two main options – with other options used in more advanced electromagnetic theory. The most useful are the ‘dot,’ or inner product, and the cross product. **THESE TWO OPTIONS ARE USED BECAUSE THEY FOLLOW WHAT IS OBSERVED IN THE PHYSICAL WORLD. CERTAINLY OTHER ‘MULTIPLICATION’ METHODS ARE DEFINABLE BUT THEY ARE NOT USEFUL.**

Dot (inner) product:
\[ \mathbf{A} \cdot \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \cos(\alpha) \]
\[ = \sqrt{a_x^2 + a_y^2 + a_z^2} \sqrt{b_x^2 + b_y^2 + b_z^2} \cos(\alpha) \]
\[ = a_x b_x + a_y b_y + a_z b_z \]

where \( \alpha \) is the angle between vector \( \mathbf{A} \) and vector \( \mathbf{B} \). Note that there are two forms of the equation and that they are equivalent. This provides an easy method for determining angles etc. Note also that a scalar is the result of this type of multiplication. The law of cosines, and similar trigonometric laws, can be readily derived from the dot product.

The second important type of multiplication is the cross product.

\[ \mathbf{A} \times \mathbf{B} = |\mathbf{A}| |\mathbf{B}| \sin(\alpha) \mathbf{\hat{a}}_n \]
\[ = \begin{vmatrix} \mathbf{\hat{x}} & \mathbf{\hat{y}} & \mathbf{\hat{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \]
\[ = (a_y b_z - a_z b_y) \mathbf{\hat{x}} + (a_z b_x - a_x b_z) \mathbf{\hat{y}} + (a_x b_y - a_y b_x) \mathbf{\hat{z}} \]
\[ = c \mathbf{\hat{a}}_n \]

where \( \mathbf{\hat{a}}_n \) is the unit vector orthogonal to \( \mathbf{A} \) and \( \mathbf{B} \). This new vector is out of the plane defined by the vectors \( \mathbf{A} \) and \( \mathbf{B} \). We can play games and prove how these two types of multiplication interact. This however will be left to homework problems. Note that the cross product is related to the ‘outer’ product multiplication of two Tensors (e.g. n dimensional matrices). In mathematical textbooks, this multiplication is often denoted with a ‘wedge’ symbol (\( \wedge \)). We will use that same symbolism, rather than the \( \times \) or \( \otimes \) symbols to avoid confusion with the letter \( x \).

Now a scalar field is simply a standard function. As freshmen, we learned how to go beyond adding subtracting, multiplying and dividing and moved on to differentiation and integration. We can do the same thing with vector fields.
1.6.1.4. Differentiation

Let

\[ \mathbf{J}(x,y,z,t) = j_x(x,y,z,t)\hat{x} + j_y(x,y,z,t)\hat{y} + j_z(x,y,z,t)\hat{z} \]

then

\[ \partial_x \mathbf{J}(x,y,z,t) = \partial_x j_x(x,y,z,t)\hat{x} + \partial_x j_y(x,y,z,t)\hat{y} + \partial_x j_z(x,y,z,t)\hat{z} \]

is an example of scalar differentiation. Note that \( \partial_x \) is short hand for \( \frac{\partial}{\partial x} \)

\[ (\hat{x}\partial_x) \cdot \mathbf{J}(x,y,z,t) = \partial_x j_x(x,y,z,t)\hat{x} \cdot \hat{x} + \partial_x j_y(x,y,z,t)\hat{x} \cdot \hat{y} + \partial_x j_z(x,y,z,t)\hat{x} \cdot \hat{z} \]

is an example of inner-product differentiation and

\[ (\hat{x}\partial_x) \wedge \mathbf{J}(x,y,z,t) = \partial_x j_x(x,y,z,t)\hat{x} \wedge \hat{x} + \partial_x j_y(x,y,z,t)\hat{x} \wedge \hat{y} + \partial_x j_z(x,y,z,t)\hat{x} \wedge \hat{z} \]

is an example of cross-product differentiation. The most common differential vector is the ‘del’ vector, where \( \nabla = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z \). Other forms of this vector can be found in other coordinate systems and are derived such that identical results are found independent of coordinate system.

There are two major operations that the del operator has on vector fields, the Divergence

\[ \nabla \cdot \mathbf{J} = \partial_x j_x(x,y,z,t) + \partial_y j_y(x,y,z,t) + \partial_z j_z(x,y,z,t) \]

and the curl

\[ \nabla \wedge \mathbf{J} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ J_x & J_y & J_z \end{vmatrix} \]

\[ = \hat{x}(\partial_y j_z - \partial_z j_y) - \hat{y}(\partial_z j_x - \partial_x j_z) - \hat{z}(\partial_x j_y - \partial_y j_x) \]

As we will show below, these two operations have physical significance. \( \nabla \cdot \) describes the source of a diverging vector field while \( \nabla \wedge \) describes the source of a twisting vector field. **WITH ONE EXCEPTION, ALL VECTOR FIELDS ARE ‘PRODUCED BY’ A TWISTING SOURCE OR A DIVerging SOURCE OR SOME COMBINATION OF SUCH SOURCES.**

The one exception is a uniform (non zero) field, but as will be seen later in the semester, this is

type of vector field not physically possible for the electric of magnetic fields. (While uniform

fields are nice to use in homework problems and as approximations, a truly uniform field across

all space results in infinite energy – perhaps we can solve the energy crisis – not.)

The final major operation of the del operator is an operation on a scalar field, which is known as

the gradient

\[ \nabla j = (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z)j(x,y,z,t) = \partial_x j(x,y,z,t)\hat{x} + \partial_y j(x,y,z,t)\hat{y} + \partial_z j(x,y,z,t)\hat{z} \]

We will return to the Del vector shortly.

1.6.1.5. LINE Integrals:

The work that a force does on an object is “W=Fr” where r is the distance moved. In reality, the

work done depends on the direction of the force. For example gravity does not do work on a box

that moves across a flat floor. So in fact \( W = F \cdot r \). If the force changes with position, ie is a

vector field, then the work becomes \( W = \int F \cdot dr \), where \( dr \) is a vector of infinitesimal length

along the \( r \) direction.

\[ \int \]

Example 1.2

Let \( F = xy\hat{x} - y^2\hat{y} \) and \( dr = \hat{x}dx + \hat{y}dy \) so that
\[ W = \int xy\,dx - y^2\,dy \]

Let us examine 4 paths from (0,0) to (2,1):

1: \( y = \frac{1}{2}x \) (line)
2: \( y = \frac{1}{4}x^2 \) (parabola)
3: \( x = 2t^3, \ y = t^2 \) (parabola)
4: \( y|0 \to 1 \ then \ x|0 \to 2 \) (broken line)

Path 1: \( y = \frac{1}{2}x \) and \( dy = \frac{1}{2}dx \) so that

\[
W = \int_{0}^{2} \left( \frac{1}{2}x^2\,dx - \frac{1}{8}x^3\,dx \right)
= \frac{3}{8} x^3 \bigg|_{0}^{2} = 1
\]

Path 2: \( y = \frac{1}{4}x^2 \) and \( dy = \frac{1}{2}xdx \) so that

\[
W = \int_{0}^{1} \left( 2y\,dy - y^2\,dy \right)
= y^2 \bigg|_{0}^{1} = \frac{2}{3}
\]

Path 3: \( x = 2t^3 \ and \ dx = 6t^2\,dt \)
\( y = t^2 \) and \( dy = 2tdy \) so that
\[ W = \int_0^1 \left( 2t^5 - 6t^2 dt - t^4 + 2tdt \right) = \frac{7}{6} \]

Path 4:
\[ W = W = \int_0^2 x \ y \ dx - \int_0^1 y^2 \ dy = \frac{5}{3} \]

Thus we find that amount of work done by the force in this field depends on the direction of travel. This is not unlike sliding a very heavy object across a floor. **Distance and direction, e.g. path, do matter.**

In some instances – for example if there were no friction – then the path that we took would not matter. This is an example of a conservative force, e.g. the energy is conserved. (The example above is a non-conservative force.) This idea of conservative/non-conservative can be extended to general vector fields. Further, we will come up with generalized rules for conservative/non-conservative field. To do this we will start with

\[ \mathbf{F} = -\nabla \Phi = -\mathbf{x} \partial_x \Phi - \mathbf{y} \partial_y \Phi - \mathbf{z} \partial_z \Phi \quad \text{Then} \]

\[ \nabla \wedge \mathbf{F} = -\begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ \partial_x & \partial_y & \partial_z \\ \partial_x \Phi & \partial_y \Phi & \partial_z \Phi \end{vmatrix}, \quad \text{noting} \quad \partial_x \partial_y = \partial_y \partial_x \]

\[ = 0 \]

The converse of the above is also true, if

\[ \nabla \wedge \mathbf{F} = 0 \quad \text{then} \]

\[ 0 = \partial_x F_y - \partial_y F_x = \partial_x F_z - \partial_z F_x = \partial_x F_z - \partial_z F_x \]

\[ \Rightarrow F_y = \partial_y \int F_x \ dx = \partial_y \int F_z \ dz \Rightarrow F_z = -\partial_z \Phi, \ F_x = -\partial_x \Phi \quad \text{etc.} \]

\[ \mathbf{F} = -\nabla \Phi \]
For purely historical reasons in electromagnetism, we have chosen to use \( F = -\nabla \Phi \) rather than \( F = +\nabla \Phi \) to arrive at our potential \( \Phi \). In either case, with the exception of the minus sign, the algebra and the results are the same. (In gravity related potentials, the positive sign is used.)

So now let us consider the work of a conservative force when moving from point a to point b

\[
W = \int_a^b \mathbf{F} \cdot d\mathbf{r} \\
= -\int_a^b \nabla \Phi \cdot d\mathbf{r} \\
= -\int_a^b \left( \hat{x} \partial_x \Phi + \hat{y} \partial_y \Phi + \hat{z} \partial_z \Phi \right) \left( \hat{x} dx + \hat{y} dy + \hat{z} dz \right) \\
= -\int_a^b \partial_x \Phi dx = \int_a^b \partial_y \Phi dy = \int_a^b \partial_z \Phi dz \\
= -(\Phi(b) - \Phi(a))
\]

which is clearly independent of path. This was our original definition of a conservative force. Gravity, and electrostatic forces are two examples of conservative forces.

**Example 1.3**

\[
\mathbf{F} = (2xy + z^3)\hat{x} + (x^3)\hat{y} + (3xz^2 + 1)\hat{z}
\]

Is this a conservative field? It is if \( \nabla \wedge \mathbf{F} = 0 \).

\[
\nabla \wedge \mathbf{F} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial_x & \partial_y & \partial_z \\
(2xy + z^3) & (x^3) & (3xz^2 + 1)
\end{vmatrix} \\
= (0 - 0)\hat{x} + (3z^2 - 3z^2)\hat{y} + (2x - 2x)\hat{z} \\
= 0
\]

This implies that \( \mathbf{F} = \nabla \Phi \). So what is \( \Phi \)?
\[ W = \int \mathbf{F} \cdot d\mathbf{r} \]
\[ = \int \nabla \Phi \cdot d\mathbf{r} = \Phi \]
\[ = \int \left( \mathbf{\hat{x}} \partial_x \Phi + \mathbf{\hat{y}} \partial_y \Phi + \mathbf{\hat{z}} \partial_z \Phi \right) \cdot \left( \mathbf{\hat{x}} dx + \mathbf{\hat{y}} dy + \mathbf{\hat{z}} dz \right) \]
\[ = \int \partial_x \Phi \, dx = \int \partial_y \Phi \, dy = \int \partial_z \Phi \, dz \]
\[ = \int (2xy + z^3) \, dx = \int (x^2) \, dy = \int (3xz^2 + 1) \, dz \]
\[ = x^2 y + xz^3 + \text{const in } x = x^3 y + \text{const in } y = xz^3 + z + \text{const in } z \]
\[ \Rightarrow \Phi = x^2 y + xz^3 + z + \text{const} \tan t \]

**Example 1.4**

The electric field is given by \( \mathbf{E} = \frac{1}{4\pi \varepsilon} \frac{q}{r^2} \mathbf{\hat{r}} \). What is the potential?

\[-\Phi = \int \mathbf{E} \cdot d\mathbf{r} = \int -\nabla \Phi \cdot d\mathbf{r} \]
\[ = \int \frac{1}{4\pi \varepsilon} \frac{q}{r^2} \mathbf{\hat{r}} \mathbf{\cdot} d\mathbf{r} \mathbf{\hat{r}} = \int \frac{1}{4\pi \varepsilon} \frac{q}{r^2} \, dr = -\frac{1}{4\pi \varepsilon} \frac{q}{r} \]

which is our standard potential introduced in earlier classes.

### 1.7. Describing Space

1.7.1.1. **Coordinate systems**

1.7.1.1.1. **Cylindrical**

1.7.1.1.2. **Spherical**

1.7.1.1.3. **Generalized curvilinear**

1.7.1.2. **Points, curves, surfaces and volumes**

Understanding how points, curves, surfaces and volumes are described is important for understanding electromagnetism and for solving problems.
Using one of our coordinate systems, we can describe any point in real space. The standard notation is to give the positions as \((x,y,z)\), \((r,\phi,z)\) or \((p,\theta,\phi)\). For example, the point at \(x=5\) on the \(x\)-axis would be given as \((5,0,0)\). The point at \(x=0, y=2, z=6\) would be given as \((0,2,6)\). These points can also be described in cylindrical and spherical coordinates as shown in example XX.

A curve or path is a one-dimensional object that can be written as \((x(s), y(s), z(s))\), where \(s\) is a free parameter. You will note that \(x\), \(y\), and \(z\) are now functions of this free parameter and thus they are allowed to vary as \(s\) changes. While this object might follow a path through our three-dimensional space, the path has only one free parameter and is thus a one-dimensional object. (In a class on topology, one would learn that one can transfer the curve to a space in which only one of the components changes with \(s\). The transformation from Cartesian to cylindrical or spherical is a manifestation of such a topological transformation.)

Surfaces and volumes follow naturally from the definition of a curve. A surface is a two-dimensional object that can be written as \((x(s,t), y(s,t), z(s,t))\), where both \(s\) and \(t\) are free parameters. (Here, \(t\) does not necessarily stand for time; however, either \(s\) or \(t\) may represent time.) Finally volumes are three-dimension objects that can be written as \((x(s,t,u), y(s,t,u), z(s,t,u))\). Like points and curves, surfaces and volumes can be described in several different coordinate systems. An example of a coordinate system translation for a surface is given in example XX.

1.7.1.3. Tangents and Normals

Now that we have developed our basic mathematical operations, we need to return to our study of points, curves, surface and volumes. The first thing to note is that point in space \((x,y,z)\) has a natural vector associated with it, namely the vector \((x,y,z)\) that runs from the origin, \((0,0,0)\), to the point \((x,y,z)\).
Here, \( \mathbf{L} \) is the vector describing the position along the curve at parameter \( s \), while \( \mathbf{L}' \) is the vector at \( s + \Delta s \). It is obvious that as \( \mathbf{L}' \) comes closer to \( \mathbf{L} \), then \( \Delta \mathbf{L} \) comes closer to the tangent, \( \mathbf{T} \), of the curve at point described by the vector \( \mathbf{L} \). Thus

\[
\mathbf{T} = \lim_{\Delta s \to 0} \frac{\Delta \mathbf{L}}{\Delta s} = \frac{d\mathbf{L}}{ds}
\]

For a two dimensional object, a surface, we have two free parameters. From this, it is easy to extend our description of tangents to two dimensions. Thus, we have

\[
\mathbf{T}_1 = \lim_{\Delta s \to 0} \frac{\Delta \mathbf{L}}{\Delta s} = \frac{d\mathbf{L}}{ds} \quad \text{and} \quad \mathbf{T}_2 = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{L}}{\Delta t} = \frac{d\mathbf{L}}{dt}
\]

where here our second free parameter is ‘\( t \)’. (There are a couple of additional requirements which we will gloss over. 1) That \( t \) and \( s \) are independent, i.e., \( t \neq f(s) \), 2) That the tangents are not parallel, i.e., \( \mathbf{T}_1 \parallel \mathbf{T}_2 \), 3) That \( s \) and \( t \) describe the surface in a single valued manner, i.e., The same point in space is not described with two different values of \( s \) or \( t \).) These two tangents describe a planar surface that is locally tangent to the surface. Now, if we want to determine the normal to the surface at that point, we simply take the normalized cross products of the tangents. (Remember that the tangents have magnitude as well as direction.) Thus,

\[
\mathbf{n} = \frac{\mathbf{T}_1 \times \mathbf{T}_2}{|\mathbf{T}_1 \times \mathbf{T}_2|}
\]

Notice that reversing the order of the cross product will give you the opposite sign for the normal. By convention, we will pick the outside normal on closed surfaces in electromagnetism.
At this point, we need to look at integration over curves and surfaces. Most of you can perform an integral of the forms

\[ \int f(x) \, dx \]

or

\[ \int f(x,y) \, dx \, dy \]

In the above, the free parameters are assumed to be \(x\) and \(y\). Now however, we are dealing with curves and surfaces that are not always so simple. More general forms of the integrals are:

\[ \int f(x(s)) \, ds \]

or

\[ \int f(x(s),y(s)) \, ds \, dt \]

To convert to our standard form, we must know the transformation between \(s\) and \(t\) and \(x\) and \(y\).

Example:

Now, what happens if our function is a vector field as opposed to the scalar field given above. Then we want to look at

\[ \int \mathbf{F}(x(s)) \cdot ds \]

or

\[ \int \mathbf{F}(x(s),y(s)) \cdot ndst \]

or similar forms.
1.7.2. Imaginary numbers, Phase and notation

1.7.3. Field lines

1.7.4. Delta functions

1.7.5. SIDE NOTE ON VECTORS

We know that in matrix notation the inner product is

\[ \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \cdot \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \]

\[ = (A_x A_y A_z) \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \]

\[ = A_x B_x + A_y B_y + A_z B_z \]

What we have above is the form

\[ \mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \cdot \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} \]

1.7.5.1. Dirac Delta function

1.7.5.2. Kronicker Delta function

1.7.6. Greens’ Theorem and Greens’ Functions

1.7.6.1. Greens’ Theorems

Form 1

\[ \oint \text{Edge of surface} \quad u \, dx + v \, dy = \int_{\text{Surface}} \left( \partial_y v - \partial_x u \right) \, dx \, dy \]

Form 2

\[ \iiint_{\text{Volume}} \left( u \nabla \cdot \nabla v - v \nabla \cdot \nabla u \right) \, d\tau = \iiint_{\text{Volume}} \left( u \nabla v - v \nabla u \right) \cdot \, ds \]

Form 3

\[ \iiint_{\text{Volume}} \left( u \nabla \cdot \nabla v + v \nabla \cdot \nabla u \right) \, d\tau = \iiint_{\text{Edge of volume}} \left( u \nabla v \right) \cdot \, ds \]
All three forms of Green’s Theorem are generalizations of the Divergence theorem and the fundamental theorem of calculus.

