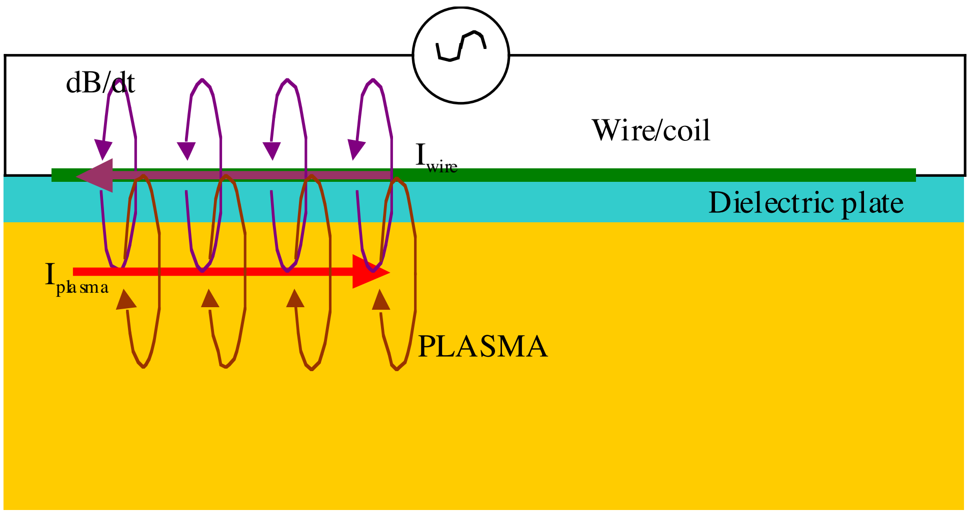
Lecture 10 Capacitively Coupled Plasmas

New homework problems:
Lieberman 12.2 and 12.3 Due April 25th, 2001. – Last Set!



Basic setup.

How do we find out how much current is being pushed about in the plasma? Well the place to start is Maxwell's equations.

$$\nabla \wedge \mathbf{E} = -\partial_t \mathbf{B}$$

$$\nabla \wedge \mathbf{H} = \mathbf{J}_{free} + \partial_t \mathbf{D}$$

$$\nabla \cdot \mathbf{D} = \rho_{free}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{J} = -\partial_t \rho$$

If we take the curl of the first (second) equation we find

$$\nabla \wedge \left[\nabla \wedge \mathbf{E} = -\partial_t \mathbf{B}\right]$$

$$\nabla \wedge (\nabla \wedge \mathbf{E}) = -\partial_t \nabla \wedge \mathbf{B}$$

$$\nabla \wedge (\nabla \wedge \mathbf{E}) = -\partial_t \mu \nabla \wedge \mathbf{H}$$

$$\nabla \wedge (\nabla \wedge \mathbf{E}) = -\partial_t \mu \left(\mathbf{J}_{free} + \partial_t \mathbf{D}\right)$$

$$\nabla \wedge (\nabla \wedge \mathbf{E}) = -\mu \partial_t \mathbf{J}_{free} - \mu \partial_t^2 \mathbf{D}$$

$$\nabla \wedge (\nabla \wedge \mathbf{E}) = -\mu \partial_t \sigma \mathbf{E} - \mu \partial_t^2 \varepsilon \mathbf{E}$$

$$\nabla \wedge (\nabla \wedge \mathbf{E}) = -\mu \sigma \partial_t \mathbf{E} - \mu \varepsilon \partial_t^2 \mathbf{E}$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu \sigma \partial_t \mathbf{E} - \mu \varepsilon \partial_t^2 \mathbf{E}$$

$$\underbrace{\nabla\left(\rho_{free}/\varepsilon\right)}_{assume\ =\ 0} - \nabla^2 \mathbf{E} = -\mu \sigma \partial_t \mathbf{E} - \mu \varepsilon \partial_t^2 \mathbf{E}$$

$$\nabla^2 \mathbf{E} = \mu \sigma \partial_t \mathbf{E} + \mu \varepsilon \partial_t^2 \mathbf{E}$$

$$\nabla \wedge \left[\nabla \wedge \mathbf{H} = \mathbf{J}_{free} + \partial_t \mathbf{D}\right]$$

$$\nabla \wedge (\nabla \wedge \mathbf{H}) = \nabla \wedge \mathbf{J}_{free} + \partial_t \nabla \wedge \mathbf{D}$$