Prove of Form 1

From the fundamental theorem of calculus, we know that,

$$\int_a^b \frac{\partial}{\partial x} f(x) \, dx = f(b) - f(a).$$

First, let us consider two functions \(u(x,y)\) and \(v(x,y)\). Now we want to look at the surface integral of \(\frac{\partial}{\partial x} v(x,y)\) over an enclosed area \(A\) in the \(x\)-\(y\) plane.

$$\int_c^d \int_a^b \frac{\partial}{\partial x} v(x,y) \, dx \, dy = \int_c^d v(b,y) - v(a,y) \, dy$$

by the fundamental theorem of calculus. Now let us integrate all the way around the edge of the area.

$$\oint_{\partial A} v(x,y) \, dy = \int_c^d v(b,y) \, dy - \int_c^d v(a,y) \, dy$$

$$= \int_c^d v(b,y) - v(a,y) \, dy$$

$$= \int_c^d \int_a^b \frac{\partial}{\partial x} v(x,y) \, dx \, dy$$

Likewise, we can show that
\[ \oint_{\partial A} u(x,y) \, dx = - \int_c^d \int_a^b \frac{\partial}{\partial y} u(x,y) \, dx \, dy \]

Thus,

\[ \oint_{\partial A} u(x,y) \, dx + v(x,y) \, dy = \int_c^d \int_a^b \frac{\partial}{\partial x} v(x,y) - \frac{\partial}{\partial y} u(x,y) \, dx \, dy \]

While what we have done is for a rectangle, it also applies to any arbitrarily shaped area. This can be seen by piecing together the area as in the figure below, and noting that adjacent sides cancel as they are in opposite directions.

We find that the sides of two adjacent rectangles will cancel as the direction of integration is opposed. This allows us to ‘build’ our arbitrary structure out of a set of rectangles. What we have just proven is known as **Green’s Theorem**. (There is also a way to prove Green’s Theorem for a general shape.) The surface is some area in the x-y plane and the enclosing curve C is in the counter-clock-wise direction. The only requirement, which we have brushed over, is that \( u(x,y) \) and \( v(x,y) \) must have continuous first partial derivatives at every point in A. (This is a requirement found in the Fundamental Theorem of Calculus.)

The other two forms of Green’s Theorem can be derived from combinations of the chain rule

\[ \nabla \cdot (u \nabla v) = \nabla v \cdot \nabla u + u \nabla^2 v \]

\[ - \left( \nabla \cdot (v \nabla u) = \nabla v \cdot \nabla u + v \nabla^2 u \right) \]

\[ \nabla \cdot (u \nabla v) - \nabla \cdot (v \nabla u) = u \nabla^2 v - v \nabla^2 u \]

and the Divergence theorem. **The exact derivations of these proofs are left as problems.**

**Example 1.5**
Using our example from a non-conservative force, \( \mathbf{F} = xy \hat{x} - y^2 \hat{y} \) along path 2: \( y = \frac{1}{4} x^2 \) (parabola) and back path 4: \( y[0 \to 1 \to 0 \to 2] \) (broken line). Then the work required to make the loop is,

\[
W = \oint_{\partial A} xy \, dx - y^2 \, dy
\]

\[
= \iint_{A} \left( \frac{\partial}{\partial x} (y^2) - \frac{\partial}{\partial y} (xy) \right) \, dx \, dy
\]

\[
= \iint_{A} (-x) \, dx \, dy
\]

\[
= \int_{y=0}^{y=2} dy \int_{x=0}^{x=2} dx (-x)
\]

\[
= -\int_{y=0}^{y=2} dy \left( 2\sqrt{y} \right)^2 = -1
\]

This is the same result as one would find from before for the path PATH 2 – PATH 4 in the earlier example.

**Example 1.6**

Now let us consider

\[ v = V_x, \text{and } u = -V_y \text{ with} \]

\[ \vec{V} = V_x \hat{x} + V_y \hat{y} \]

Therefore

\[ \partial_x u - \partial_y v = (\hat{x} \partial_x + \hat{y} \partial_y) \cdot (V_x \hat{x} + V_y \hat{y}) = \nabla \cdot \vec{V} \]
By Green’s theorem
\[ \int_A \partial_x u(x,y) - \partial_y v(x,y) \, dx \, dy = \int_A \nabla \cdot \mathbf{V} \, dx \, dy \] and
\[ \oint_{\partial A} u(x,y) \, dx + v(x,y) \, dy = \oint_{\partial A} \mathbf{V} \cdot d\mathbf{r} \]

Now \( d\mathbf{F} = \hat{x} \, dx + \hat{y} \, dy \) and the differential normal in the x-y plane is \( d\mathbf{l} = \hat{x} \, dy - \hat{y} \, dx \) from earlier. Therefore we find that
\[ \oint_{\partial A} \nabla \cdot \mathbf{V} \, d\mathbf{r} = \int_A \nabla \cdot \mathbf{V} \, dx \, dy. \]

This is the two-dimensional version of the Divergence theorem.

**Example 1.7**

Now, if we pick
\[ u = V_y, \text{ and } v = V_x \] with
\[ \mathbf{V} = V_x \hat{x} + V_y \hat{y} \]

then
\[ \hat{z} \cdot \nabla \wedge \mathbf{V} = \partial_x V_y - \partial_y V_x \] and
\[ \nabla \cdot d\mathbf{r} = V_x \, dx + V_y \, dy \]

Now using Green’s Theorem we find
\[ \oint_{\partial A} v(x,y) \, dx + u(x,y) \, dy = \int_c^d \int_a^b \partial_x u(x,y) - \partial_y v(x,y) \, dx \, dy \]
\[ \oint_{\partial A} \nabla \cdot d\mathbf{r} = \int_A (\nabla \wedge \mathbf{V}) \cdot \hat{z} \, dx \, dy \]

This is Stokes’ Theorem in two-dimensions.

1.7.6.2. Greens’ Functions
1.7.7. Helmholtz’ Theorem (independence of ‘∇ ∧’ and ‘∇ •’ fields)

Helmholtz’s Theorem states that any arbitrary vector field is a combination of diverging, twisting and constant vector fields. This means that physically our sources for the electric and magnetic fields have to have divergence and Stokes’ like components. From our understanding of each t
Section 2. Overview of Fluid Mechanics

We “know” that there are four different types of matter in the universe, solids, liquids, gases and plasmas. Each have distinct physical characteristics. For example, solids will retain their shape under most conditions. Liquids will fill a volume up to the point at which they would have to expand. Gases will expand to fill a volume. Plasma have gas-like characteristics while also being subject to electromagnetic fields. These additional forces cause plasmas to support waves and other phenomena that are not possible in gases. Liquids, gases and plasmas all fit within “Fluids.” Fluids are distinct from solids in how they respond to shear forces and the fact that they “flow”.

2.1. “Lines” of flow

We can examine flow in a fluid a number of ways – here we are specifically concerned with ‘streams’ or ‘lines’. We have all seen ads in which a smoke stream is seen to pass over a car (or plane or…).

- **Laminar flow**
- **Turbulent flow**

There are effectively three main types of lines.
1. “Pathline” – The path of an individual particle in a flow.
2. “Streamline” – Path tangent to local direction of flow.
3. “Streakline” – The path that is formed by accounting for all particles that pass through a single point.

A fourth line type that is used occasionally is the ‘timeline’

Under steady state conditions, the first three lines are identical. If the flow is changing, they can be different. Mathematically we can define them as follows:

**Pathline.**

Clearly a particle will follow the standard equation of motion
\[
\Delta \mathbf{r} (x, y, z, t) = \mathbf{v} (x, y, z, t) \Delta t
\]

Thus the change in position is determined by the current particle velocity. Likewise the particle velocity will change in response to external forces.

**Streamline.**
Here the path that is determined by following the tangent to the local flow velocities. A simple picture gives one an understanding of this line. First, we plot all of the local velocities in the flow field. Then we simply connect the lines…

\[ T = \frac{\Delta L}{\Delta s} \left|_{\lim_{s \to 0}} \right. = \frac{dL}{ds} \]

By definition the tangent of the pathline is parallel to the velocity.
\[ T = \gamma V \]

Thus
\[ \frac{dL}{dt} = \gamma V \]

\[ dx = \gamma V_x (x, y, z, t) dt \]
\[ dy = \gamma V_y (x, y, z, t) dt \]
\[ dz = \gamma V_z (x, y, z, t) dt \]

When these last three equations are solved simultaneously, one finds the pathline.

**Streakline.**
A streakline is what you always observed with smoke. Here we are following the location of particles that all come from the same position – but at different starting times. Thus
\[ r(t_0) = r \left( (x_0, y_0, z_0, t_0) \text{ starting point, } t_0 \right) \]

Here at a given time “t” we have a curve that is traced out with the free parameter “t_0” – and “r” is the current position vector of the particle that started at t_0.

### 2.2. Stress Tensor

Forces act throughout fluids, often with the force varying smoothly from point to point. If we have a given surface area across which a force is acting, we can have a number of things happen. If the force is parallel to the surface normal, the area generally gets pushed forward. If the force is perpendicular to the normal, the area will in general get twisted. In addition, as these vary
‘smoothly’ across the surface, and the bigger the area the bigger the total force, we generally normalize by surface area. We define these as the Normal and Shear stress. (And as you might expect, they have units of pressure!)

\[ \tau_n = \frac{dF \cdot \hat{n}}{dA} \equiv \tau_{n,n} \]
\[ \tau_{n,t} = \frac{dF \cdot \hat{T}}{dA} \]

Here we have defined the shear along the normal and along a unit tangent. Now, we know that any surface has two distinct tangents AND at a given point we can have surfaces pointing in multiple directions. Let’s deal with the multiple normals first. We know that any normal that we do have, we must be able define in terms of a coordinate system having three independent directions. For simplicity, we will use the Cartesian coordinate system – implying we have \( \tau_{x,x}, \tau_{y,y}, \tau_{z,z} \). Likewise, for each of the three main directions, we must be able to divide our tangents along at most two other directions – and again we will use the Cartesian coordinate system to subdivide things

\( \tau_{x,y}, \tau_{x,z} \)
\( \tau_{y,x}, \tau_{y,z} \)
\( \tau_{z,x}, \tau_{z,y} \)

As you might guess, we need to be able to apply all of these within vector equation and thus we define a stress tensor

\[ \bar{\tau} = \begin{bmatrix} \tau_{x,x} & \tau_{y,y} & \tau_{z,z} \\ \tau_{x,y} & \tau_{y,x} & \tau_{z,y} \\ \tau_{x,z} & \tau_{y,z} & \tau_{z,z} \end{bmatrix} \]

(and as you might guess, pressure also is a tensor).

### 2.3. Velocity Distribution

As gas atoms/molecules (or particles) move through out a volume they collide and randomly distribute their energy. A basic tenant of statistics is that random processes result in Normally distributed results. This is know as the central limit theorem, see for example Box Hunter and Hunter, “Statistics for experimenters” (This title might not be quite correct.) The normal distribution is given by

\[ p(y) = \frac{1}{\sigma} \exp \left[ \frac{-(y - \eta)^2}{\sigma^2} \right] \]

Here \( \sigma^2 \) is the population variance, \( \sigma \) is the population standard deviation, and \( \eta \) is the central value. The constant is usually set such that the total probability is 1. The distribution looks like
2.4. Concept of Temperature

Maxwell and Boltzmann proposed that this same distribution can be used to model the velocity distribution of particles in thermal equilibrium. (This is known as the Maxwell Distribution, Boltzman Distribution and the Maxwell-Boltzman Distribution.) Using this assumption we have that the velocity distribution is

\[ f(v) = \text{const} \times \exp\left(\frac{-m(v)^2}{2kT}\right), \]

where \( v \) is the velocity, \( v_0 \) is the average or drift velocity (which is often zero), \( k \) is Boltzmann’s constant, \( T \) is the gas temperature in Kelvin and \( m \) is a particle mass. Notice that the temperature is often stated in units of energy (typically eV). When this is done you are using \( kT \) not just \( T \). Note also that the temperature is a measure of the standard deviation of the velocity distribution – hence it is a measure of the variation in the velocity. We can also rewrite this as an energy distribution

\[ f(\varepsilon) = \text{const} \times \exp\left[\frac{-\varepsilon}{kT}\right], \]

where the two constants are different.

To get the constant for the velocity distribution, we set the constant such that

\[ n = \iiint f(v) dv \]

where \( n \) is the particle density then we find that for our gas particles we get a velocity distribution of

\[ f(v) = \frac{1}{\sqrt{2\pi kT}} \exp\left(\frac{-m(v)^2}{2kT}\right), \]
\[ f(v) = n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -\frac{1}{2} \frac{m(v - v_0)^2}{kT} \right) \]

The velocity distribution is such that the peak is at the average velocity and is bell shaped.

**Problem 1**

Show that if in one dimension, \( \int f(v) \, dv = n \), then \( \text{const} = n \left( \frac{m}{2\pi kT} \right)^{1/2} \).

Also show that if in three dimensions, \( \int f(\mathbf{v}) \, d\mathbf{v} = n \), then \( \text{const} = n \left( \frac{m}{2\pi kT} \right)^{3/2} \).

Typically, the velocity distributions are well behaved and we have distributions as such:

The Maxwellian Distribution

![Maxwellian Distribution](image)

Where

\[ f(\mathbf{v}) = n \left( \frac{m}{2\pi kT} \right)^{1/2} \exp \left[ -\frac{m\mathbf{v}^2}{2kT} \right]. \]

This is what happens if there are enough collisions to evenly distribute the energy. In other cases, however, we may be working in with a system in which the rate of collisions is not large enough to evenly distribute the energy. Some typical examples are:
The bi-Maxwellian Distribution

Where

\[ f(v) = (1 - \alpha)n \left( \frac{m}{2\pi k T_1} \right)^{1/2} \exp \left( \frac{-m(v)^2}{2k T_1} \right) \]

\[ + (\alpha)n \left( \frac{m}{2\pi k T_2} \right)^{1/2} \exp \left( \frac{-m(v)^2}{2k T_2} \right) . \]

The drifting Maxwellian Distribution
Where

\[ f(v) = n \left( \frac{m}{2 \pi kT} \right)^{1/2} \exp \left[ -\frac{m(v - v_0)^2}{2kT} \right] \]

and the ‘bump-on-tail’
Where
\[ f(v) = f_{\text{Maxwell}} + f_{\text{drift}}. \]

While other distributions are certainly possible, these four comprise most of what is observed in most fluids.

At this point we can examine the velocity integral in more detail.
\[ n = \iiint f(\mathbf{v})d\mathbf{v} \]
\[ = \iiint f(\mathbf{v})d\mathbf{v}_x d\mathbf{v}_y d\mathbf{v}_z \quad \text{(Cartesian)} \]
\[ = \iiint f(\mathbf{v})v^2d\mathbf{v} \sin \theta d\theta d\phi \quad \text{(Spherical)} \]
\[ = 4\pi \int_0^\infty f(\mathbf{v})v^2dv \]
\[ = 4\pi \int_0^\infty n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left[ \left( -\frac{1}{2} \frac{mv^2}{kT} \right) \right] v^2dv \]
\[ = \int_0^\infty 4\pi n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -\frac{1}{2} \frac{mv^2}{kT} \right) v^2dv \quad \text{speed distribution} \]
\[ \Rightarrow \frac{dn}{dv} = f_{\text{speed}}(v) = 4\pi n v^2 \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -\frac{1}{2} \frac{mv^2}{kT} \right) \]

Likewise, we can convert this into energy space to get the energy distribution
\[ n = \int_0^\infty 4\pi n \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left( -\frac{1}{2} \frac{mv^2}{kT} \right) v^2dv \quad \text{noting } E = \frac{1}{2}mv^2, \ dE = mvdv \]
\[ = \int_0^\infty 2\pi n (\pi kT)^{-3/2} \sqrt{E} \exp \left( -\frac{E}{kT} \right) dE \quad \text{energy distribution} \]
\[ = \int_0^\infty \frac{dn}{dE} = g(E) = 2\pi n (\pi kT)^{-3/2} \sqrt{E} \exp \left( -\frac{E}{kT} \right) \]

which of course gives us our energy distribution constant from above

We can further use these equations to determine the most probable speed,
\[
\frac{d}{dv} f_{\text{speed}}(v) = 0
\]
\[
\downarrow
\]
\[
v_p = \left(\frac{2kT}{m}\right)^{1/2}
\]

Most probable energy

\[
\frac{d}{dE} g(E) = 0
\]
\[
\downarrow
\]
\[
E_p = \frac{1}{2} kT
\]

Mean (average) velocity

\[
\int v f_{\text{speed}}(v) dv = v_{\text{ave}} = 0
\]

Mean (average) speed

\[
\int_0^\infty v f_{\text{speed}}(v) dv = v_{\text{ave}} = \left(\frac{8kT}{\pi m}\right)^{1/2}
\]

Mean (average) energy

\[
\int_0^\infty E g(E) dE = E_{\text{ave}} = \frac{3}{2} kT
\]

RMS velocity

\[
\int |v| f_{\text{speed}}(v) dv = v_{\text{RMS}} = \sqrt{\frac{3kT}{m}}
\]

2.5. Gas pressure

2.5.1. Ideal Gas Law
Lets assume that we have a particle in a box of sides $l$. Assume that the particle is reflected with no loss of energy (why?) when it hits wall A1. Thus $\Delta E = 0$ but

$$\Delta M = -\frac{mv_x}{\text{final momentum}} - \frac{mv_x}{\text{initial momentum}} = -2mv_x$$

Where we have assumed that $v_y = 0 = v_z$. Now let us assume that the particle travels across the volume without striking anything else and reflects off side A2. The round trip time is $\Delta t = \frac{2l}{v_x}$

The force applied to side A1 is

$$F = -\frac{\Delta M}{\Delta t} = \frac{2mv_x}{2l/v_x} = \frac{mv_x^2}{l}$$

and the pressure is

$$P = \frac{F}{l^2} = \frac{mv_x^2}{l^3}$$

Putting in more particles we find

$$P = \sum_{\text{all particles}} \frac{mv_x^2}{l^3} = m\frac{N}{l^3} \sum_{\text{all particles}} \frac{mv_x^2}{N} = mn \langle v_x^2 \rangle = 2n \frac{1}{2} n \langle v_x^2 \rangle = 2n \langle v_x^2 \rangle = nkT - \text{the ideal gas law!}$$

(Note that this is done in 1-D. If we were to do it in 3-D we of course get the same result.)

2.5.2. XYZ Gas Law

Now if we add van der Wall’s Force and account for the size of the particles, we get the more complex version of the gas law

$$P + \left( \frac{a}{\sqrt{V^2}} \right) \left( V - \frac{b}{\text{particle size}} \right) = NkT$$

For the most part, the ideal gas law is sufficient to accurately describe our systems.

2.6. Concept of Mean-free path

As atoms/molecules pass through a volume, they collide with other atoms/molecules. A prime example of this is Brownian motion – the motion of a dust particle in air that moves in disjointed fashion.
We can calculate the mean-free path, mfp, if we consider the following picture.

If we were to repeatedly fire test particles at the target, a fraction of the test particles will be scattered by collisions with atoms in the target. The fraction scattered is simply the area ratio.

\[ \text{Fraction scattered} = \frac{N \sigma}{A} \]

where \( N \) is the number of atoms in the target. Assuming that our target is a part of a larger piece of material, in which we know the density, \( n \), of the material then

\[ \text{Fraction scattered} = \frac{nA dx \sigma}{A} = n \sigma dx . \]
Now let us send a continuous flux, \( \Gamma \), of test particle – or a current density \( J \), at the target then the change in \( J \) across the target is

\[
\Delta J = J_{\text{after}} - J_{\text{before}} = (1 - n \sigma dx) J_{\text{before}} - J_{\text{before}} = -J n \sigma dx
\]

or

\[
\frac{dJ}{dx} = -J n \sigma
\]

\[
\downarrow
\]

\[
J = J_0 \exp \left( -\frac{x}{\lambda_{mfp}} \right)
\]

where \( \lambda_{mfp} = \frac{1}{n \sigma} \).

We can make approximations of the cross section of the target atom, based on the radius of the target atom. This is known as the hard-sphere approximation, which is at best imprecise. The reality is that the interaction, and hence collisions, is related to the target atom electron orbitals and particle energies, i.e. it is quantum mechanical in nature. From our above discussion, we can calculate the Mean frequency of collisions

\[
v = \frac{\nu}{\lambda_{mfp}} = v n \sigma
\]

And the mean period of collisions

\[
\tau = \frac{1}{v}
\]

Of course the average collision frequency for a distribution of velocities is

\[
\langle \nu \rangle = n \langle v \sigma \rangle \text{ noting } \sigma = \sigma(v).
\]

Collisions with walls

We are not really interested in the collision rate of a particle with a wall but rather we interested in how many particles are hitting the wall. (This may seem like it is just semantics but it is not.) To determine this we must first consider the flux of the particles in one given direction.

\[
\Gamma = \langle n v_z \rangle = \int v_z f(v) dv
\]

\[
= \frac{1}{4} n v_{ave}
\]

\[
= \frac{1}{4} n \sqrt{\frac{8 kT}{\pi m}}
\]

(The \( 1/4^{th} \) arises from integrating over all the angles.)

Now we can consider how long it might take to completely cover the wall with a monolayer of particles. This is simply
\[ \tau_{\text{monolayer}} = \frac{1}{\Gamma A_{\text{particle}}} \]

where \( A_{\text{particle}} \) is the area of a particle.

[Note that when the mean-free path is on the order of the size of the chamber or longer, the collision rate of the particles with walls can be given by the effective chamber size divided by the average speed.]

### 2.7. Gas Flow

Conservation of Flux (which we prove later)

If a gas passes through a series of pipes with various cross sectional areas, then the mass flux is conserved. Physically this should be obvious. Think of this as the same as saying that we do not have rarefaction/compression of the gas, nor do we have a source/sink for the gas.

\[
A_1 \Gamma = A_1 n_1 \bar{v}_1 = A_2 n_2 \bar{v}_2 = A_3 n_3 \bar{v}_3 \ldots
\]

Flow resistance/conductance
If a gas flow through a pipe, we expect some pressure differential to cause the flow. The ratio of the pressure to the flow is known as the resistance, \( R \).

\[
R = \frac{\Delta P}{\Gamma}.
\]

The conductance is simply the inverse of the \( R \),

\[
C = \frac{1}{R} = \frac{\Gamma}{\Delta P} \frac{\text{#/s/area}}{\text{Force/area}}
\]

These equations are very similar to the resistance in a circuit, where we replace voltage with pressure. Thus for a series of pipes
\[
\Delta P_{\text{total}} = \frac{\Gamma}{C_{\text{total}}} \quad \text{but} \quad \Delta P_{\text{total}} = \sum_i \Delta P_i \quad \text{and} \quad \text{so}
\]
\[
\Delta P_{\text{total}} = \frac{\Gamma}{C_{\text{total}}} = \sum_i \frac{\Gamma}{C_i} = \Gamma \sum_i \frac{1}{C_i}
\]
\[
\downarrow
\]
\[
\frac{1}{C_{\text{total}}} = \sum_i \frac{1}{C_i}
\]
or in a similar vane,
\[
R_{\text{total}} = \sum_i R_i.
\]

For Parallel pipes we find
\[
C_{\text{total}} = \sum_i C_i \quad \text{and}
\]
\[
\frac{1}{R_{\text{total}}} = \sum_i \frac{1}{R_i}
\]

Pumps

We can think of pumps as simply another conductance, S. After all we are trying to have a certain \(\Gamma\) go through the pump and the pump acts as if the back side has a pressure of zero. Here however, the pump conductance is given a special name, S, for ‘speed’.

If a pump has a pipe connecting it to the chamber, we get an effective pump speed from
\[
\frac{1}{S_{\text{eff}}} = \frac{1}{S} + \frac{1}{C_{\text{pipe}}}
\]

Example:

Given a pipe with a conductance \(C = 100\ \text{L/s}\) and a pump speed of \(S = 200\ \text{L/s}\) we find that the effective pump speed is 66.6 L/s. This means that we using only 33% of our possible pumping ability. This can become critical in the design of a process system.

2.8. Types of gas flow

There are three radically different types of gas flow. They are:

- Laminar fluid (viscous) flow
- Turbulent fluid flow
Molecular flow

The distinction between Laminar (‘layer’) and turbulent is often seen in wind tunnel experiments on new cars.

Laminar flow

Turbulent flow

One can determine if one is operating in fluid flow or molecular flow by determining the Knudsen number, $K_n$.

\[
K_n = \frac{\lambda_{\text{mfp}}}{d} = \begin{cases} 
> 1 & \text{Molecular (most interactions with walls)} \\
\sim 1 & \text{Transition or 'slip' region} \\
<< 1 & \text{Fluid (most interactions with other particles)} 
\end{cases}
\]

In addition we use Reynold’s number to describe the smooth or turbulent nature of the flow

\[
R = \frac{u \rho d}{\eta} = \begin{cases} 
< 1200 & \text{Laminar (smooth flow streams)} \\
\text{else} & \text{Transition} \\
> 2200 & \text{Turbulent (chaotic flow streams)} 
\end{cases}
\]

Kinetic theory gives

\[
\eta \approx \frac{1}{2} n m v_{\text{ave}} \lambda_{\text{mfp}}
\]

This gives rise to the following diagram:
We can calculate the conductance in molecular and fluid flow regimes from the following equations. (Note these equations are at least partly experimental approximations.)

**EXAMPLE**

Fluid Flow through a round tube

\[ C = \frac{\pi d^4}{128\eta} \frac{P_1 + P_2}{L} \quad ; \quad \eta = \text{gas viscosity} \]

Here \( P_1 \) and \( P_2 \) are in mbar and \( L \) and \( d \) are in cm, giving \( C \) in L/s.

**Turbulent Flow**

\[ C = \frac{d}{P_1 - P_2} \left( \frac{4}{\pi \eta} \right)^{1/7} \left[ \frac{\pi^2}{3.2} \frac{5}{4} d^3 \frac{P_1^2 - P_2^2}{2L} \right]^{4/7} \left( \frac{\text{gas constant} \ R}{M_{\text{molar}}} \right)^{3/7} T \]
Here, $M_{\text{molar}}$ is in g/mole, $R$ is 83.14 mBar L/Mole/K and $T$ is in K.

Fluid Flow through an orifice

\[
C = \frac{P_1}{P_1 - P_2} \frac{\pi d^2}{4} \left( \frac{2RT}{M_{\text{molar}}} \right)^{1/2} \left( \frac{P_2}{P_1} \right)^{1/2} \left[ \frac{\chi}{\chi - 1} \left( 1 - \frac{P_2}{P_1} \right)^{(1-\chi)/\chi} \right]^{1/2}
\]

where $\chi = \frac{C_p}{C_v}$ is the adiabatic constant

Molecular flow through a round tube

\[
C = \frac{\pi d^2}{4} \left( \frac{RT}{2\pi M_{\text{molar}}} \right)^{1/2} \left( 1 + \frac{3L}{4d} \right)^{3/2} \zeta; \text{ where } 1 \leq \zeta \leq 1.12 \text{ is a fudge factor}
\]

Momentum flow

Heat flow

2.9. **Solving basic equations and viewpoints**

We can examine fluids in ways that would not be entirely natural in other areas of Mechanical Engineering. Specifically we can look at how individual (or small group of) particles move OR we can examine what happens within a ‘control’ volume. These are known as Lagrange and Euler descriptions. In the end these two methods will reveal the same answer BUT sometimes one is easier than another. In Fluid Mechanics, the second, Euler description, is often easier to understand and ultimately arrive at a useful answer.

(I NEED TO MOVE THIS TO SOMEPLACE ELSE EARLIER IN THE NOTES!)

2.9.1. Lagrange description

In the Lagrange description one considers a particle at point P. Time $t_0$ the particle is at $(x_0, y_0, z_0)$. It then follows a path given by:
A second particle, Q, would start at point \((x_1, y_1, z_1, t_0)\), at the same time, \(t_0\). It would follow a path given by:

\[
x = x(x_1, y_1, z_1, t)
y = y(x_1, y_1, z_1, t)
z = z(x_1, y_1, z_1, t)
\]

Here we find that the particles trajectories are dependent on starting position and one free parameter, time. From this, we can arrive at velocity and acceleration.

\[
v_x = \partial_x x; \quad a_x = \partial_x^2 x
\]
\[
v_y = \partial_y y; \quad a_y = \partial_y^2 y
\]
\[
v_z = \partial_z z; \quad a_z = \partial_z^2 z
\]

While clearly we can follow a particle or two, we cannot follow the \(10^{23}\) particles that move ‘independently’ in a fluid. Thus the Lagrange description is rarely used in Fluid Mechanics.

### 2.9.2. Euler description – and the “Substantive Derivative”

Note – this is the same as our total derivatives – where in the velocity of a group of particles all act as if they have the same velocity – with the velocity changing from location to location. Thus,

\[
v_x = g_x(x, y, z, t)
v_y = g_y(x, y, z, t)
v_z = g_z(x, y, z, t)
\]
where \( g_{x/y/z} \) is some functional form that describes the (average) velocity as a function of position and time. Let us now move forward in time, \( \Delta t \). Then

\[
v_x + \Delta v_x = g_x \left( x + v_x \Delta t, y + v_y \Delta t, z + v_z \Delta t, t + \Delta t \right)
\]

\[
v_y + \Delta v_y = g_y \left( x + v_x \Delta t, y + v_y \Delta t, z + v_z \Delta t, t + \Delta t \right)
\]

\[
v_z + \Delta v_z = g_z \left( x + v_x \Delta t, y + v_y \Delta t, z + v_z \Delta t, t + \Delta t \right)
\]

expanding (Taylor series) these terms, we find

\[
v_x + \Delta v_x = g_x (x,y,z,t) + \left( v_x \partial_x g_x + v_y \partial_y g_x + v_z \partial_z g_x + \partial_t g_x \right) \Delta t + O(\Delta t^2)
\]

\[
v_y + \Delta v_y = g_y (x,y,z,t) + \left( v_x \partial_x g_y + v_y \partial_y g_y + v_z \partial_z g_y + \partial_t g_y \right) \Delta t + O(\Delta t^2)
\]

\[
v_z + \Delta v_z = g_z (x,y,z,t) + \left( v_x \partial_x g_z + v_y \partial_y g_z + v_z \partial_z g_z + \partial_t g_z \right) \Delta t + O(\Delta t^2)
\]

or cancelling \( v_x = g_x (x,y,z,t) \) etc, we find

\[
a_x = \frac{dv_x}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v_x}{\Delta t} = \left( v_x \partial_x g_x + v_y \partial_y g_x + v_z \partial_z g_x + \partial_t g_x \right) = (\partial_t + \mathbf{v} \cdot \nabla) g_x
\]

\[
a_y = \frac{dv_y}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v_y}{\Delta t} = \left( v_x \partial_x g_y + v_y \partial_y g_y + v_z \partial_z g_y + \partial_t g_y \right) = (\partial_t + \mathbf{v} \cdot \nabla) g_y
\]

\[
a_z = \frac{dv_z}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v_z}{\Delta t} = \left( v_x \partial_x g_z + v_y \partial_y g_z + v_z \partial_z g_z + \partial_t g_z \right) = (\partial_t + \mathbf{v} \cdot \nabla) g_z
\]

(This is sometimes referred to as the “Substantive Derivative.”)
Section 3.  BASIC PRINCIPLES OF FLUIDS

3.1. Overview

A fluid is a soup of particle that interact and as such have collective properties that individual particles would not have. This means that we must have the particles interacting often enough to produce this collective behavior.

If we were to try to model the fluid precisely, we would have to follow the “$10^{23}$” particles and all of their interactions. Each of the particles can be represented as delta functions in velocity and position space. While we can follow the motion of individual particles, which we have done in class, it is simply impossible, even with the most powerful computer, to follow all of the particles. Thus, we at first assume that instead of discrete points in our 6-D space we will assume that the particles can be modeled as a continuous function in the 6-D space. This is known as kinetic theory. Kinetic theory can be employed in computer models of plasmas, as has been done successfully. However, it is very difficult to use kinetic theory to derive analytic equations describing fluids. To do this we must go to a simpler model, Fluid Theory. We get fluid theory by integrating over all velocities. There are several moments of the kinetic equations that we can obtain. The first two are the most important – giving us conservation of particles and conservation of momentum (energy). We can picture what we are studying in the following manner:
From this we can get ‘global’ models (0-D!) that give overall or averaged processes across the system.

### 3.2. Kinetic Theory to Fluid Theory

Because the best that we can mathematically model is kinetic theory, we will begin by looking at the velocity distribution function $f(r,v,t) = f(x,y,z,v_x,v_y,v_z,t)$. The number of particles that are inside a volume of $dx dy dz dv_x dv_y dv_z$ is simply $f(x,y,z,v_x,v_y,v_z,t) dx dy dz dv_x dv_y dv_z$. Often the six coordinates are considered to be independent. Then the rate of change of particles at point is

$$\frac{df}{dt} = \frac{\partial f}{\partial r} \frac{dr}{dt} + \frac{\partial f}{\partial v} \frac{dv}{dt} + \frac{\partial f}{\partial t}$$

$$= (\nabla_r f) \cdot v + (\nabla_v f) \cdot a + \frac{\partial f}{\partial t}$$

$$= v \cdot (\nabla_r f) + \frac{F}{m} \cdot (\nabla_v f) + \frac{\partial f}{\partial t}$$

This is the Boltzmann equation or the **Navier-Stokes** equation. (In fluid mechanics it is the Navier-Stokes… although this depends on which book you read. In plasma theory it is known as the Boltzmann equation.) If particles are not sourced/sunk at the point then the total number does not change and hence

$$\frac{df}{dt} = 0 = (\nabla_r f) \cdot v + (\nabla_v f) \cdot a + \frac{\partial f}{\partial t}$$
This is known as the collisionless Boltzmann equation or the Vlasov equation. In the last section of this book, we related the velocity distribution, \( f(v) \), to a probability distribution, \( p(y) \). Now, we want to return to that concept.

### 3.2.1. Expectation values

Let us assume that we can randomly pick a ‘\( y \)’ from a set, having a probability \( p(y_0) \) of picking \( y_0 \). What value would we expect to get – on average – for \( y \)? If I were to draw 100 \( y \)’s from a basket, we would expect \( p(y_0)\cdot100 \) to be \( y_0 \). Assume that I can also pick \( y_1 \), \( y_2 \), and \( y_3 \). Then the average of my 100 picks would be

\[
\langle y \rangle = \frac{y_0 p(y_0) + y_1 p(y_1) + y_2 p(y_2) + y_3 p(y_3)}{p(y_0) + p(y_1) + p(y_2) + p(y_3)}
\]

\[
= \frac{\sum_i y_i p(y_i)}{\sum_i p(y_i)}
\]

If I were to have a continuous set of \( y \)’s then the equation becomes

\[
\langle y \rangle = \frac{\int y p(y) dy}{\int p(y) dy}
\]

as we have already used. (Often we normalize the probability such that \( \int p(y) dy = 1 \) ) This averaging is important as when we make a measurement, we are typically making a series of measurements – even if we don’t know it – and our result is the average of our series of measurements.

\( f(r, v, t) \) is our velocity and density distribution function and it is in a very real sense a probability distribution – with 6 dimensions \( (r, v) \) and one free parameter \( t \)! (Because of this definition, \( r \) and \( v \) are truly independent quantities – \( v \) does not depend on the location only \( f \). Thus, if we where to desire to know the average velocity then we look at

\[
\langle v \rangle = \frac{\int v f(r, v, t) dv}{\int f(r, v, t) dv}
\]

Likewise the average of the \( v^2 \) is simply

\[
\langle v^2 \rangle = \frac{\int v^2 f(r, v, t) dv}{\int f(r, v, t) dv}
\]

We can do this for all sorts of averages

\[
\frac{\int r^0 f(r, v, t) dr}{\int f(r, v, t) dr}; \quad \frac{\int r^1 f(r, v, t) dr}{\int f(r, v, t) dr}; \quad \frac{\int r^2 f(r, v, t) dr}{\int f(r, v, t) dr}; \quad \ldots;
\]

\[
\frac{\int v^0 f(r, v, t) dv}{\int f(r, v, t) dv}; \quad \frac{\int v^1 f(r, v, t) dv}{\int f(r, v, t) dv}; \quad \frac{\int v^2 f(r, v, t) dv}{\int f(r, v, t) dv}; \quad \ldots;
\]

However, we are not interested in most of them. In fact we typically only look at
Thus, to solve this, we must deal with some vector identity.  Since the force is dependent on velocity, it is typically just to first order as in the magnetic force.  Then the distribution goes to zero for physical reasons (strictly speaking, this is not true as directed kinetic energy).