$$\nabla \wedge (\nabla \wedge \mathbf{H}) = \nabla \wedge \sigma \mathbf{E} + \partial_t \nabla \wedge \varepsilon \mathbf{E}$$

$$\nabla \wedge (\nabla \wedge \mathbf{H}) = \nabla \wedge \sigma \mathbf{E} + \varepsilon \partial_t \left(-\partial_t \mathbf{B}\right)$$

$$\nabla \wedge (\nabla \wedge \mathbf{H}) = \sigma\left(-\partial_t \mathbf{B}\right) - \varepsilon \partial_t^2 \mathbf{B}$$

$$\nabla \wedge (\nabla \wedge \mathbf{H}) = -\sigma \mu \partial_t \mathbf{H} - \varepsilon \mu \partial_t^2 \mathbf{H}$$

$$\nabla \wedge (\nabla \wedge \mathbf{H}) = -\sigma \mu \partial_t \mathbf{H} - \varepsilon \mu \partial_t^2 \mathbf{H}$$

$$\nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = -\sigma \mu \partial_t \mathbf{H} - \varepsilon \mu \partial_t^2 \mathbf{H}$$

$$\nabla\left(\mu \underbrace{\nabla \cdot \mathbf{B}}_{=0}\right) - \nabla^2 \mathbf{H} = -\sigma \mu \partial_t \mathbf{H} - \varepsilon \mu \partial_t^2 \mathbf{H}$$

$$\nabla^2 \mathbf{H} = \sigma \mu \partial_t \mathbf{H} + \varepsilon \mu \partial_t^2 \mathbf{H}$$

This leads directly to the general electromagnetic wave equation

$$\nabla^2\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \sigma\mu\partial_t\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} + \varepsilon\mu\partial_t^2\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

$$|\mathbf{E}| = \pm\frac{|\mathbf{H}|}{\eta};\ \eta = \frac{i\mu\omega}{\gamma} = \sqrt{\frac{i\mu\omega}{\sigma + i\varepsilon\omega}}\ \left(= \sqrt{\frac{\mu}{\varepsilon}}\ \ if\ \ \sigma = 0\right)$$

$$sign\ \ determined\ \ by\ \ growth/decay$$

$$(growth \Rightarrow -, decay \Rightarrow +)$$

$$\beta = \frac{2\pi}{\lambda} = \frac{v_{ph}}{\omega}; v_{ph} = \frac{1}{\sqrt{\varepsilon\mu}}$$

This leaves us with a need to know what ε and σ are…

First, we know that the plasma is driven with an rf electric field. We can model this field as a sinusoidal variation,

$$E = \mathrm{Re}\,\tilde{E}e^{i\omega t}.$$

This field will accelerate the electrons

$$m\frac{dv}{dt} = qE - m\nu_m v$$

where ν_m is the electron-neutral collision frequency. Hence the last term is simply the resistive drag term that we need to have to transfer the power from the electrons to the neutrals and the ions. Now assuming that the electron velocity is also sinusoidal, e.g. they follow the electric field.

$$v = \mathrm{Re}\,\tilde{v}e^{i\omega t}$$

Then we find that

$$mi\omega\,\mathrm{Re}\,\tilde{v}e^{i\omega t} = q\,\mathrm{Re}\,\tilde{E}e^{i\omega t} - m\nu_m\,\mathrm{Re}\,\tilde{v}e^{i\omega t}$$

$$\Downarrow$$

$$\tilde{v} = \frac{q}{m}\frac{1}{(i\omega + \nu_m)}\tilde{E}$$

This is of course related to the free current density

$$\tilde{J}_{free} = qn\tilde{v} = \frac{nq^2}{m}\frac{1}{(i\omega + \nu_m)}\tilde{E}$$

$$= \varepsilon_0\omega_{ps}^2\frac{1}{(i\omega + \nu_m)}\tilde{E}$$

$$= \sigma_p\tilde{E}$$

Where $\sigma_p = \varepsilon_0\omega_{ps}^2\frac{1}{(i\omega + \nu_m)}$ is the plasma conductivity.