Now what happens to the central term on the right hand side?  First let us assume that force, \( \mathbf{F} \), is independent of the velocity. (Strictly speaking, this is not true as \( \mathbf{F} = q(\mathbf{v} \wedge \mathbf{B}) \).) Then the integral becomes
\[
\int \left( \frac{df}{dt} \right) d\mathbf{v} = \text{Gain} - \text{Loss} = f|_c = \int \left( \mathbf{v} \cdot (\nabla_r f) + \frac{\mathbf{F}}{m} \cdot \nabla f + \frac{\partial f}{\partial t} \right) d\mathbf{v}
\]

\[
= \left( \int \left( \mathbf{v} \cdot (\nabla_r f) \right) d\mathbf{v} + \int \left( \frac{\mathbf{F}}{m} \cdot \nabla f \right) d\mathbf{v} + \int \left( \frac{\partial f}{\partial t} \right) d\mathbf{v} \right)
\]

\[
= \left( \nabla_r \cdot \int (\mathbf{v} f) d\mathbf{v} + \int \left( \frac{\mathbf{F}}{m} \cdot \nabla f \right) d\mathbf{v} + \frac{\partial}{\partial t} \int f d\mathbf{v} \right)
\]

\[
= \left( \nabla_r \cdot m \langle \mathbf{v} \rangle + \int \left( \frac{\mathbf{F}}{m} \cdot \nabla f \right) d\mathbf{v} + \frac{\partial n}{\partial t} \right)
\]

Now what happens to the central term on the right hand side?  First let us assume that force, \( \mathbf{F} \), is independent of the velocity. (Strictly speaking, this is not true as \( \mathbf{F} = q(\mathbf{v} \wedge \mathbf{B}) \).) Then the integral becomes
\[
\int \left( \frac{\mathbf{F}}{m} \cdot \nabla f \right) d\mathbf{v} = \frac{\mathbf{F}}{m} \cdot \int (\nabla f) d\mathbf{v}
\]

\[
= \frac{\mathbf{F}}{m} \left| f \right|_\infty = \frac{\mathbf{F}}{m} (0 - 0) = 0
\]

The distribution goes to zero for physical reasons – we don’t want an infinite density.  If the force is dependent on velocity, it is typically just to first order as in the magnetic force.  Then
\[
\int \left( \frac{\mathbf{F}}{m} \cdot \nabla f \right) d\mathbf{v} = \frac{q}{m} \int ((\mathbf{v} \cdot \mathbf{B}) \cdot \nabla f) d\mathbf{v}
\]

To solve this, we must deal with some vector identities. First, \( \mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = \mathbf{B} (\mathbf{A} \cdot \mathbf{C}) - \mathbf{C} (\mathbf{A} \cdot \mathbf{B}) \) and \( \mathbf{A} \cdot (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{A} \wedge \mathbf{B}) \cdot \mathbf{C} = -(\mathbf{B} \wedge \mathbf{A}) \cdot \mathbf{C} \). Thus,
\((\mathbf{v} \wedge \mathbf{B}) \cdot \nabla, f = (\mathbf{v} \wedge \mathbf{B}) \cdot \nabla, [f] \)
\[= \nabla, [f] \cdot (\mathbf{v} \wedge \mathbf{B}) \]
\[= (\nabla, [f] \wedge \mathbf{v}) \cdot \mathbf{B} \]
\[= (\nabla, [f \mathbf{v}] \cdot \mathbf{B} - (f \nabla, [\mathbf{v}] \cdot \mathbf{B}) \]
\[= \nabla, \cdot ([f \mathbf{v}] \wedge \mathbf{B}) - f \nabla, \cdot (\mathbf{v} \wedge \mathbf{B}) \]

First we will look at the second term on the right.
\[\nabla, \cdot (\mathbf{v} \wedge \mathbf{B}) = \partial_{v_x} (V_y B_z - V_z B_y) + \partial_{v_y} (V_x B_z - V_z B_x) + \partial_{v_z} (V_x B_y - V_y B_x) = 0 \]

Now we can go back to the integral and we find

\[
\int \left( \frac{\mathbf{F}}{m} \cdot (\nabla, f) \right) dv = -\frac{q}{m} \int (\mathbf{v} \wedge \mathbf{B}) \cdot \nabla, f dv \\
= -\frac{q}{m} \int (\nabla, \cdot ([f \mathbf{v}] \wedge \mathbf{B}) dv \\
= -\frac{q}{m} \oint_{\text{surface at } \infty} ([f \mathbf{v}] \wedge \mathbf{B}) dv \\
= 0
\]

Here \(f \mathbf{v}\) goes to zero on the surface for physical reasons – we don’t want an infinite energy so \(f\) must go to zero faster than \(\mathbf{v}\).

This leaves the continuity equation

\[
\int_{\text{particle gain/loss}} f = (\nabla, \cdot (n \langle \mathbf{v} \rangle + \partial, n) \\
\]

### 3.2.3. First moment of the Boltzmann equation – Momentum Conservation

Now we multiple Boltzmann’s equation
\[
\frac{df}{dt} = \mathbf{v} \cdot (\nabla, f) + \frac{\mathbf{F}}{m} \cdot (\nabla, f) + \frac{\partial f}{\partial t} \\
\]
by \(m\mathbf{v}\) and integrate over velocity to get
\[
m \int \mathbf{v} \left( \frac{df}{dt} \right) dv = \Delta \text{Momentum} = m \int \mathbf{v} \cdot (\nabla, f) + \frac{\mathbf{F}}{m} \cdot (\nabla, f) + \frac{\partial f}{\partial t} dv \\
= m \left( \int \mathbf{v} (\nabla, f) dv + \int \mathbf{v} \left( \frac{\mathbf{F}}{m} \cdot (\nabla, f) \right) dv + \int \mathbf{v} \left( \frac{\partial f}{\partial t} \right) dv \right)
\]

First we will look at the third part of the right-hand side
Finally, we can deal with the first term...

\[
\int v \frac{\partial f}{\partial t} \, dv = \frac{\partial}{\partial t} \left[ \int v(f) \, dv \right] - \int \frac{\partial v}{\partial t}(f) \, dv
\]

\[
= \frac{\partial}{\partial t} \left[ \int v(f) \, dv \right] - \int a(f) \, dv
\]

Which is zero from before

\[
= \frac{\partial}{\partial t} \left[ \int v(f) \, dv \right]
\]

\[
= \frac{\partial}{\partial t}[n(v)]
\]

Now for the second term

\[
\int v \left( \frac{\mathbf{F}}{m} \cdot \nabla f \right) \, dv = \frac{q}{m} \int v \left[ \left( \frac{m}{q} \mathbf{g} + E + v \wedge \mathbf{B} \right) \cdot \nabla f \right] \, dv
\]

- using the chain rule

\[
= \frac{q}{m} \int \nabla \cdot \left[ f v \left( \frac{m}{q} \mathbf{g} + E + v \wedge \mathbf{B} \right) \right] \, dv
\]

This operates on the force

Notice that this notation is correct
This is correct (matrix)

\[
- \frac{q}{m} \int f v \nabla \cdot \left[ \left( \frac{m}{q} \mathbf{g} + E + v \wedge \mathbf{B} \right) \right] \, dv
\]

\[
= \frac{q}{m} \int f v \left( \frac{m}{q} \mathbf{g} + E + v \wedge \mathbf{B} \right) \, dv
\]

\[
= \frac{q}{m} \int f v \left( \frac{m}{q} \mathbf{g} + E + v \wedge \mathbf{B} \right) \, dv
\]

\[
= -\frac{q}{m} \int f \left( \frac{m}{q} \mathbf{g} + E + v \wedge \mathbf{B} \right) \cdot \mathbf{I} \, dv
\]

\[
= -\frac{q}{m} \int f \left( \frac{m}{q} \mathbf{g} + E + v \wedge \mathbf{B} \right) \, dv
\]

\[
= -\frac{q}{m} \int \left( \frac{m}{q} \mathbf{g} + E + \langle v \rangle \wedge \mathbf{B} \right) \, dv
\]

Finally, we can deal with the first term...

(HOLD ON THIS GETS EASIER!)

\[
\int v(v \cdot (\nabla f)) \, dv = \nabla v \cdot \int f v v \, dv
\]

\[
= \nabla v \cdot (n(vv))
\]
What precisely is this last term? Well let us consider what \( \mathbf{v} \) is...

\[ \mathbf{v} = \mathbf{v}_{\text{directed}} + \mathbf{v}_{\text{random}} \]

Thus

\[
\nabla_r \cdot [n(\mathbf{v} \mathbf{v})] = \nabla_r \cdot [n(\mathbf{v}_{\text{dir}} + \mathbf{v}_{\text{rnd}})(\mathbf{v}_{\text{dir}} + \mathbf{v}_{\text{rnd}})] \\
= \nabla_r \cdot [n(\mathbf{v}_{\text{dir}} + \mathbf{v}_{\text{dir}} \mathbf{v}_{\text{rnd}} + \mathbf{v}_{\text{rnd}} \mathbf{v}_{\text{dir}} + \mathbf{v}_{\text{rnd}} \mathbf{v}_{\text{rnd}})] \quad \text{but}
\]

\[
\langle \mathbf{v}_{\text{dir}} \rangle = \mathbf{v}_{\text{dir}} \quad \text{so}
\]

\[
= \nabla_r \cdot (n \mathbf{v}_{\text{dir}} \mathbf{v}_{\text{dir}}) + \nabla_r \cdot \left( 2n \mathbf{v}_{\text{dir}} \langle \mathbf{v}_{\text{rnd}} \rangle \right) + \nabla_r \cdot (n \langle \mathbf{v}_{\text{rnd}} \mathbf{v}_{\text{rnd}} \rangle)
\]

The last term we have dealt with before. We know for a Maxwellian that

\[
\langle E \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} kT = \frac{4}{3} \frac{P}{n} \quad \text{but for our calculations}
\]

\[ \mathbf{v} = \mathbf{v}_{\text{rnd}} ! \]

Thus we find

\[ \mathbf{P} = mn \langle \mathbf{v}_{\text{rnd}} \mathbf{v}_{\text{rnd}} \rangle \]

(Notice that \( \mathbf{P} \) is a tensor – as might make sense – Pressure is direction dependent)

Now going back to our momentum conservation equation we find

\[
\Delta \text{Momentum} \equiv \mathbf{\Delta M}_c = m \left[ \int \mathbf{v} \cdot (\nabla_r f) d\mathbf{v} + \int \mathbf{v} \left( \frac{\mathbf{F}}{m} \cdot (\nabla_r f) \right) d\mathbf{v} + \int \mathbf{v} \left( \frac{\partial f}{\partial t} \right) d\mathbf{v} \right] \\
= m \left[ \nabla_r \cdot (n \langle \mathbf{v} \rangle \langle \mathbf{v} \rangle) + \nabla_r \cdot \frac{\mathbf{P}}{m} + \frac{\partial}{\partial t} \left( \frac{q}{m} \mathbf{E} + \langle \mathbf{v} \rangle \mathbf{B} \right) + \frac{\partial}{\partial t} \left[ n \langle \mathbf{v} \rangle \right] \right] \\
= \left( m \langle \mathbf{v} \rangle \nabla_r \cdot (n \langle \mathbf{v} \rangle) + mn \langle \mathbf{v} \rangle \cdot \nabla_r \langle \langle \mathbf{v} \rangle \rangle + \nabla_r \cdot \mathbf{P} - mng - qn (\mathbf{E} + \langle \mathbf{v} \rangle \times \mathbf{B}) + m \frac{\partial}{\partial t} \left[ n \langle \mathbf{v} \rangle \right] \right)
\]

(Note sign error on \( \mathbf{P} \))

using the continuity equation

\[ f_1 = \left( \nabla_r \cdot (n \langle \mathbf{v} \rangle) + \frac{\partial n}{\partial t} \right) \]

and rearranging...

\[
\Delta \mathbf{M}_c = \left( m \langle \mathbf{v} \rangle \nabla_r \cdot (n \langle \mathbf{v} \rangle) + mn \langle \mathbf{v} \rangle \cdot \nabla_r \langle \langle \mathbf{v} \rangle \rangle + \nabla_r \cdot \mathbf{P} - mng - qn (\mathbf{E} + \langle \mathbf{v} \rangle \times \mathbf{B}) + m \frac{\partial}{\partial t} \left[ n \langle \mathbf{v} \rangle \right] \right) \\
= \left( m \langle \mathbf{v} \rangle \left( f_1 - \frac{\partial n}{\partial t} \right) + mn \langle \mathbf{v} \rangle \cdot \nabla_r \langle \langle \mathbf{v} \rangle \rangle + \nabla_r \cdot \mathbf{P} - mng - qn (\mathbf{E} + \langle \mathbf{v} \rangle \times \mathbf{B}) + m \frac{\partial}{\partial t} \left[ n \langle \mathbf{v} \rangle \right] \right) \\
\]

↓

\[
\text{Noting that}
\]
\[
\text{we find}
\]
\[
\text{This is also known as the fluid equation of motion (NOTE – some authors also call THIS the Navier-Stokes Equation.)}
\]

**3.2.4. Energy Conservation (Heat flow equation)**

Now we multiply Boltzmann’s equation
\[
\frac{df}{dt} = v \cdot (\nabla f) + \frac{F}{m} \cdot (\nabla f) + \frac{\partial f}{\partial t}
\]
by \( \frac{1}{2} m v^2 \) and integrate over velocity to get
\[
\frac{1}{2} m \int v^2 \left( \frac{df}{dt} \right) dv = \Delta \text{Energy} = \frac{1}{2} m \left( \int v^2 (v \cdot (\nabla f)) dv + \int v^2 \left( \frac{F}{m} \cdot (\nabla f) \right) dv + \int v^2 \left( \frac{\partial f}{\partial t} \right) dv \right).
\]

This is trivial and is left to the reader (yeah right!)

Rather than go through this derivation, we will accept the formula in the book. (We don’t have time to do this in class now anyway and the first two formula are more important.)
Section 4. Fluid Statics

Our fluid equations that we developed before are:

\[
\begin{aligned}
\frac{f}{\text{particle gain/loss}}_{\text{particle gain/loss}} = & \left( \nabla \cdot (n \langle v \rangle + \partial_t n) \right) \\
\frac{nm}{dt} \frac{d \langle v \rangle}{dt} = & \text{momentum change via collisions} + \text{momentum change via particle gain/loss} \\
= & \Delta M_{\text{collision}} - m \langle v \rangle f_{\text{collision}} + \nabla \cdot \mathbf{P} + \text{mng} + qn (\mathbf{E} + \langle v \rangle \wedge \mathbf{B})
\end{aligned}
\]

Under static conditions all of the time derivatives are zero, as are all of the collision terms. Ignoring all electromagnetic fields, and the term \( \nabla \cdot \langle v \rangle \), leaves:

\[
\nabla \cdot (n \langle v \rangle) = 0
\]

\[
\nabla \cdot \mathbf{P} = -\text{mng}
\]

Why should we keep just these two terms in the 1st moment? Well they occur at all points on earth. The other terms occur on occasion.

For fluids, pressure is isotropic implying that we can write the latter equation as:

\[
-\nabla \cdot \mathbf{P} = \nabla \cdot p \mathbf{I}
\]

\[
= \mathbf{I} \cdot \nabla \cdot p = \nabla \cdot p = \text{mng}
\]

4.1. General concepts

4.1.1. Conservation of flux

We can use the first of the above equations to show

\[
\nabla \cdot (n \langle v \rangle) = 0
\]

\[
\downarrow
\]

\[
n \langle v \rangle = \Gamma = \text{Constant (Flux)}
\]

This means that the flux is constant. We can also make use of the Divergence theorem (and the physical interpretation) to see that there is no source or sink of flux in a small volume.
4.1.2. Pressure variations

4.1.2.1. Constant density

We can use the second of the above equations to show

\[ \nabla_r P = m g \]

Usually we set the height in the 'z' direction

and noting g is pointed down - gives

\[ \partial_z P = -mg = -\rho g \]

or

\[ \Delta P = -mgz = -\rho gz \]

This means that the pressure will decrease linearly with height above ground.

4.1.2.2. Variable density

Sometimes we have the situation in which the density changes with position. An example of this

is the atmosphere around the earth. Making use of the ideal gas law we can come up with an

approximate measure of the pressure of the atmosphere.

\[ \frac{1}{P} \int dP = \frac{-g}{kT} \int dz \]

giving

\[ \ln(P) = \frac{-gz}{kT} \implies P = P_0 e^{-gz/kT} \]

We can do the same thing over again – but this time assume that the temperature changes as we
go higher in elevation. “Halley’s Law” makes the assumption that the temperature decreases
linearly with height from sea level to ~45,000 feet. Here the temperature is assumed to be:

\[ T(z) = T_0 - \alpha z \]

where \( T_0 \) is the temperature a sea level (\( z=0 \)) and \( \alpha \sim 6.5 \, ^\circ C / km \). Thus
\[ \frac{\partial}{\partial z} P = -mg = -\rho g = -\frac{P g}{kT} \]

or

\[ \int P \frac{dP}{g} = \int \frac{kT}{1 - \frac{\alpha z}{T}} dz \]

giving

\[ \ln(P) = \frac{g}{\alpha k} \ln \left(1 - \frac{\alpha z}{T} \right) \]

\[ P = P_0 \left(1 - \frac{\alpha z}{T} \right)^{g/\alpha k} \]

This is known as the logarithmic pressure law.

### 4.2. Examples and problem solving

**Example 1: Manometers**

Manometers may use any type of liquid to help measure pressure within a closed vessel. Sometimes, one might use more than one ‘measuring’ liquid. Historically this had been mercury (Hg) but in more recent times other fluids have gained favor due to the poisoning hazard from Hg. (Hg was favored because it is very stable, it is fairly dense, it has a very low vapor pressure and it is plentiful.)

In the above figure we have a liquid (#1 in green) connected through a pipe to a second liquid (#2 in grey). We find that the liquids meet at a height B, which is below the center of the green liquid by H₁. The other side of the grey liquid meets vacuum at a height C.

The difference between the pressures on the two sides of the grey liquid is given by:

\[ P = P_0 \left(1 - \frac{\alpha z}{T} \right)^{g/\alpha k} \]
\[ \Delta P = -mngz = -\rho gz \]

or

\[ P_{C\rightarrow B} = \rho_1 gH_1 \]

Now to get the pressure inside liquid 1 we do the same thing.

\[ \Delta P = -mngz \]

\[ P_{B\rightarrow A} = -\rho_1 gH_1 \]

thus

\[ P_{C\rightarrow A} = +\rho_2 gH_2 - \rho_1 gH_1 = P_1 \]

QUESTION: If the top of the tube had not been vacuum, we would have needed to add that pressure as well to determine the pressure of liquid 1.