Likewise, we have the displacement current density

$$\tilde{J}_{displace} = \varepsilon_0\partial_t\tilde{E} = i\varepsilon_0\omega\tilde{E}e^{i\omega t}.$$

Thus the total current density is simply

$$
\begin{aligned}
\nabla \wedge \tilde{H} = \tilde{J}_{total} &= \tilde{J}_{free} + \tilde{J}_{displace} \\
&= \left[\frac{nq^2}{m} \frac{1}{(i\omega + \nu_m)} + i\varepsilon_0 \omega \right] \tilde{E} \\
&= i\omega\varepsilon_0 \left[1 - \frac{\omega_{ps}^2}{(\omega^2 - i\omega\nu_m)} \right] \tilde{E} \\
&= \left[i\omega\varepsilon_0 + \frac{\varepsilon_0 \omega_{ps}^2}{(i\omega + \nu_m)} \right] \tilde{E} \\
&= \left[i\omega\varepsilon_0 + \sigma_p \right] \tilde{E} \\
&\quad - \text{ or } - \\
&= i\omega\varepsilon_p \tilde{E} = i\omega\varepsilon_0 \kappa_p \tilde{E}
\end{aligned}
$$

where ε_p is know as the plasma dielectric constant.

Now let us go back to the wave equation

$$\nabla^2 \mathbf{E} = \mu\sigma\partial_t \mathbf{E} + \mu\varepsilon\partial_t^2 \mathbf{E}$$

Further, we will use $\mathbf{E} = \mathrm{Re}\,\tilde{\mathbf{E}} e^{i\omega t}$ so that

$$
\begin{aligned}
\nabla^2 \mathbf{E} &= \underbrace{i\omega\mu\sigma\mathbf{E}}_{free\ \ current} - \underbrace{\omega^2 \mu\varepsilon\mathbf{E}}_{disp\ \ current} \\
&= i\omega\mu_0 \sigma_p \mathbf{E} - \omega^2 \mu_0 \varepsilon_0 \mathbf{E} \\
&= i\omega\mu_0 (\sigma_p + i\omega\varepsilon_0) \mathbf{E} \\
&= -\omega^2 \mu_0 \varepsilon_p \mathbf{E} \\
&= -\omega^2 \mu_0 \varepsilon_0 \kappa_p \mathbf{E} \\
&= \frac{-\omega^2}{c^2} \kappa_p \mathbf{E}
\end{aligned}
$$

Now we can assume a one-dimensional problem. So that

$$
\begin{aligned}
\nabla^2 \mathbf{E} &= \partial_z \mathbf{E} \\
&= \frac{-\omega^2}{c^2} \kappa_p \mathbf{E} \\
&= \alpha^2 \mathbf{E} \\
&\Downarrow \\
\mathbf{E} &= \left(E_+ e^{+\alpha z} + E_- e^{-\alpha z} \right)
\end{aligned}
$$

Because we cannot physically have the electric field grow as it goes to infinity than the first term must be zero and

$$\alpha = i\frac{\omega}{c} \kappa_p^{1/2} = \frac{1}{\delta}$$

where δ is the skin depth. Further noting that α must be real (as well as ω and c!), we find

$$\alpha = \frac{\omega}{c}\mathrm{Im}\,\kappa_p^{1/2}$$

So… what is $\kappa_p^{1/2}$? From above

$$i\omega\varepsilon_0\kappa_p = \left(i\omega\varepsilon_0 + \sigma_p\right)$$

$$\begin{aligned}
\kappa_p &= \left(1 - \frac{i}{\omega\varepsilon_0}\sigma_p\right) \\
&= \left(1 - \frac{i}{\omega\varepsilon_0}\varepsilon_0\omega_{ps}^2\frac{1}{\left(i\omega + \nu_m\right)}\right) \\
&= \left(1 - \frac{\omega_{ps}^2}{\omega^2}\frac{1}{\left(1 - i\nu_m/\omega\right)}\right) \\
&\approx -\frac{\omega_{ps}^2}{\omega^2}\frac{1}{\left(1 - i\nu_m/\omega\right)}
\end{aligned}$$

where we have made the assumption that $\omega_{ps}^2 >> \omega^2$. This assumption is reasonable as frequencies above the plasma frequency will be cut off. Further the plasma frequency is typically several GHz compared to our typical 13.56 MHz driving frequency. Thus

$$\kappa_p^{1/2} \approx i\frac{\omega_{ps}}{\omega}\frac{1}{\left(1 - i\nu_m/\omega\right)^{1/2}}$$