QUESTION: How would one make a barometer?

Example 2: Forces on plane surfaces
We can next examine the force on a planar surface. Examples of planar surfaces include locks (on dams), dams, ship hulls, propellers and other similar objects. Here we will use the force on a sluice gate door as an example.
\[ \Delta P = -mgz = -\rho g z \]

The depth is given by
\[ z = -ysin(\alpha) \]

thus
\[ P(y) = P(0) + \rho g y \sin(\alpha) \]

Now the force on the other side of the door is atmospheric pressure, \( P(0) \). Thus the net pressure on the door (and accounting for direction) is
\[ P_{net}(y) = -\rho g y \sin(\alpha) \hat{x} \]

The force is simply
\[ \mathbf{F} = \int P_{net}(y) \, dA \]
\[ = -\hat{x} \int_0^L \int_A^B \rho g y \sin(\alpha) \, dy \, dA \]
\[ = -\hat{x} \rho g \sin(\alpha) L \left( B^2 - A^2 \right) \]

where we have made the assumption that the gate is length \( L \) out of the page.

**Example 3: Forces on curved surfaces**

Here we will examine the force that is applied to the hull of a sailboat – noting that this could be any curved surface. First, we note that the pressure acts normal to the surface at every point. Thus, we can break the pressure into three components, along each of the axes. For the time being, we are going to reduce this to two by careful choice of coordinate system, as is shown in the figure below. Then
\[ \mathbf{P} = P_x \hat{x} + P_y \hat{y} + P_z \hat{z} \]
\[ = P \left( \cos(\theta) \sin(\theta) \hat{x} + \sin(\theta) \sin(\theta) \hat{y} + \cos(\theta) \hat{z} \right) \]
\[ = P_x \hat{x} + P_z \hat{z} \]
\[ = P \left( \sin(\theta) \hat{x} + 0 \sin(\theta) \hat{y} + \cos(\theta) \hat{z} \right) \]
\[ = P \left( \sin(\theta) \hat{x} + \cos(\theta) \hat{z} \right) \]

Now we can determine the force by integrating the pressure over all surfaces.
\[ \mathbf{F} = \int \mathbf{P} \, dA \]
\[ = \int P \left( \sin(\theta) \hat{x} + \cos(\theta) \hat{z} \right) \, dA \]
\[ = F_x \hat{x} + F_z \hat{z} \]
The force in the horizontal direction (X) is:

\[ F_x = \int P_x dA = \int P \sin(\theta) dA \]

\[ = \int P dA_x \]

where \( A_x \) is the effective area along the z axis. (Note that we are only looking at the force on one side here.) Thus

\[ F_x = \int d\lambda \int_0^\tau dz \rho g z \]

\[ = \frac{1}{2} \rho g L z^2 \]

Likewise we can do the same for the other component

\[ F_z = \int P_z dA = \int P \cos(\theta) dA \]

\[ = \int P dA_z \]

where \( A_z \) is the effective area along the x axis. Thus

\[ F_x = \int d\lambda \int_0^\tau dx \rho g z \]

\[ = \rho g L x z \]

Now we note that \( L x z \) is the volume displaced by the object – thus the water is putting an overall force equal upward to the weight of the water displaced.

**Example 4: Accelerating containers**

Now let us consider the case in which a container holding a liquid is being accelerated. We will consider two cases – with all other cases being effecting linear combinations of the ones presented here. Specifically we will consider a container accelerating in one direction, and a second container that is spinning. For the case of linear acceleration we find:

\[ \nabla_x P = mg + ma \]
For now, will assume \( \mathbf{a} \) has a component along \( z \) and another along \( x \). This gives
\[
\nabla \mathbf{r} P = \rho \left( a_z - g \right) \hat{z} + \rho a_x \hat{x} \\
= \partial_x P \hat{x} + \partial_z P \hat{z}
\]
or
\[
\partial_x P = \rho a_x \\
\partial_z P = \rho \left( a_z - g \right)
\]
then integrating gives
\[
\Delta P = \rho a_x \left( x_2 - x_1 \right) \\
\Delta P = \rho \left( a_z - g \right) \left( z_2 - z_1 \right)
\]

In the figure above, the pressure will be constant along the line – so
\[
0 = \rho a_x \left( x_2 - x_1 \right) + \rho \left( a_z - g \right) \left( z_2 - z_1 \right)
\]
\[
\tan \alpha = \frac{\left( z_2 - z_1 \right)}{\left( x_2 - x_1 \right)} = \frac{a_x}{a_z - g}
\]

Rotating acceleration

NEED TO ADD
Section 5.  Point (differential) form of the Fluid equations and general solutions.

5.1. Control Volumes and general balance of properties.

One can define a property of a material as either intensive or extensive.

**Intensive:** This is a property that does not depend on the volume of material that you have. Examples include density, energy density, thermal conductivity, specific gravity, heat capacity, viscosity, etc.

**Extensive:** This is a property that DOES depend on how much material you have. Examples include: mass (or particle #), Total energy, volume, etc.

Mathematically these properties are related by

\[ \Phi = \int \int \int \phi d\tau \]

or if given in terms of something per unit mass \((\phi)\)

\[ \Phi = \int \int \int \rho \phi d\tau \]

(Examples of the latter would be heat capacity.)

Here have defined (without really thinking about it!) a volume over which we are examining material property. We call this volume the **CONTROL VOLUME.** As a fluid flows through the control volume our extensive values can change.

Fluid at time \(t=t_0\). The control volume is denoted by the dashed green line (e.g. the central yellow volume plus the red dashed volume (A)).

Fluid at time \(t=t_1\). Here the fluid has moved to the right a bit – Now the green dashed line is still the same fluid we had before – it is just moved over a bit and occupies a slightly different volume.
We can see that at time $t=t_0$, the control volume (denoted by the dashed green line) contains the fluid marked in yellow and the volume mark with red dashes. A short time later, $t=t_1$, the fluid has moved to the right, and is now contained in the same yellow volume plus the green dashed volume. So what happens to the extensive property?

\[
\Phi_{\text{Green dash}}(t = t_0) = \int\int\int A \phi \, d\tau + \int\int\int B \phi \, d\tau = \Phi_{\text{Central}}(t = t_0) + \Phi_A(t = t_0)
\]

\[
\Phi_{\text{Green dash}}(t = t_1) = \int\int\int A \phi \, d\tau + \int\int\int B \phi \, d\tau = \Phi_{\text{Central}}(t = t_1) + \Phi_B(t = t_1)
\]

giving rise to

\[
\Delta \Phi = \Phi(t_1) - \Phi(t_0)
\]

\[
= \left( \Phi_{\text{Central}}(t_1) + \Phi_B(t_1) \right) - \Phi_{\text{Central}}(t_0)
\]

\[
= \left( \Phi_{\text{Central}}(t_1) - \Phi_A(t_1) + \Phi_B(t_1) \right) - \Phi_{\text{Central}}(t_0)
\]

Rearranging and dividing by $\Delta t$ gives,

\[
\frac{\Delta \Phi_{\text{Central}}}{{\Delta t}} = \frac{\Phi_{\text{Central}}(t_1) - \Phi_{\text{Central}}(t_0)}{\Delta t} + \frac{\Phi_B(t_1) - \Phi_A(t_1)}{\Delta t}
\]

The first term in this equation (I) is simply the total derivative

\[
\frac{d\Phi_{\text{Central}}}{dt} = \lim_{{\Delta t \to 0}} \frac{\Delta \Phi_{\text{Central}}}{\Delta t}
\]

The second term is simply the partial derivative

\[
\frac{\partial \Phi_{\text{Central}}}{\partial t} = \lim_{{\Delta t \to 0}} \frac{\Phi_{\text{Central}}(t_1) - \Phi_{\text{Central}}(t_0)}{\Delta t}
\]

Now what is the third term?

\[
\frac{\Phi_B(t_1) - \Phi_A(t_1)}{\Delta t} = \frac{\int\int\int B \phi \, d\tau - \int\int\int A \phi \, d\tau}{{\Delta t}}
\]

\[
= \frac{1}{{\Delta t}} \left( \int\int B \phi v \Delta t \cdot ds - \int\int A \phi v \Delta t \cdot ds \right)
\]

\[
= \int\int B \phi v \Delta t \cdot ds - \int\int A \phi v \Delta t \cdot ds
\]

\[
= \int\int \phi v \Delta t \cdot ds
\]

OK what we did above is the traditional method to get to what we want. It is a difficult mess – and we already proved it in a much simpler fashion. Basically:
\[ \frac{d\Phi}{dt} = \partial_t \Phi + \nabla \cdot \Phi \mathbf{v} \]

\[ = \partial_t \iiint \phi \, d\tau + \iint_{\text{control surface}} \phi \mathbf{v} \cdot ds \quad (\text{where we have made use of Stokes' theorem}) \]

or going to the intensive value

\[ \frac{d\phi}{dt} = \partial_t \phi + \nabla \cdot \phi \mathbf{v} \]

**THIS IS KNOWN AS THE “REYNOLDS TRANSPORT THEOREM”**

NOTE – we have defined the “CONTROL SURFACE” as the surface surrounding the control volume. It is the surface through which our material passes. Sometimes these surface areas are not quite the same – see for example the rocket ship below.

![Control surface](image)

**EXAMPLE:** Use the continuity equation to describe the a) mass and b) energy flow in a system

**ANSWER**

\[ \rho = \frac{dm}{d\tau}; \quad e = \frac{dE}{d\tau} \]

therefore

a)

\[ \Phi = \iiint \phi \, d\tau = \iiint \rho \, d\tau \]

\[ = M \]

thus

\[ \frac{dM}{dt} = \partial_t M + \iint_{\text{control surface}} \rho \mathbf{v} \cdot ds \]

noting that mass is often conserved, e.g. \( \frac{dM}{dt} = 0 \) so,

\[ \partial_t M = -\iint_{\text{control surface}} \rho \mathbf{v} \cdot ds \]

b)

\[ \Phi = \iiint \phi \, d\tau = \iiint e \, d\tau \]

\[ = E \]
thus
\[
\frac{dE}{dt} = \partial_t E + \oint_{\text{control surface}} e v \cdot ds
\]

5.1.1. Continuity equation for steady incompressible fluids

Under the condition that we have a steady flow with an incompressible fluid, the continuity equation becomes very simple. Steady flow implies that all of the time derivatives are zero. Thus,
\[
\frac{d\phi}{dt} = 0
\]
\[
\partial_t \phi + \nabla \cdot \phi v = 0
\]
\[
\nabla \cdot \phi v = v \cdot \nabla \phi + \phi \nabla \cdot v = 0
\]
If the fluid is incompressible, this implies that the density is constant. Thus our intrinsic variable is the density, \( \rho \). Thus
\[
v \cdot \nabla \rho + \rho \nabla \cdot v = 0
\]
\[
D \equiv \nabla \cdot v = 0
\]
This last equation is used often in fluid mechanics – and is known as the “Dilatation”, D. [From Latin dilatare, "to spread wide"] (It is used often because there are many physical systems that follow the steady state – incompressible model.) If we break this down into component, we get:
\[
\nabla \cdot v \equiv D = \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y + \frac{\partial}{\partial z} v_z = 0
\]
Physically – what does this imply? If the velocity on one side of our control volume in inward, then the velocity at the other sides must be outward, so as to balance.

EXAMPLE: Consider a steady two-dimensional incompressible fluid with \( v_x = \sinh(x) \).

a) What is the general \( y \) component of the velocity? b) Assuming boundary conditions of \( v_y = 2 \cosh(x) \) at \( y = 0 \), what is the specific solution?

ANSWER

a) As the fluid is steady and incompressible \( D \equiv \nabla \cdot v = 0 \)

Therefore:
\[
\nabla \cdot v = \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y + \frac{\partial}{\partial z} v_z = 0
\]
\[
\nabla \cdot v = 0
\]
\[
\frac{\partial}{\partial x} v_x = -\frac{\partial}{\partial y} v_y
\]
\[
= \cosh(x)
\]
We can now integrate to get $v_y$.
\[
\int \partial_y v_y \, dy = -\int \cosh(x) \, dy \\
\downarrow
\]
\[
v_y = -y\cosh(x) + g(x)
\]
Here $g(x)$ is some arbitrary function of $x$. We can get at this function by applying boundary conditions.
b) From above
\[
v_y = -y\cosh(x) + g(x)
\]
At $y=0$,
\[
|_{y=0} v_y = -0\cosh(x) + g(x) = 2\cosh(x)
\]
\[
\downarrow
\]
\[
g(x) = 2\cosh(x)
\]
Now our total velocity is
\[
\mathbf{v} = \sinh(x)\hat{x} + (2 - y)\cosh(x)\hat{y} + \hat{z}
\]

5.1.2. Momentum Equation

5.1.2.1. Angular velocity

Before we start down this path, we need to remember the definition of angular velocity – and its relation to linear velocity.
\[
\omega = \frac{\mathbf{r} \wedge \mathbf{v}}{r^2}
\]
\[
= \frac{1}{r^2} \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
x & y & z \\
v_x & v_y & v_z
\end{vmatrix}
\]
\[
= \frac{1}{r^2}[ (yv_z - zv_y)\hat{x} - (xv_z - zv_x)\hat{y} + (xv_y - yv_x)\hat{z}]
\]
\[
= \omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}
\]
This means that the angular velocity and velocity are related by:
\[
\omega_x = \frac{1}{r^2}(yv_z - zv_y)
\]
\[
\omega_y = -\frac{1}{r^2}(xv_z - zv_x)
\]
\[
\omega_z = \frac{1}{r^2}(xv_y - yv_x)
\]
Let’s assume that we have only one component of the angular velocity.

\[ \omega_x = \frac{1}{r^2} (y v_z - z v_y) \]

From this, we can back out our velocity

\[ \mathbf{v} = 0 \hat{x} - \omega_x z \hat{y} + \omega_y y \hat{z} \]

and

\[ \mathbf{r} = 0 \hat{x} + y \hat{y} + z \hat{z} \quad \text{noting that } \mathbf{v} \perp \mathbf{r} \perp \omega \]

Thus

\[
\dot{\omega} = \frac{1}{r^2} \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial_x & \partial_y & \partial_z \\
0 & -\omega_z & \omega_y \\
0 & \omega_z & -\omega_y \\
\end{vmatrix}
\]

\[
= \frac{1}{r^2} \left[ \hat{x} (y \omega_y + z \omega_x) - \hat{y} (0 v_z - z 0) + \hat{z} (0 \omega_z - y 0) \right]
\]

\[
= \frac{1}{r^2} \left[ \hat{x} \omega_x (y^2 + z^2) \right] = \hat{x} \omega_x
\]

Now let us consider

\[ \nabla \wedge \mathbf{v} = ? \]

\[
= \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial_x & \partial_y & \partial_z \\
0 & -\omega_z & \omega_y \\
0 & \omega_z & -\omega_y \\
\end{vmatrix}
\]

\[
= (\omega_x + \omega_y) \hat{x} - (0) \hat{y} + (0) \hat{z} = 2 \omega_x \hat{x}
\]

\[
\nabla \wedge \mathbf{v} = 2 \dot{\omega}
\]

We note that this works for any of the directions – thus we have in general that

\[ \dot{\omega} = \frac{1}{2} \nabla \wedge \mathbf{v} \]

\[
= \frac{1}{2} \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial_x & \partial_y & \partial_z \\
v_x & v_y & v_z \\
\end{vmatrix}
\]

\[
= \frac{1}{2} \left[ (\partial_x v_z - \partial_z v_x) \hat{x} - (\partial_y v_z - \partial_z v_y) \hat{y} + (\partial_x v_y - \partial_y v_x) \hat{z} \right]
\]

This makes our definition of the curl much more meaningful!

Why did we go to all of this trouble? Well let’s now consider what happens to the momentum in a fluid.
We already know that fluid velocity can change from location to location. Thus we are going to look at how that change occurs by performing a Taylor’s series expansion of the velocity around a point
\[ \mathbf{v} = (v_x, v_y, v_z) \]
\[
= v_{x0} + (x-x_0) \partial_x v_x + O(\partial^2_x v_x) + (y-y_0) \partial_y v_x + O(\partial^2_y v_x) + (z-z_0) \partial_z v_x + O(\partial^2_z v_x) \hat{x} \\
+ \left[ v_{y0} + (x-x_0) \partial_x v_y + O(\partial^2_x v_y) + (y-y_0) \partial_y v_y + O(\partial^2_y v_y) + (z-z_0) \partial_z v_y + O(\partial^2_z v_y) \right] \hat{y} \\
+ \left[ v_{z0} + (x-x_0) \partial_x v_z + O(\partial^2_x v_z) + (y-y_0) \partial_y v_z + O(\partial^2_y v_z) + (z-z_0) \partial_z v_z + O(\partial^2_z v_z) \right] \hat{z}
\]

Here we have eliminated the higher order terms – as they are very small. Now we are going to split the derivative terms and create all of the cross over terms...
\[ \mathbf{v} = (v_x, v_y, v_z) \]
\[
= \left[ v_{x0} + \frac{1}{2} \left[ (x-x_0) \left( \partial_x v_x - \partial_x v_x \right) + (y-y_0) \left( \partial_y v_x - \partial_y v_x \right) + (z-z_0) \left( \partial_z v_x - \partial_z v_x \right) \right] \right] \hat{x} \\
+ \left[ \text{Similar terms} \right] \hat{y} \\
+ \left[ \text{Similar terms} \right] \hat{z}
\]

We can now simplify these terms... The first term is easy – it is a cross product:
\[
(\mathbf{r} - \mathbf{r}_0) \times \nabla \times \mathbf{v} = -\frac{1}{2} \left[ (x-x_0) \left( \partial_x v_x - \partial_x v_x \right) + (y-y_0) \left( \partial_y v_x - \partial_y v_x \right) + (z-z_0) \left( \partial_z v_x - \partial_z v_x \right) \right] \\
= (\mathbf{r} - \mathbf{r}_0) \times \alpha
\]

What is the second term? It looks like an inner product.
\[
(\mathbf{r} - \mathbf{r}_0) \cdot \partial \mathbf{S} = \frac{1}{2} \left[ (x-x_0) \left( \partial_x v_x + \partial_x v_x \right) + (y-y_0) \left( \partial_y v_x + \partial_y v_x \right) + (z-z_0) \left( \partial_z v_x + \partial_z v_x \right) \right]
\]

where
\[
\partial \mathbf{S} = \frac{1}{2} \left[ (\partial_x v_x + \partial_x v_x) \hat{x} + (\partial_y v_x + \partial_y v_x) \hat{y} + (\partial_z v_x + \partial_z v_x) \hat{z} \right]
\]
Now this is repeated for each of the three directions – indicating that we really want a tensor and not a vector. Additionally, unit analysis indicates that this needs to be a time derivative of the tensor $\ddot{S}$. Thus:

$$\ddot{S} = \frac{1}{2} \begin{bmatrix} 2(\partial_x v_x) & (\partial_y v_x + \partial_z v_x) & (\partial_z v_x + \partial_y v_x) \\ (\partial_y v_x + \partial_z v_y) & 2(\partial_y v_y) & (\partial_z v_y + \partial_y v_y) \\ (\partial_z v_x + \partial_y v_z) & (\partial_z v_y + \partial_z v_z) & 2(\partial_z v_z) \end{bmatrix} \\ = \frac{1}{2} \begin{bmatrix} 2(\partial_r v_r) & \left(r \partial_r \left(\frac{v_\phi}{r}\right) + \frac{1}{r} \partial_\phi v_z\right) & (\partial_z v_z + \partial_z v_z) \\ \left(r \partial_r \left(\frac{v_\phi}{r}\right) + \frac{1}{r} \partial_\phi v_z\right) & 2\left(\frac{1}{r} \partial_\phi v_\phi + \frac{v_z}{r}\right) & (\partial_z v_\phi + \frac{1}{r} \partial_\phi v_z) \\ (\partial_z v_z + \partial_z v_z) & (\partial_z v_\phi + \frac{1}{r} \partial_\phi v_z) & 2(\partial_z v_z) \end{bmatrix}$$

**NEED TO PROVE IN CYLINDRICAL COORDINATES**

This is known as the **Strain Rate Dyadic (Tensor)**. In total this leaves:

$$\mathbf{v} = (v_x, v_y, v_z)$$

$$= \mathbf{v} - (\mathbf{r} - \mathbf{r}_0) \wedge \omega + (\mathbf{r} - \mathbf{r}_0) \cdot \partial \ddot{S}$$

### 5.1.2.2. Physical interpretation of the Strain Rate Tensor

$$\ddot{S} = ? = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

$$\downarrow$$

$$\ddot{S} = \begin{bmatrix} \dot{e}_{xx} & \dot{e}_{xy} & \dot{e}_{xz} \\ \dot{e}_{yx} & \dot{e}_{yy} & \dot{e}_{yz} \\ \dot{e}_{zx} & \dot{e}_{zy} & \dot{e}_{zz} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2(\partial_x v_x) & (\partial_y v_x + \partial_z v_x) & (\partial_z v_x + \partial_y v_x) \\ (\partial_y v_x + \partial_z v_y) & 2(\partial_y v_y) & (\partial_z v_y + \partial_y v_y) \\ (\partial_z v_x + \partial_y v_z) & (\partial_z v_y + \partial_z v_z) & 2(\partial_z v_z) \end{bmatrix}$$

First thing that we note is that

$$\dot{e}_{ij} = \dot{e}_{ji}$$

The next thing to note is that diagonal of $S$ is the Dilatation.

$$D = \nabla \cdot \mathbf{v} = \dot{e}_{xx} + \dot{e}_{yy} + \dot{e}_{zz}$$
5.2. Shape changes for control volumes

In general we can have a combination of four motions on our control volume. They are translation, rotation, Dilatation (shear), and Angular deformation (shear).

5.2.1.1. Translation

When our volume moves, the center moves
\[ \text{dr} = dx\hat{x} + dy\hat{y} + dz\hat{z} \]
Now this occurs in a time \( dt \) - so
\[
\frac{d\mathbf{r}}{dt} = \frac{dx}{dt} \hat{x} + \frac{dy}{dt} \hat{y} + \frac{dz}{dt} \hat{z} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}
\]

Now the distance traveled is

\[
PP' \hat{r} = v_x dt \hat{x} + v_y dt \hat{y} + v_z dt \hat{z}
\]

5.2.1.1.2. Dilatation (shear)

Dilatation is expansion or contraction of the volume. The increase (or decrease) in size can be determined by examining the motion of a single particle at point \(P\), \((x,y,z)\). There, the fluid has a velocity \((v_x,v_y,v_z)\). At some time later, the particle has moved to \(P'\), \((x',y',z')\). The velocity has now changed to

\[
v_{P'} = (v_x + \partial_x v_x dx) \hat{x} + (v_y + \partial_y v_y dy) \hat{y} + (v_z + \partial_z v_z dz) \hat{z}
\]

The dilatation of the volume then now given by:

\[
\Delta x = \partial_x v_x dx dt
\]
\[
\Delta y = \partial_y v_y dy dt
\]
\[
\Delta z = \partial_z v_z dz dt
\]

(Noting that the rest of the velocity acts to move the center…)

So,

\[
\Delta \mathbf{r}_{\text{Dilatation}} = \Delta x \hat{x} + \Delta y \hat{y} + \Delta z \hat{z}
\]

\[
= (\partial_x v_x dx dt) \hat{x} + (\partial_y v_y dy dt) \hat{y} + (\partial_z v_z dz dt) \hat{z}
\]

or

\[
\frac{d\mathbf{r}}{dt}_{\text{Dilatation}} = \partial_x v_x dx \hat{x} + \partial_y v_y dy \hat{y} + \partial_z v_z dz \hat{z}
\]

\[
= (\dot{e}_{xx} dx) \hat{x} + (\dot{e}_{yy} dy) \hat{y} + (\dot{e}_{zz} dz) \hat{z}
\]
5.2.1.1.3. Rotation

The rotation in size can be determined by examining the motion of a single particle at point P, (x,y,z). There, the fluid has a velocity $(v_x, v_y, v_z)$. At some time later, the particle has moved to $P'$, $(x', y', z')$. The velocity has now changed to

$$v_{P'} = (v_x + \partial_y v_x dy + \partial_z v_x dz) \hat{x} + (v_y + \partial_x v_y dx + \partial_z v_y dz) \hat{y} + (v_z + \partial_x v_z dx + \partial_y v_z dy) \hat{z}$$

Now we can look down one of the axis, here we have picked the y axis but the same thing works with the other axis. Remember that

$$\omega = \frac{r \wedge \mathbf{v}}{r^2} = \frac{1}{2} \nabla \wedge \mathbf{v} = \omega_x \hat{x} + \omega_y \hat{y} + \omega_z \hat{z}$$

$$= \frac{1}{r^2} \left[ (yv_z - zv_y) \hat{x} - (xv_z - zv_x) \hat{y} + (xy - yx) \hat{z} \right]$$

$$= \frac{1}{2} \left[ (\partial_y v_z - \partial_z v_y) \hat{x} - (\partial_x v_z - \partial_z v_x) \hat{y} + (\partial_x v_y - \partial_y v_x) \hat{z} \right]$$

For this case,
\[ \ddot{\omega} = \omega_z \hat{z} \]
\[ = \frac{1}{r^2} \left[ (x v_y - y v_x) \hat{z} \right] \]
\[ = \frac{1}{2} \left[ (\partial_x v_y - \partial_y v_x) \hat{z} \right] \]

For point P in the picture above, \( r = dy \) and \( v_y = 0 \), so
\[ \frac{v_x}{dy} = \frac{1}{2} \left( -\partial_y v_x \right) \]
\[ \downarrow \]
\[ \frac{\Delta x}{\Delta t} = -\frac{1}{2} \partial_y v_x dy \]
\[ \downarrow \]
\[ \Delta x = -\frac{1}{2} \partial_y v_x dy \Delta t \]

We can now repeat this with Point Q. Here \( r = dx \) and \( v_x = 0 \), so
\[ \frac{v_y}{dx} = \frac{1}{2} \left( \partial_x v_y \right) \]
\[ \downarrow \]
\[ \frac{\Delta y}{\Delta t} = \frac{1}{2} \partial_x v_y dx \]
\[ \downarrow \]
\[ \Delta y = \frac{1}{2} \partial_x v_y dx \Delta t \]

Thus in general we arrive at
\[ \Delta x = \left( \partial_y v_x dy + \partial_z v_x dz \right) dt \]
\[ \Delta y = \left( \partial_x v_y dx + \partial_z v_y dz \right) dt \]
\[ \Delta z = \left( \partial_x v_y dx + \partial_y v_y dy \right) dt \]

If this were a rotating around the center, the change would be in a sinusoidal fashion.
(Noting that the rest of the velocity acts to move the center…)

So,
\[ \Delta r = \Delta \hat{x} + \Delta \hat{y} + \Delta \hat{z} \]
\[ = \left( \partial_y v_x dy + \partial_z v_x dz \right) dt \hat{x} + \left( \partial_x v_y dx + \partial_z v_y dz \right) dt \hat{y} + \left( \partial_x v_y dx + \partial_y v_y dy \right) dt \hat{z} \]

For what is pictured above,
\[ \Delta r = \Delta \hat{x} + \Delta \hat{y} + \Delta \hat{z} \]
\[ = \left( \partial_y v_x dy \right) dt \hat{x} + \left( \partial_x v_y dx \right) dt \hat{y} \]

MINUS SIGNS ARE MESSED UP AND MISSING \( \frac{1}{2} \).
Example
Rotation around the center point.

5.2.1.1. Angular Deformation (shear)

At the center of our control volume, the velocity is zero. As we move away from the center, we will assume that the velocity increases linearly, \( \partial_y v_x \) in the y direction and \( \partial_x v_y \) in the x direction.

Thus at the edge of the control volume, the velocity is
\[
\frac{\Delta x}{\Delta t} = \partial_y v_x dy, \\
\downarrow \\
\Delta x = \partial_y v_x dy \Delta t \\
\frac{\Delta y}{\Delta t} = \partial_x v_y dx, \\
\downarrow \\
\Delta y = \partial_x v_y dx \Delta t
\]

In addition, we can also consider the angles, \( \alpha \) and \( \beta \). We can discover these from the tangent of the angle – at small angles, e.g. small times. Under these conditions,

\[
\alpha = \tan(\alpha) = \frac{\Delta x}{dy} = \frac{\partial_y v_x dy \Delta t}{dy} \\
\partial_\alpha = \partial_y v_x \\
\beta = \tan(\beta) = \frac{\Delta y}{dx} = \frac{\partial_x v_y dx \Delta t}{dx} \\
\partial_\beta = \partial_x v_y
\]

But this can now be related to Strain Rate Tensor.

\( \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}(\partial_1 \alpha + \partial_1 \beta) \)

### 5.3. The Strain Rate Tensor, \( \dot{S} \), and the Stress tensor, \( P \)

It can be shown that the pressure and the Strain Rate Tensor are related. For an incompressible fluid,

\[
\dot{P} = 2\mu \dot{S} - p \mathbf{I}
\]

For a compressible fluid,

\[
\dot{P} = 2\mu \dot{S} - \left(p + \frac{\gamma}{2} \mu \nabla \cdot \mathbf{v}\right) \mathbf{I}.
\]

Here \( p \) is the isotropic pressure and \( \mu \) is the dynamic viscosity.

(Note that the latter equation for a compressible fluid collapses to the incompressible form if \( \nabla \cdot \mathbf{v} = 0 \).

We will not prove this in class...

(I will add the proof at a later point.)
5.4. **Surface Forces on Control Volumes**

We can now look at the forces on the surfaces of our control volume. This is simply

\[ \mathbf{F} = \oint \mathbf{P} \cdot d\mathbf{s} \]

We can further break this up into the components normal to the surfaces

\[ \mathbf{F}_n = \oint \mathbf{P}_{ij} \cdot \hat{n} \, d\sigma \quad i = j \]

and those tangential to the surface

\[ \mathbf{F}_T = \oint \mathbf{P}_{ij} \cdot (\hat{T}_i + \hat{T}_j) \, d\sigma \quad i \neq j \]

Looking at these in terms of individual components we find,

\[ F_{n_x} = \oint p_x \, dy \, dz \]

\[ F_{n_y} = \oint p_y \, dx \, dz \]

\[ F_{n_z} = \oint p_z \, dx \, dy \]

\[ F_{T_x} = \oint \left( p_{yx} \, dx \, dz + p_{zx} \, dx \, dy \right) \]

\[ F_{T_y} = \oint \left( p_{xy} \, dy \, dz + p_{zy} \, dy \, dx \right) \]

\[ F_{T_z} = \oint \left( p_{xz} \, dy \, dz + p_{yz} \, dx \, dz \right) \]

We can simplify the normal component somewhat. To do this, let us look at a picture of what is going on.
\[ F_{n nett} = F_{n x} + F_{n y} \]
\[ = \int\int_{x=x} p_{xx} \, dy \, dz - \int\int_{x=0} p_{xx} \, dy \, dz \]
\[ = \int\int_{x=0} (p_{xx} + \partial_x P_{xx}) \, dy \, dz - \int\int_{x=0} p_{xx} \, dy \, dz \]
\[ = \int\int\partial_x P_{xx} \, dx \, dy \, dz \]
\[ F_{n nett_y} = \int\int\partial_y P_{yy} \, dx \, dy \, dz \]
\[ F_{n nett_z} = \int\int\partial_z P_{zz} \, dx \, dy \, dz \]

We can repeat the same thing for the tangential forces and we find
\[ F = \nabla_r \cdot \hat{P} \]

(I will add the rest of the proof at a later point.)

We can now add this to Navier-Stokes to give
\[ \frac{f_r}{\text{particle gain/loss}} = (\nabla_r \cdot (n \langle v \rangle) + \partial_v n) \]

\[ \frac{nm}{dt} \frac{d\langle v \rangle}{d\langle v \rangle} = nm(\partial_v \langle v \rangle + \langle v \rangle \cdot \nabla_r, \langle v \rangle) \]

\[ = \Delta M_{\text{mom}} - m \langle v \rangle f_r^{\text{coll}} + \nabla_r \cdot \left( 2\mu \dot{S} - \left( p + \frac{2}{3} \mu \nabla \cdot v \right) \hat{I} \right) + mng + qn(\mathbf{E} + \langle v \rangle \wedge \mathbf{B}) \]
Section 6. Incompressible Inviscid Flow

Our main formulas are the continuity equation (particle conservation)

\[ \frac{f_c}{\text{particle gain/loss}} = (\nabla \cdot (n \langle v \rangle) + \partial_n) \]

and the conservation of momentum equation (e.g. Navier-Stokes)

\[
\begin{align*}
nm \frac{d \langle v \rangle}{dt} &= nm (\partial_v \langle v \rangle + \langle v \rangle \cdot \nabla \langle v \rangle) \\
&= \frac{\Delta M_{\text{collisions}}}{\text{momentum change via collisions}} - m \langle v \rangle f_{\text{collisions}} + \nabla \cdot \left(2 \mu \dot{S} - (p + \frac{2}{3} \mu \nabla \cdot v) \hat{I}\right) + mng + qn (E + \langle v \rangle \wedge B)
\end{align*}
\]

For most fluids, the fluid particles do not change and there are no collisions with other particles. This allows us to set the following terms to zero.

\[ \frac{\Delta M_{\text{collisions}}}{\text{momentum change via collisions}} = 0 \]

\[ f_{\text{collisions}} = 0 \]

Additionally, there are often no charge carriers in the fluid. (Caution – as some chemical fluids you will run into in life DO have charge carriers – examples are acids and bases.) This allows us to further simplify our equation

\[ qn (E + \langle v \rangle \wedge B) = 0 \]

This gives a greatly simplified set of basic equations

\[ 0 = (\nabla \cdot (n \langle v \rangle) + \partial_n) \]

\[
\begin{align*}
nm \frac{d \langle v \rangle}{dt} &= nm (\partial_v \langle v \rangle + \langle v \rangle \cdot \nabla \langle v \rangle) \\
&= \nabla \cdot \left(2 \mu \dot{S} - (p + \frac{2}{3} \mu \nabla \cdot v) \hat{I}\right) + mng
\end{align*}
\]

6.1. Euler's Equation.

Incompressible

Incompressible implies that the density is constant, spatially and temporally. We can apply this to our continuity equation.
\[ 0 = \left( \nabla_r \cdot (n\langle v \rangle) + \frac{\partial}{\partial t} n \right) \]

\[ n\nabla_r \cdot \langle v \rangle = \nabla_r \cdot (n\langle v \rangle) \]

\[ \nabla_r \cdot \langle v \rangle = 0 \]

**Inviscid**

Inviscid implies that the viscosity \( \mu \) is zero. So we can now simplify Navier-Stokes to

\[ nm \frac{d\langle v \rangle}{dt} = nm \mathbf{A} = nm (\partial_r \langle v \rangle + \langle v \rangle \cdot \nabla_r \langle v \rangle) \]

\[ = -\nabla_r \cdot (p\mathbf{I}) + mng \]

\[ = -\nabla_r p + mng \]

We now have a very simple formula that we can use to examine our fluid flow.

**6.2. Euler’s Equation along a streamline.**

Examining an element of fluid following a streamline, we can calculate the forces on that volume.

We can use our shortened version of Navier-Stokes to arrive at the forces on each of the sides shown in the figure above.
\[ \rho A = -\nabla_p - \rho g \]

\[ \downarrow \]

\[ \rho A_s = -\partial_s p - \rho g \sin(\beta) \]

\[ \rho A_n = -\partial_n p - \rho g \cos(\beta) \]

where \(s\) is along the streamline and \(n\) normal to the streamline.

**Along the streamline**

To figure out what happens along the streamline we first need to know \(\sin(\beta)\). A short examination of the angle between \(s\) and \(y\), tells us that this is simply

\[ \frac{\partial z}{\partial s} = \sin(\beta) \]

This can be plugged into the first of the two equations above to give

\[ A_s = -\frac{1}{\rho} \partial_s p - g \partial_s z \]

\[ = \partial_s \langle v \rangle + \langle v \rangle \cdot \partial_s \langle v \rangle \]

Often we are in steady flow and the gravitational force is small compared to the other forces. Making these assumptions further simplifies our equation to

\[ -\frac{1}{\rho} \partial_s p = \langle v \rangle \cdot \partial_s \langle v \rangle \]

This tells us that if we decrease the velocity, the pressure must increase – and vise-a-versa.

**Normal to the streamline**

To figure out what happens along the streamline we first need to know \(\cos(\beta)\). A short examination of the angle between \(n\) and \(z\), tells us that this is simply

\[ \frac{\partial z}{\partial n} = \cos(\beta) \]

This can be plugged into the second of the two equations above to give

\[ \rho A_n = -\partial_n p - \rho g \cos(\beta) \]

\[ = -\partial_n p - \rho g \partial_n z \]

\[ = \partial_n \langle v \rangle + \langle v \rangle \cdot \partial_n \langle v \rangle \]

Again, we are often in steady flow and the gravitational force is small compared to the other forces. Making these assumptions further simplifies our equation to

\[ \rho A_n = -\partial_n p \]

\[ = \langle v \rangle \cdot \partial_n \langle v \rangle \]

We can solve this using centripetal acceleration around the point \(R\). Here (Will add proof later)

\[ \rho A_n = \rho \frac{\langle v \rangle^2}{R} = -\partial_n p \]

This tells us two things.