Now we can examine the different cases

<u>Collisionless case: $\nu_m/\omega << 1$</u>

$$\begin{aligned}
\kappa_p^{1/2} &\approx i\frac{\omega_{ps}}{\omega}\frac{1}{\left(1 - i\nu_m/\omega\right)^{1/2}} \\
&\approx i\frac{\omega_{ps}}{\omega}
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\alpha &= \frac{1}{\delta} \\
&= \frac{\omega}{c}\mathrm{Im}\,\kappa_p^{1/2} \\
&= \frac{\omega}{c}\frac{\omega_{ps}}{\omega} = \frac{\omega_{ps}}{c}
\end{aligned}$$

$$\Downarrow$$

$$\delta = \frac{c}{\omega_{ps}}$$

Collisional case: $\nu_m/\omega >> 1$

$$\kappa_p^{1/2} \approx i\frac{\omega_{ps}}{\omega}\frac{1}{(1-i\nu_m/\omega)^{1/2}}$$

$$\approx i\frac{\omega_{ps}}{\omega}\frac{1}{(-i\nu_m/\omega)^{1/2}}$$

$$= i\frac{\omega_{ps}}{\omega^{1/2}}\frac{1}{i^{3/2}\nu_m^{1/2}}$$

$$= \frac{\omega_{ps}}{i^{1/2}\nu_m^{1/2}\omega^{1/2}}$$

$$\Downarrow$$

$$\alpha = \frac{1}{\delta}$$

$$= \frac{\omega}{c}\operatorname{Im}\kappa_p^{1/2}$$

$$= \frac{\omega}{c}\operatorname{Im}\left(\frac{\omega_{ps}}{i^{1/2}\nu_m^{1/2}\omega^{1/2}}\right)$$

$$= \frac{\omega^{1/2}\omega_{ps}}{\nu_m^{1/2}c}\operatorname{Im}\left(\frac{1}{i^{1/2}}\right)$$

$$\Downarrow$$

$$\delta = \frac{\nu_m^{1/2}c}{\omega^{1/2}\omega_{ps}}\operatorname{Im}\left(i^{1/2}\right)$$

Now what is $\operatorname{Im}\left(i^{1/2}\right)$?

$$\operatorname{Im}\left(i^{1/2}\right) = \operatorname{Im}\left[\left(e^{i\pi/2}\right)^{1/2}\right]$$

$$= \operatorname{Im}\left[e^{i\pi/4}\right]$$

$$= \operatorname{Im}\left[\cos(\pi/4) + i\sin(\pi/4)\right]$$

$$= \operatorname{Im}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}}$$

so

$$\alpha = \frac{\omega^{1/2}\omega_{ps}}{v_m^{1/2} c\sqrt{2}}$$

and

$$\delta = \frac{v_m^{1/2} c}{\omega^{1/2}\omega_{ps}\sqrt{2}}$$

So now we need to ask how much power is being deposited into the discharge?

The power deposited is

$$P_{ave} = \frac{1}{2}\iiint_{Vol} J\,\Sigma E^* d\tau$$

For a planar system, this is very complex. If on the other hand, we where to assume that we have a solenoidal coil, we end up with a simpler system. First, the both the current and electric field are approximately azimuthal. Second, both the current and the electric field are approximately constant. Thus,

$$\begin{aligned} P_{ave} &= \frac{1}{2}\iiint_{Vol} J\,\Sigma E^* d\tau \\ &\approx \frac{1}{2} J\,\Sigma E^* \iiint_{Vol} d\tau \\ &= \frac{1}{2} J_\phi\,\Sigma E_\phi^* \iiint_{Vol} d\tau \\ &\approx \frac{1}{2} J_\phi\,\Sigma E_\phi^* \pi\left(r^2\right)\Big|_{R-\delta}^{R} (z)\Big|_0^l \\ &= \frac{1}{2} J_\phi\,\Sigma E_\phi^* \pi\left(R^2 - (R-\delta)^2\right) l \\ &= \frac{1}{2} J_\phi\,\Sigma E_\phi^* \pi\left(2R\delta - \delta^2\right) l \\ &\approx \frac{1}{2} J_\phi\,\Sigma E_\phi^* 2\pi R\delta l \end{aligned}$$