1) The pressure get higher the further one is from the center of rotation
2) There is no pressure in the normal direction of streamlines which do not bend. (\(R = \infty\))
Now let's go back to our equation for acceleration along the streamline

\[ A_s = -\frac{1}{\rho} \partial_s p - g \partial_s z \]

= \partial_t \langle v \rangle + \langle v \rangle \cdot \partial_s \langle v \rangle

If we have a situation in which we have steady state flow, the equation reduces to

\[ A_s = -\frac{1}{\rho} \partial_s p - g \partial_s z \]

= \partial_t \langle v \rangle + \langle v \rangle \cdot \partial_s \langle v \rangle

\[ \frac{1}{2} \langle v \rangle^2 = -\frac{p}{\rho} - gz + \text{Const} \]

or

\[ \frac{1}{2} \langle v \rangle^2 + \frac{p}{\rho} + gz = \text{Const} \]

This is **Bernoulli's** Equation. Remember it is only for

1) Steady flow
2) Incompressible flow
3) Frictionless flow
4) Flow along a streamline.

Please do not overuse it! (or abuse it!)
Section 7. Incompressible viscid Flow

Our main formulas are the continuity equation (particle conservation)
\[ \int_{\text{particle}} \left\{ \nabla \cdot (n \langle v \rangle) + \partial_n \right\} \Delta t = 0 \]

and the conservation of momentum equation (e.g. Navier-Stokes)
\[ \int_{\text{motion change via collisions}} \int_{\text{momentum change via particle gain/loss}} \frac{\partial \langle v \rangle}{\partial t} = \int_{\text{momentum change via collisions}} \nabla \cdot \left( \int_{\text{momentum change via particle gain/loss}} \left( 2 \mu \mathbf{\dot{S}} - \left( \frac{3}{2} \mu \nabla \cdot v \right) \mathbf{I} \right) + m \mathbf{g} \right) \] + qn (E + \langle v \rangle \wedge \mathbf{B})

For most fluids, the fluid particles do not change and there are no collisions with other particles. This allows us to set the following terms to zero.
\[ \int_{\text{momentum change via collisions}} \Delta M = 0 \]
\[ \int_{\text{particle gain/loss}} f = 0 \]

Additionally, there are often no charge carriers in the fluid. (Caution – as some chemical fluids you will run into in life DO have charge carriers – examples are acids and bases.) This allows us to further simplify our equation
\[ qn (E + \langle v \rangle \wedge \mathbf{B}) = 0 \]

This gives a greatly simplified set of basic equations
\[ 0 = \left( \nabla \cdot (n \langle v \rangle) + \partial_n \right) \]
\[ \int_{\text{motion change via collisions}} \int_{\text{momentum change via particle gain/loss}} \frac{\partial \langle v \rangle}{\partial t} = \int_{\text{momentum change via collisions}} \nabla \cdot \left( \int_{\text{momentum change via particle gain/loss}} \left( 2 \mu \mathbf{\dot{S}} - \left( \frac{3}{2} \mu \nabla \cdot v \right) \mathbf{I} \right) + m \mathbf{g} \right) \]
7.1. **Flow fields entering a tube**

![Diagram of flow fields entering a tube]

The length over which this happens is related to the diameter of the tube. Typically it is about 140 times the diameter.

7.2. **Flow fields between two fixed plates**

We can now calculate the flow fields between two fixed plates. Here we will assume that we have fully formed laminar flow.

![Diagram of flow fields between two fixed plates]

Based on the picture, we will assume that there is no change in the x and y directions. (The plates are infinitely big in both of those directions.) Additionally, we will assume that there is no time dependence – e.g. the velocity field is fully formed in the location of interest and there are no other changes.

Assumptions

1) Steady state \( \Rightarrow \frac{\partial}{\partial t} = 0 \)

2) Incompressible \( \Rightarrow n = const \)

3) The plates are infinite in the y direction and that there are no flow variations along the direction of the flow \( \Rightarrow \frac{\partial}{\partial y} v = 0 \)
4) The plates are infinite in the x direction and that there are no flow variations along that direction \( \Rightarrow \frac{\partial}{\partial x} v = 0 \). Further there is no flow in the x direction \( \Rightarrow v_x = 0 \) – nor any pressure variations along x \( \Rightarrow \partial_x p = 0 \). (Otherwise there would be flow along x…)

5) There is no flow in the z-direction \( \Rightarrow v_z = 0 \).

**Velocities**

Based on these assumptions, our equations simplify to:

\[
0 = (\nabla \cdot (n \langle v \rangle) + \partial_t n)
\]

\[
\downarrow
\]

\[
0 = \nabla \cdot (\langle v \rangle) = \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y + \frac{\partial}{\partial z} v_z
\]

and

\[
nm \frac{d\langle v \rangle}{dt} = nm (\partial_t \langle v \rangle + \langle v \rangle \cdot \nabla \cdot \langle v \rangle)
\]

\[
= \nabla \cdot \left( 2\mu \dot{\mathbf{S}} - (p + \frac{\mu}{\rho} \nabla \cdot \mathbf{v}) \mathbf{I} \right) + mg
\]

\[
\downarrow
\]

\[
0 = \nabla \cdot \left( 2\mu \dot{\mathbf{S}} - \left( p + \frac{\mu}{\rho} \nabla \cdot \mathbf{v} \right) \mathbf{I} \right) + mg
\]

\[
= 2\mu \nabla \cdot \dot{\mathbf{S}} - \nabla \cdot \mathbf{p} + mg
\]

\[
= \mu \nabla \cdot \left[ 2 (\partial_x v_x) \mathbf{I} + 2 (\partial_y v_y) \mathbf{I} + 2 (\partial_z v_z) \mathbf{I} \right] - \nabla \cdot \mathbf{p} - mg
\]

\[
= \mu \left[ 2 \partial_x \left( \frac{\partial_x v_x}{\partial x} = 0 \right) + \partial_y \left( \frac{\partial_y v_y}{\partial y} = 0 \right) + \partial_z \left( \frac{\partial_z v_z}{\partial z} = 0 \right) \right] - \nabla \cdot \mathbf{p} - mg
\]

Simplifying
\[ nm \frac{d\langle v \rangle}{dt} = 0 \]

\[ = \mu \left[ \begin{array}{c}
\partial_x (\partial_x v_y) \\
\partial_y (\partial_y v_x) \\
\partial_z (\partial_z v_y)
\end{array} \right] - \rho \left[ \begin{array}{c}
\partial_x p \\
\partial_y p \\
\partial_z p
\end{array} \right] - \rho \left[ \begin{array}{c}
g_z
\end{array} \right] \]

Now we need to add that there is no variation in the x and y directions (making the derivatives zero). So

\[ nm \frac{d\langle v \rangle}{dt} = 0 \]

\[ = \mu \left[ \begin{array}{c}
0 \\
\partial_x^2 v_y \\
0
\end{array} \right] - \rho \left[ \begin{array}{c}
0 \\
\partial_y p \\
\partial_z p
\end{array} \right] - \rho \left[ \begin{array}{c}
g_z
\end{array} \right] \]

We can now look at each of the parts.

The x-component tells us that
\[ p \neq p(x) \]

The z-component tells us that
\[ \partial_z p = -\rho g_z \]

\[ \downarrow \]

\[ p = p(y) - \rho g_z \]

The y-component tells us that
\[ \mu \partial_y^2 v_y = \partial_y p \]

\[ \downarrow \]

\[ \mu \partial_y v_y = \int \partial_y \left[ f(y) - \frac{1}{2} \rho g_z z^2 \right] dz \]

\[ = \partial_y \left[ \frac{1}{2} z f(y) - \frac{1}{4} \rho g_z z^2 \right] + C_i \]

\[ \downarrow \]

\[ \mu v_y = \int \left[ \partial_y \left[ \frac{1}{2} z^2 f(y) - \frac{1}{2} \rho g_z z^3 \right] + C_i \right] dz \]

\[ = \partial_y \left[ \frac{1}{2} z^2 f(y) - \frac{1}{6} \rho g_z z^3 \right] + C_i z \]

We can integrate over z the last of these equations to get
\[ \mu \partial_z^2 v_y = \partial_y p \]

\[ \downarrow \]

\[ \mu \partial_y v_y = \int \partial_z \left[ p(y) - \rho g z \right] dz \]

\[ = \partial_y \left[ z p(y) \right] + C_1 \]

\[ \downarrow \]

\[ \mu v_y = \int \left\{ \partial_y \left[ z p(y) \right] + C_1 \right\} dz \]

\[ = \partial_y \left[ \frac{1}{2} z^2 p(y) \right] + C_1 z + C_2 \]

noting that the \( y \) derivative of the gravity term is zero. We can now apply limits – assuming that the velocity is no slip at the top, \( z = a \), and bottom, \( z = 0 \), of the flow region to find

\[ \mu v_y \bigg|_{z=0} = \partial_y \left[ \frac{1}{2} z^2 p(y) \right] + C_1 \frac{z}{a} + C_2 \]

\[ = 0 \]

\[ \downarrow \]

\[ C_2 = 0 \]

\[ \mu v_y \bigg|_{z=a} = \partial_y \left[ \frac{1}{2} a^2 p(y) \right] + C_1 a \]

\[ = 0 \]

\[ \downarrow \]

\[ C_1 = -\partial_y \left[ \frac{1}{2} a p(y) \right] \]

or

\[ \mu v_y = \partial_y \left[ \frac{1}{2} z^2 p(y) \right] - \partial_y \left[ \frac{1}{2} a p(y) \right] z \]

\[ = \left[ \frac{1}{2} z^2 - \frac{1}{2} a z \right] \partial_y p(y) \]

One thing that needs to be pointed out – as \( z < a \), this implies that the velocity is in the negative \( y \)-direction unless the pressure is decreasing as we move in the positive \( y \)-direction...

**Flow rate**

Is there anything else that we can take away from this? One thing that we do know is that the net flux of fluid must be a constant as we move in the \( y \) direction. (If this were not the case, we would have the fluid piling up or areas where the fluid disappeared. We have made the assumption that the fluid is incompressible – hence this cannot happen here.) So we need to know the flux of material through an area (also known as the flow rate).

\[ Q = \left( \rho_m / n / \rho_q \right) \int v \cdot ds \]

where often the mass density, particle density or charge density \( \left( \rho_m / n / \rho_q \right) \) are used.

(Common flow rates are Standard Cubic Centimeters per Minute, Standard Liters per Second, Gallons per minute, Liters per second, Amps. ‘Standard’ is used for gaseous fluids and implies the number of particles found in that volume of air at sea-level density...
and room temperature. This number is Loschmidt’s number (1 amagat) - 2.6867774 × 10^{25} m^{-3}.

Here we will consider just the ‘flow’ of the control volume… thus

\[ Q = \int (v \cdot ds) \]

\[ = \int_0^{\Delta x} dx \int_0^{\Delta y} \frac{v_y}{2\mu} \left( z^2 - az \right) \partial_y p(y) dy dz \]

\[ = \Delta x \int_0^{\Delta y} \frac{1}{2\mu} \left( z^2 - az \right) \partial_y p(y) dy \]

\[ = \Delta x \left[ \frac{1}{2} z^3 - \frac{1}{2} az^2 \right] \partial_y p(y) \bigg|_0^y \]

\[ = -\Delta x \frac{1}{12\mu} a^3 \partial_y p(y) \]

Again, we note that the flow rate is negative unless the pressure decreases as we move in the positive y-direction.

**Pressure drop**

We can now use our flow rate to determine the functionality of the pressure variation. To make sure that Q is a constant,

\[ \partial_y p(y) = \text{const} = K_p \]

thus

\[ p = K_p y + p_c \]

We can determine the value of the constant by measuring the pressure at two points a distance \( \Delta y \) apart.

\[ p_1 = K_p y_1 + p_c \]

\[ p_2 = K_p y_2 + p_c \]

\[ K_p = \frac{p_2 - p_1}{y_2 - y_1} = \frac{\Delta p}{\Delta y} \]

\[ = -\frac{|\Delta p|}{\Delta y} \]

Here we note that the pressure must decrease as we move in the positive y-direction for the flow to go in the positive y-direction. Thus the pressure must decrease – giving the minus sign we see above. We can now go back and add this to our velocity and flow rate equations.

\[ v_y = \frac{1}{2\mu} \left( az - z^2 \right) \frac{|\Delta p|}{\Delta y} \]

\[ Q = \Delta x \frac{1}{12\mu} a^3 \frac{|\Delta p|}{\Delta y} \]

**Sheer stress**

The sheer stress can be calculated from the velocity.
\[ v_y = \frac{1}{2\mu} \left[ az - z^2 \right] \frac{\Delta p}{\Delta y} \]

Here we note that the velocity only changes in the \( z \) direction – and hence the sheer must be along that direction…

\[ \tau_{zy} = \frac{1}{2} \partial_z v_y = \partial_z \left\{ \frac{1}{4\mu} \left[ az - z^2 \right] \frac{\Delta p}{\Delta y} \right\} \]
\[ = \frac{1}{4\mu} \left[ a - 2z \right] \frac{\Delta p}{\Delta y} \]

**Average velocity**
The average velocity can be calculated in two manners – one via flow rate and the other via direct averaging of the velocity. We will do both.

\[ \langle v_y \rangle = \frac{1}{a} \int_0^a v_y \, dz \]
\[ = \frac{1}{a} \int_0^a \frac{1}{2\mu} \left[ az - z^2 \right] \frac{\Delta p}{\Delta y} \, dz \]
\[ = \frac{1}{12\mu} a^2 \frac{\Delta p}{\Delta y} \]
\[ \langle v_y \rangle = \frac{Q}{A} = \frac{Q}{a\Delta x} \]
\[ = \frac{1}{12\mu} a^2 \frac{\Delta p}{\Delta y} \]

**Maximum velocity**
The maximum velocity can be calculated by setting the derivative of the velocity to zero.

\[ \partial_z v_y = \partial_z \left\{ \frac{1}{2\mu} \left[ az - z^2 \right] \frac{\Delta p}{\Delta y} \right\} \]
\[ = \frac{1}{2\mu} \left[ a - 2z \right] \frac{\Delta p}{\Delta y} \]
\[ = 0 \]
\[ \downarrow \]
\[ z_{\text{max}} = \frac{1}{2} a \]
\[ \downarrow \]
\[ v_{y\text{-max}} = \frac{1}{4\mu} a^2 \frac{\Delta p}{\Delta y} \]
7.3. Flow fields between two moving plates

We now consider what happens when one of the plates moves. We could consider three situations,

1) Where the plate moves in the same direction as the flow and
2) Where the plate moves at right angles to the fluid flow.
   a. \( x \)-direction movement of the plate
   b. \( z \)-direction movement of the plate

If the plate moves at right angle to the flow – and we are in a no-slip condition, the fluid must also flow in the direction of the plate movement. (At least partially.) This causes some of our approximations in the previous section to go away – making the problem much more difficult to solve. Thus, we will drop those situations and focus on the first. In that case, we have the same assumptions:

1) Steady state \( \Rightarrow \frac{\partial}{\partial t} = 0 \)
2) Incompressible \( \Rightarrow n = \text{const} \)
3) The plates are infinite in the \( y \) direction and that there are no flow variations along the direction of the flow \( \Rightarrow \frac{\partial}{\partial y} v = 0 \)
4) The plates are infinite in the \( x \) direction and that there are no flow variations along that direction \( \Rightarrow \frac{\partial}{\partial x} v = 0 \). Further there is no flow in the \( x \) direction \( \Rightarrow v_x = 0 \) – nor any pressure variations along \( x \Rightarrow \partial_x p = 0 \). (Otherwise there would be flow along \( x \)…)
5) There is no flow in the \( z \)-direction \( \Rightarrow v_z = 0 \).

Velocity

This leads to the same general velocity profile:

\[ \mu v_y = \frac{1}{2} c^2 p(y) + C_1 z + C_2 \]

We can now apply limits – assuming that the velocity is no slip at the top, \( z = a \), and bottom, \( z = 0 \), of the flow region to find
\[ \mu v_y, \mid_{z=0} = \partial_y \left[ \frac{1}{2} \frac{z^2}{r_m} p(y) \right] + C_1 \frac{z}{r_m} + C_2 \]
\[ = 0 \]
\[ \downarrow \]
\[ C_2 = 0 \]
\[ \mu v_x, \mid_{z=a} = \partial_y \left[ \frac{1}{2} a^2 p(y) \right] + C_1 a \]
\[ = \mu v_p \]
\[ \downarrow \]
\[ C_1 = \frac{\mu}{a} v_p - \partial_y \left[ \frac{1}{2} a p(y) \right] \]
\[ \text{or} \]
\[ v_y = \frac{1}{\mu} \partial_y \left[ \frac{1}{2} z^2 p(y) \right] + \frac{1}{\mu} \left[ \frac{\mu}{a} v_p - \partial_y \left[ \frac{1}{2} a p(y) \right] \right] z \]
\[ = \frac{1}{2\mu} \left[ z^2 - az \right] \partial_y p(y) + \frac{z}{a} v_p \]

Again as \( z < a \), this implies that the velocity is in the negative \( y \)-direction unless the pressure is decreasing as we move in the positive \( y \)-direction...

**Flow rate**

Is there anything else that we can take away from this? One thing that we do know is that the net flux of fluid must be a constant as we move in the \( y \) direction. (If this were not the case, we would have the fluid piling up or areas where the fluid disappeared. We have made the assumption that the fluid is incompressible – hence this cannot happen here.) So we need to know the flux of material through an area (also known as the flow rate).

\[ Q = \left( \frac{\rho_m}{n / \rho_q} \right) \int v \cdot ds \]

where often the mass density, particle density or charge density \( \left( \rho_m / n / \rho_q \right) \) are used.

(Common flow rates are Standard Cubic Centimeters per Minute, Standard Liters per Second, Gallons per minute, Liters per second, Amps. ‘Standard’ is used for gaseous fluids and implies the number of particles found in that volume of air at sea-level density and room temperature. This number is Loschmidt’s number (1 amagat) - 2.6867774×10²⁵ m⁻³.)