Now we can use the plasma conductivity, $\sigma_p = \varepsilon_0 \omega_{ps}^2 \frac{1}{(i\omega + v_m)}$, to get

$$P_{ave} = \frac{2\pi R\delta l}{2\sigma_p} J_\phi^2.$$

For high pressure discharge $\omega << v_m$, so that

$$\begin{aligned} \sigma_p &\approx \varepsilon_0 \omega_{ps}^2 \frac{1}{v_m} \\ &= \frac{e^2 n}{m v_m} \end{aligned}$$

and

$$P_{ave} = \frac{m v_m \pi R\delta l}{e^2 n} J_\phi^2$$

Further noting that

$$I_\phi = J_\phi l\delta$$

so that

$$P_{ave} = \frac{m\nu_m \pi R}{e^2 n l \delta} I_\phi^2$$

$$= \frac{1}{2} R_p I_\phi^2$$

so

$$R_p = \frac{2m\nu_m \pi R}{e^2 n l \delta}$$

is the resistance. The other part of the picture is the inductance. By definition the induction is the total magnetic flux Φ contained by the structure divided by the current I.

$$L_p = \Phi / I$$

For a single turn of the coil, the flux is

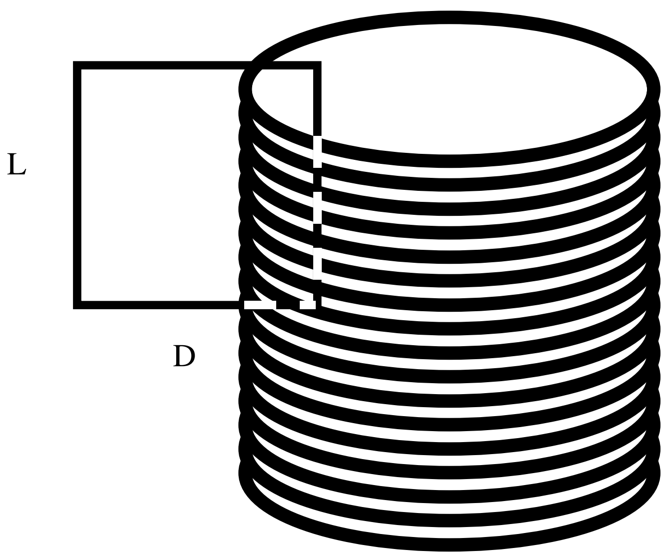
$$\Phi = \pi R^2 B_z$$

$$= \mu_0 \pi R^2 H_z$$

further the magnetic field produced by the induced surface current is simply

$$H_z = J_\phi \delta$$

If we assume that the source is a long solinoid then we can determine the magnetic field.



By symmetry in z and ϕ, the magnetic field is only dependent on the radial position.

$B = B(r)$ Further

$$\mathbf{B} = B\hat{\mathbf{z}}$$

Then by Maxwell's equations

$$\mu_0^{-1} \oint \mathbf{B} \Sigma d\mathbf{l} = \iint \mathbf{J} \Sigma d\mathbf{s}$$

With the exception of the curve shown above, the current through a given surface is zero. (Ok you can pick some 'fun' surfaces that YOU can work with…) By letting D -> 0, with the coil still passing through surface, we find

$$\mu_0^{-1}\oint \mathbf{B}\,\Sigma d\mathbf{l} = In'L$$

$$\mu_0^{-1} B_z L =$$

$$\Downarrow$$

$$B_z = \begin{cases} \mu_0 In' & inside \\ 0 & outside \end{cases}$$

where n' is the number of turns per length L.
The flux of the magnetic field from the coil (a) through a surface (b) is

$$\Phi_{ab} = \iint \mathbf{B}_a\,\Sigma d\mathbf{s}_b$$

Further the induction is given by

$$L_{ab} = \frac{N\Phi_{ab}}{I}$$

(Often L is used for self induction and M is used for mutual induction. Note that the mutual induction is the such that $M_{ab} = M_{ba}$. This can be shown from

$$\begin{aligned}\Phi_{ab} &= \iint \mathbf{B}_a\,\Sigma d\mathbf{s}_b \\ &= \iint \left(\nabla \wedge \mathbf{A}_a\right)\Sigma d\mathbf{s}_b \\ &= \oint_b \mathbf{A}_a\,\Sigma d\mathbf{l}_b \\ &= \oint_b \left(\frac{\mu_0 I_a}{4\pi}\oint_a \frac{1}{r}\,d\mathbf{l}_a\right)\Sigma d\mathbf{l}_b \\ &= \frac{\mu_0 I_a}{4\pi}\oint_b\oint_a \frac{1}{r}\,d\mathbf{l}_a\,\Sigma d\mathbf{l}_b \\ &= M_{ab} I_a\end{aligned}$$

)

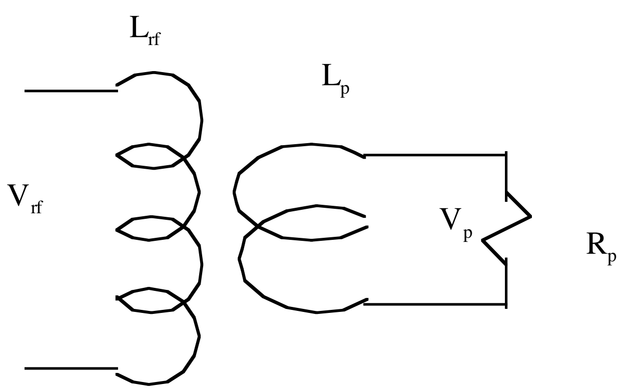
Now let us assume that the plasma is a single turn coil of radius R inside the power coil of radius b. Thus,

$$\begin{aligned}\Phi_{rf\ \ p} &= \Phi_{rf\ \ rf}\ \overbrace{\frac{\pi R^2}{\pi b^2}}^{area\ \ ratio} \\ &= \frac{\mu_0 N I_{rf}}{L}\pi R^2\end{aligned}$$

and

$$L_{rf\ \ p} = \frac{N\Phi_{rf\ \ p}}{I}$$
$$= \frac{\mu_0 N^2}{L}\pi R^2$$
$$L_{rf\ \ rf} = \frac{N\Phi_{rf\ \ rf}}{I_{rf}}$$
$$= \frac{\mu_0 N^2}{L}\pi b^2$$
$$L_{p\ \ p} = \frac{\overbrace{N}^{=1}\Phi_{p\ \ p}}{I_p}$$
$$= \frac{\mu_0}{L}\pi R^2$$

Now, we need to consider the circuit, which looks like



$$V_{rf} = i\omega L_{rf\ \ rf} I_{rf} + i\omega L_{rf\ \ p} I_p$$
$$V_p = R_p I_p = i\omega L_{rf\ \ p} I_{rf} + i\omega L_{p\ \ p} I_p$$

Now the impedance of the source is

$$
\begin{aligned}
\frac{V_{rf}}{I_{rf}} &= Z_{rf} \\
&= i\omega L_{rf\ \ rf} - \frac{\omega^2 L_{rf\ \ p}^2}{\left(R_p - i\omega L_{p\ \ p}\right)} \\
&= i\omega \frac{\mu_0 N^2}{L}\pi b^2 - \frac{\omega^2 \frac{\mu_0^2 N^2}{L^2}\pi^2 R^4}{\left(\frac{2m\nu_m \pi R}{e^2 n l \delta} - i\omega \frac{\mu_0}{L}\pi R^2\right)} \quad but \quad R_p << \omega L_{p\ \ p} \\
&\approx i\omega \frac{\mu_0 N^2}{L}\pi b^2 - i\frac{\omega \mu_0 N^2 \pi R^2}{L} \\
&= i\omega \frac{\mu_0 N^2}{L}\pi\left(b^2 - R^2\right) \\
&= i\omega L_s
\end{aligned}
$$

or to get the real part

$$
\begin{aligned}
Z_{rf} &= i\omega \frac{\mu_0 N^2}{L}\pi b^2 - \frac{\omega^2 \frac{\mu_0^2 N^2}{L^2}\pi^2 R^4}{\left(\left(\frac{2m\nu_m \pi R}{e^2 n L \delta}\right)^2 + \left(\omega \frac{\mu_0}{L}\pi R^2\right)^2\right)}\left(\frac{2m\nu_m \pi R}{e^2 n L \delta} + i\omega \frac{\mu_0}{L}\pi R^2\right) \\
&\approx i\omega \frac{\mu_0 N^2}{L}\pi b^2 - i\frac{\omega \mu_0 N^2 \pi R^2}{L} - \frac{2m\nu_m \mu_0 N^2 \pi R}{e^2 n \delta L} \\
&= i\omega L_s + R_s
\end{aligned}
$$