Here we will consider just the ‘flow’ of the control volume... thus
\[ Q = \int v \cdot ds \]
\[ = \int_0^{\Delta x} dx \int_0^a dz \left\{ v_y \hat{y} \cdot \hat{y} \right\} \]
\[ = \int_0^{\Delta x} dx \int_0^a dz \left\{ \frac{1}{2\mu} \left[ z^2 - az \right] \partial_y p(y) + \frac{z}{a} v_p \right\} \]
\[ = \Delta x \int_0^a dz \left\{ \frac{1}{2\mu} \left[ z^2 - az \right] \partial_y p(y) + \frac{z}{a} v_p \right\} \]
\[ = \Delta x \left\{ \frac{1}{2\mu} \left[ \frac{3}{2} z^3 - \frac{1}{2} a z^2 \right] \partial_y p(y) + \frac{z^2}{2a} v_p + \text{Const} \right\} \]
\[ = \Delta x \left\{ \frac{-a^3}{12\mu} \partial_y p(y) + \frac{a}{2} v_p \right\} \]

Again, we note that the flow rate might be negative unless the pressure decreases as we move in the positive y-direction.

**Pressure drop**

We can now use our flow rate to determine the functionality of the pressure variation. To make sure that \( Q \) is a constant,
\[ \partial_y p(y) = \text{const} = K_p \]

thus
\[ p = K_p y + p_0 \]

We can determine the value of the constant by measuring the pressure at two points a distance \( \Delta y \) apart.
\[ p_1 = K_p y_1 + p_0 \]
\[ p_2 = K_p y_2 + p_0 \]
\[ K_p = \frac{p_2 - p_1}{y_2 - y_1} = \frac{\Delta p}{\Delta y} \]
\[ = \frac{-|\Delta p|}{\Delta y} \]

Here we note that the pressure must decrease as we move in the positive y-direction for the flow to go in the positive y-direction. Thus the pressure must decrease – giving the minus sign we see above. We can now go back and add this to our velocity and flow rate equations.

\[ v_y = \frac{1}{2\mu} \left[ a z - z^2 \right] \frac{\Delta p}{\Delta y} + \frac{z}{a} v_p \]
\[ Q = \Delta x \left\{ \frac{a^3}{12\mu} \frac{\Delta p}{\Delta y} + \frac{a}{2} v_p \right\} \]

**Sheer stress**

The sheer stress can be calculated from the velocity.
\[ v_y = \frac{1}{2\mu} \left[ az - z^2 \right] \frac{\Delta p}{\Delta y} + \frac{z}{a} v_p \]

Here we note that the velocity only changes in the z direction – and hence the shear must be along that direction…

\[
\tau_{zy} = \frac{1}{2} \frac{\partial}{\partial z} v_y = \frac{1}{2} \left[ az - z^2 \right] \frac{\Delta p}{\Delta y} + \frac{z}{a} v_p \]

\[
= \frac{1}{4\mu} \left[ a - 2z \right] \frac{\Delta p}{\Delta y} + \frac{v_p}{a}
\]

**Average velocity**

The average velocity can be calculated in two manners – one via flow rate and the other via direct averaging of the velocity. We will do both.

\[
\langle v_y \rangle = \frac{1}{a} \int_0^a v_y \, dz
\]

\[
= \frac{1}{a} \int_0^a \left[ \frac{1}{2\mu} \left[ az - z^2 \right] \frac{\Delta p}{\Delta y} + \frac{z}{a} v_p \right] \, dz
\]

\[
= \frac{1}{12\mu} a^2 \frac{\Delta p}{\Delta y} + \frac{a}{2} v_p
\]

\[
\langle v_y \rangle = \frac{Q}{A} = \frac{Q}{a\Delta x}
\]

\[
= \frac{1}{12\mu} a^2 \frac{\Delta p}{\Delta y} + \frac{a}{2} v_p
\]

**Maximum velocity**

The maximum velocity can be calculated by setting the derivative of the velocity to zero.
\[ \partial_z v_y = \partial_z \left[ \frac{1}{2\mu} \left[ az - z^2 \right] \frac{\Delta p}{\Delta y} + \frac{z}{a} v_y \right] \]
\[ = \frac{1}{2\mu} a - 2z \left[ \frac{\Delta p}{\Delta y} + \frac{1}{a} v_y \right] \]
\[ = 0 \]
\[ \downarrow \]
\[ z_{\text{max}} = \frac{4\mu v_y \Delta y}{a} \left| \frac{\Delta p}{\Delta y} \right| + 2a \]
\[ \downarrow \]
\[ \hat{v}_{y-\text{max}} = \frac{1}{2\mu} \left[ a \left( \frac{4\mu v_y \Delta y}{a} \right) + 2a \right] - \left( \frac{4\mu v_y \Delta y}{a} \right)^2 \left[ \frac{\Delta p}{\Delta y} + \frac{v_y}{a} \left( \frac{4\mu v_y \Delta y}{a} + 2a \right) \right] \]
\[ = \left( \frac{4\mu v_y \Delta y}{a} + 2a \right) \left[ \frac{a \left| \Delta p \right|}{2\mu \Delta y} - \frac{2v_y}{a} \right] + \frac{v_y}{a} \left( \frac{4\mu v_y \Delta y}{a} + 2a \right) \]
\[ = \left( \frac{4\mu v_y \Delta y}{a} + 2a \right) \left[ \frac{a \left| \Delta p \right|}{2\mu \Delta y} - \frac{v_y}{a} \right] \]

7.4. Flow fields in a cylindrical pipe

Here we consider what happens if we have non-turbulent fully formed flow in a cylindrical pipe. Again, our assumptions are:

1) Steady state \( \Rightarrow \frac{\partial}{\partial t} = 0 \)
2) Incompressible \( \Rightarrow n = \text{const} \)
3) There is no flow in the \( r \)-direction \( \Rightarrow v_r = 0 \).
4) There is no flow in the \( \phi \)-direction \( \Rightarrow v_\phi = 0 \)
5) There is cylindrical symmetry \( \Rightarrow \partial_\phi = 0 \)
6) The flow is fully formed \( \Rightarrow \partial_z v_z = 0 \)
Velocity

Based on these assumptions, our equations simplify to:

\[ 0 = \nabla_r \cdot (n \langle v \rangle) + \partial_n \]

\[ \downarrow \]

\[ 0 = \nabla_r \cdot \langle v \rangle \]

\[ = \left(r^{-1} \partial_r (r v_r) + r^{-1} \partial_\phi v_\phi + \partial_z v_z \right) \]

\[ = \partial_z v_z \]

and

\[ n m \frac{d \langle v \rangle}{dt} = n m \left( \partial_r \langle v \rangle + \langle v \rangle \cdot \nabla_r \langle v \rangle \right) \]

\[ = \nabla_r \cdot \left( 2 \mu \dot{S} - \left( p + \frac{2}{3} \mu \nabla \cdot v \right) \hat{I} \right) + m n g \]

\[ \downarrow \]

\[ 0 = \nabla_r \cdot \left( 2 \mu \dot{S} - \left( p + \frac{2}{3} \mu \nabla \cdot v \right) \hat{I} \right) + m n g \]

\[ = 2 \mu \nabla_r \cdot \dot{S} - \nabla_r p + m n g \]

\[ = \mu \nabla_r \cdot \left[ \begin{array}{ccc}
2 (\partial_r v_r) & \left( r \partial_r \left( \frac{v_\phi}{r} \right) + \frac{1}{r} \partial_\phi v_\phi \right) & \left( \partial_r v_z + \partial_z v_r \right) \\
\left( r \partial_r \left( \frac{v_\phi}{r} \right) + \frac{1}{r} \partial_\phi v_\phi \right) & 2 \left( \frac{1}{r} \partial_\phi v_\phi + \frac{v_r}{r} \right) & \left( \partial_z v_\phi + \frac{1}{r} \partial_\phi v_z \right) \\
\left( \partial_r v_z + \partial_z v_r \right) & \left( \partial_z v_\phi + \frac{1}{r} \partial_\phi v_z \right) & 2 (\partial_z v_z) \\
\end{array} \right] - \nabla_r p - m n g \]

\[ = \mu \left[ r^{-1} 2 \partial_r (r \partial_r v_r) + r^{-1} \partial_\phi \left( r \partial_r \left( \frac{v_\phi}{r} \right) + \frac{1}{r} \partial_\phi v_\phi \right) \right] + \partial_z \left( \partial_r v_z + \partial_z v_r \right) \]

\[ = \mu \left[ r^{-1} \partial_r \left( r \partial_r \left( \frac{v_\phi}{r} \right) + \frac{1}{r} \partial_\phi v_\phi \right) + 2 r^{-1} \partial_\phi \left( \frac{1}{r} \partial_\phi v_\phi + \frac{v_r}{r} \right) + \partial_z \left( \partial_z v_\phi + \frac{1}{r} \partial_\phi v_z \right) \right] \]

\[ = \mu \left[ \partial_r p - r^{-1} \partial_\phi p \right] - \rho \left[ \begin{array}{c}
\dot{\partial}_r p \\
\dot{\partial}_z p \\
\end{array} \right] \]

\[ = \rho \left[ \begin{array}{c}
0 \\
0 \\
g_z \\
\end{array} \right] \]

Applying our assumptions and simplifying
\[\frac{ nm d\langle v \rangle}{dt} = 0\]

\[= \mu \left[ r^{-1} \partial_z (r \partial_r v_r) + r^{-1} \partial_r \left( \frac{v_0^0}{r} + \frac{1}{r} \partial_{\phi} v_0 + \frac{1}{r} \partial_{\phi} v_0 \right) + \partial_z \left( \partial_r v_0 + \frac{1}{r} \partial_{\phi} v_0 \right) \right] \]

\[= \mu \left[ \partial_z p - \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial z} v_0 \right) \right] - \rho \left[ \begin{array}{c} 0 \\ 0 \\ g_z \end{array} \right] \]

\[= \mu \left[ \begin{array}{c} 0 + 0 + \partial_r \left( \frac{\partial}{\partial z} v_0 \right) \\ 0 + 0 + 0 \\ r^{-1} \partial_z v_0 + \left( \partial_r v_0 \right) + 0 + 2 \left( \partial_z \frac{\partial}{\partial \phi} v_0 \right) \end{array} \right] - \rho \left[ \begin{array}{c} \partial_z p \\ 0 \\ g_z \end{array} \right] \]

We can now look at each of the parts.

The \( r \) and \( \phi \)-component tells us that

\[ p \neq p(r, \phi) \]

The \( z \)-component tells us that

\[ \partial_z p = \mu r^{-1} \left( \partial_r r v_r + r \left( \partial_r v_r \right) \right) - \rho g_z \]

\[= \mu r^{-1} \partial_r \left( r \partial_r v_r \right) - \rho g_z \]

none of this is a function of \( z \)...

\[ p = \left[ \mu r^{-1} \partial_r \left( r \partial_r v_r \right) - \rho g_z \right] + p_0 \]

Now, we know that the pressure can at most be a function of \( z \) – therefore
\[ \mu r^{-1} \partial_r \left( r \partial_r v_z \right) - \rho g_z = c_1 = \text{const} \]  
\[ (= \partial_z p) \]

Rearranging we find

\[ \partial_r \left( r \partial_r v_z \right) = \frac{r}{\mu} (\rho g_z + c_1) \]
\[ \downarrow \]
\[ r \partial_r v_z = \frac{r^2}{2\mu} (\rho g_z + c_1) + c_2 \]
\[ \downarrow \]
\[ \partial_r v_z = \frac{r}{2\mu} (\rho g_z + c_1) + \frac{c_2}{r} \]
\[ \downarrow \]
\[ v_z = \frac{r^2}{4\mu} (\rho g_z + c_1) + c_2 \ln(r) + c_3 \]

We can now apply limits – assuming that the flow is a non-infinite flow velocity at the centerline, \( r = 0 \):

\[ v_z \big|_{r=0} = \frac{a^2}{4\mu} (\rho g_z + c_1) + c_2 \ln(0) + c_3 \]
\[ = 0 + c_2 (-\infty) + c_3 \]
\[ \neq \infty \]
\[ \Rightarrow c_2 = 0 \]

and now assuming that the velocity is no slip at the wall, \( r = a \), and

\[ v_z \big|_{r=a} = \frac{a^2}{4\mu} (\rho g_z + c_1) + c_3 \]
\[ = 0 \]
\[ \downarrow \]
\[ c_3 = -\frac{a^2}{4\mu} (\rho g_z + c_1) \]

Combining all of this, we find that

\[ v_z = \frac{r^2 - a^2}{4\mu} (\rho g_z + \partial_z p) \]

As before, one thing that needs to be pointed out – as \( r < a \), this implies that the velocity is in the negative \( z \)-direction unless the pressure is decreasing as we move in the positive \( z \)-direction…

**Flow rate**
Is there anything else that we can take away from this? One thing that we do know is that the net flux of fluid must be a constant as we move in the $y$ direction. (If this were not the case, we would have the fluid piling up or areas where the fluid disappeared. We have made the assumption that the fluid is incompressible—hence this cannot happen here.) So we need to know the flux of material through an area (also known as the flow rate).

$Q = \left(\frac{\rho_m}{n} / \rho_q\right) \int v \cdot ds$

where often the mass density, particle density or charge density $\left(\frac{\rho_m}{n} / \rho_q\right)$ are used.

(Common flow rates are Standard Cubic Centimeters per Minute, Standard Liters per Second, Gallons per minute, Liters per second, Amps. ‘Standard’ is used for gaseous fluids and implies the number of particles found in that volume of air at sea-level density and room temperature. This number is Loschmidt’s number (1 amagat) - $2.6867774 \times 10^{25}$ m$^{-3}$.)

Here we will consider just the ‘flow’ of the control volume… thus

$Q = \int v \cdot ds$

$= \int_0^{\Delta x} dx \int_0^a dz \left\{ v_y \hat{y} \cdot \hat{y} \right\}$

$= \int_0^{\Delta x} dx \int_0^a dz \left[ \frac{1}{\mu} \left( z^2 - az \right) \partial_y p(y) + \frac{z}{a} v_p \right]$

$= \Delta x \int_0^a dz \left[ \frac{1}{\mu} \left( z^2 - az \right) \partial_y p(y) + \frac{z}{a} v_p \right]$

$= \Delta x \left\{ \frac{1}{12\mu} \left[ \frac{1}{3} z^3 - \frac{1}{2} az^2 \right] \partial_y p(y) + \frac{z^2}{2a} v_p + \text{Const} \right\}_0^a$

Again, we note that the flow rate might be negative unless the pressure decreases as we move in the positive $y$-direction.

**Pressure drop**

We can now use our flow rate to determine the functionality of the pressure variation. To make sure that $Q$ is a constant,

$\partial_y p(y) = \text{const} = K_p$

thus

$p = K_p y + p_0$

We can determine the value of the constant by measuring the pressure at two points a distance $\Delta y$ apart.
\[ p_1 = K_p y_1 + p_0 \]
\[ p_2 = K_p y_2 + p_0 \]
\[ K_p = \frac{p_2 - p_1}{y_2 - y_1} = \frac{\Delta p}{\Delta y} = \frac{-|\Delta p|}{\Delta y} \]

Here we note that the pressure must decrease as we move in the positive \( y \)-direction for the flow to go in the positive \( y \)-direction. Thus the pressure must decrease – giving the minus sign we see above. We can now go back and add this to our velocity and flow rate equations.

\[ v_y = \frac{1}{2\mu} \left[ az - z^2 \right] \frac{|\Delta p|}{\Delta y} + z v_p \]
\[ Q = \Delta x \left\{ \frac{a^3}{12\mu} \frac{|\Delta p|}{\Delta y} + \frac{a}{2} v_p \right\} \]

**Sheer stress**

The sheer stress can be calculated from the velocity

\[ v_y = \frac{1}{2\mu} \left[ az - z^2 \right] \frac{|\Delta p|}{\Delta y} + z v_p \]

Here we note that the velocity only changes in the \( z \) direction – and hence the sheer must be along that direction…

\[ \tau_{yz} = \frac{1}{2} \partial_y v_y = \partial_z \left\{ \frac{1}{2\mu} \left[ az - z^2 \right] \frac{|\Delta p|}{\Delta y} + z \frac{v_p}{a} \right\} \]
\[ = \frac{1}{4\mu} \left[ a - 2z \right] \frac{|\Delta p|}{\Delta y} + \frac{v_p}{a} \]

**Average velocity**

The average velocity can be calculated in two manners – one via flow rate and the other via direct averaging of the velocity. We will do both.

\[ \langle v_y \rangle = \frac{1}{a} \int_0^a v_y \, dz \]
\[ = \frac{1}{a} \int_0^a dz \left[ \frac{1}{2\mu} \left[ az - z^2 \right] \frac{|\Delta p|}{\Delta y} + z \frac{v_p}{a} \right] \]
\[ = \frac{1}{12\mu} a^2 \frac{|\Delta p|}{\Delta y} + \frac{a}{2} v_p \]

\[ \langle v_y \rangle = \frac{Q}{A} = \frac{Q}{a\Delta x} \]
\[ = \frac{1}{12\mu} a^2 \frac{|\Delta p|}{\Delta y} + \frac{a}{2} v_p \]
**Maximum velocity**

The maximum velocity can be calculated by setting the derivative of the velocity to zero.

\[
\frac{\partial z_v}{\partial z} = \frac{1}{2\mu} \left[ a z - z^2 \right] \frac{\Delta p}{\Delta y} + \frac{z}{a} v_p
\]

\[
= \frac{1}{2\mu} \left[ a - 2z \right] \frac{\Delta p}{\Delta y} + \frac{1}{a} v_p
\]

\[
= 0
\]

\[
\downarrow
\]

\[
z_{\text{max}} = \frac{4\mu v_p}{a} \frac{\Delta y}{|\Delta p|} + 2a
\]

\[
\downarrow
\]

\[
v_{y-\text{max}} = \frac{1}{2\mu} \left[ a \left( \frac{4\mu v_p}{a} \frac{\Delta y}{|\Delta p|} + 2a \right) - \left( \frac{4\mu v_p}{a} \frac{\Delta y}{|\Delta p|} + 2a \right)^2 \right] \frac{\Delta p}{\Delta y} + \frac{v_p}{a} \left( \frac{4\mu v_p}{a} \frac{\Delta y}{|\Delta p|} + 2a \right)
\]

\[
= \left( \frac{4\mu v_p}{a} \frac{\Delta y}{|\Delta p|} + 2a \right) \left( a \frac{\Delta p}{2\mu \Delta y} - \frac{2v_p}{a} \right) + \frac{v_p}{a} \left( \frac{4\mu v_p}{a} \frac{\Delta y}{|\Delta p|} + 2a \right)
\]

\[
= \left( \frac{4\mu v_p}{a} \frac{\Delta y}{|\Delta p|} + 2a \right) \left[ a \frac{\Delta p}{2\mu \Delta y} - \frac{v_p}{a} \right]
\]