Lecture 3 "Introduction to material processing" is covered in a separate document (power point slides).

Lecture 4 Basic Plasma Equations
In our first lecture, we dealt with some simple properties of fluids. While plasmas have many of the qualities of fluids, they also have charged particles which are subject to electromagnetic forces.
In undergraduate electromagnetism, we learned that the E-M forces were governed by a set of equations known as Maxwell's Equations. (Here we also include the equation for charge conservation.)

<u>Maxwell's Equations Point Form:</u>

$$\nabla \wedge \mathbf{E} = -\partial_t \mathbf{B}$$
$$\nabla \wedge \mathbf{H} = \mathbf{J}_{free} + \partial_t \mathbf{D}$$
$$\nabla \Sigma \mathbf{D} = \rho_{free}$$
$$\nabla \Sigma \mathbf{B} = 0$$
$$\nabla \Sigma \mathbf{J} = -\partial_t \rho$$

<u>Maxwell's Equations Integral Form:</u>

$$\oint_{\substack{Enclosing\\curve}} \mathbf{E} \bullet d\mathbf{l} = -\frac{d}{dt} \iint_{Surface} \mathbf{B}\, \Sigma d\mathbf{S}$$
$$\oint_{\substack{Enclosing\\curve}} \mathbf{H} \bullet d\mathbf{l} = \iint_{Surface} \mathbf{J}_{free}\, \Sigma d\mathbf{S} + \frac{d}{dt} \iint_{Surface} \mathbf{D}\, \Sigma d\mathbf{S}$$
$$\oint\!\!\oint_{\substack{Enclosing\\surface}} \mathbf{D}\, \Sigma d\mathbf{S} = \iiint_{Volume} \rho_{free}\, d\tau$$
$$\oint\!\!\oint_{\substack{Enclosing\\surface}} \mathbf{B}\, \Sigma d\mathbf{S} = 0$$
$$\oint\!\!\oint_{\substack{Enclosing\\surface}} \mathbf{J}\, \Sigma d\mathbf{S} = -\frac{d}{dt} \iiint_{Volume} \rho\, d\tau$$

We can readily transfer from the point form of Maxwell's equations to the integral from using the Divergence and Stoke's Theorems.

<u>Divergence Theorem</u>
$$\iiint_{Volume} \nabla \bullet \mathbf{V}\, d\tau = \oint\!\!\oint_{\substack{enclosing\\surface}} \mathbf{V} \bullet \mathbf{n}\, d\sigma$$

<u>Stoke's Theorem</u>
$$\iint_{surface} (\nabla \wedge \mathbf{V}) \bullet \mathbf{n}\, d\sigma = \oint_{\substack{enclosing\\curve}} \mathbf{V} \bullet d\mathbf{r}$$

In addition, we also found **<u>Lorentz Force Equation:,</u>** $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$, and **<u>Ohm's Law:</u>**, $\mathbf{J} = \sigma \mathbf{E}$.

To keep things simple, we will at first just focus on plasma-induced fields. Of particular interest is the first equation in our list of Maxwell's equations, $\nabla \wedge \mathbf{E} = -\partial_t \mathbf{B}$ (Faraday's Law). We know that we can let $\mathbf{B} = \nabla \wedge \mathbf{A}$, where $\mathbf{A}$ is know as the magnetic vector potential. Thus, rewriting Faraday's law, we find

$$\nabla \wedge \mathbf{E} = -\partial_t \nabla \wedge \mathbf{A}$$

$$\Downarrow$$

$$\nabla \wedge (\mathbf{E} + \partial_t \mathbf{A}) = 0$$

$$\Downarrow$$

$$\mathbf{E} = -\nabla \Phi - \partial_t \mathbf{A}$$

where Φ is the scalar potential. Often in laboratory plasmas there is no time varying magnetic field and hence

$$\partial_t \mathbf{A} = 0$$

$$\mathbf{E} = \nabla \Phi$$

NOTE THAT THIS IS NOT TRUE IN FUSION PLAMSAS NOR IS IT TRUE IN SOME ARC TYPE PLASMAS! IN BOTH CASES THERE EXISTS A STRONG SELF-INDUCED TIME-VARYING MAGNETIC FIELDS.

We can now plug our scalar potential back into Gauss' Law to get

$$\begin{aligned} \nabla \cdot \mathbf{E} &= -\nabla \cdot \nabla \Phi \\ &= -\nabla^2 \Phi \\ &= \frac{\rho}{\varepsilon} \end{aligned} .$$

This is known as Poisson's equation. Assume that all of our charged particles are singly charged positive ions, Ion^+, and negatively charged electrons, e^-. (Electron-positron, Ion^+-Ion^-, etc, plasmas also exist. We are just looking at the most typical.) Then Poisson's equation becomes

$$\begin{aligned} \nabla^2 \Phi &= -\frac{\rho}{\varepsilon} \\ &= \frac{e}{\varepsilon}(n_e - n_i) \end{aligned} ,$$

where n_e is the electron density and n_i is the ion density. This tells us that a plasma can induce an electric field that depends on the local densities of the charge carrier. Because of this, we will begin to look at the velocity distribution function $f(\mathbf{r},\mathbf{v},t) = f(x,y,z,v_x,v_y,v_z,t)$. The number of particles that are inside a volume of $dxdydzdv_x dv_y dv_z$ is simply $f(x,y,z,v_x,v_y,v_z,t)dxdydzdv_x dv_y dv_z$. Often the six coordinates are considered to be independent. Then the rate of change of particles at point is

$$
\begin{aligned}
\frac{df}{dt} &= \frac{\partial f}{\partial \mathbf{r}}\frac{d\mathbf{r}}{dt} + \frac{\partial f}{\partial \mathbf{v}}\frac{d\mathbf{v}}{dt} + \frac{\partial f}{\partial t} \\
&= (\nabla_{\mathbf{r}} f)\Sigma\mathbf{v} + (\nabla_{\mathbf{v}} f)\Sigma\mathbf{a} + \frac{\partial f}{\partial t} \\
&= \mathbf{v}\,\Sigma(\nabla_{\mathbf{r}} f) + \frac{\mathbf{F}}{m}\Sigma(\nabla_{\mathbf{v}} f) + \frac{\partial f}{\partial t}
\end{aligned}
$$

This is the Boltzmann equation. If particles are not sourced/sunk at the point then the total number does not change and hence

$$
\frac{df}{dt} = 0 = (\nabla_{\mathbf{r}} f)\Sigma\mathbf{v} + (\nabla_{\mathbf{v}} f)\Sigma\mathbf{a} + \frac{\partial f}{\partial t}
$$

This is known as the collisionless Boltzmann equation or the Vlasov equation.

<u>The velocity distribution and macroscopic quantities.</u>

In the very first class, we talked about the velocity distribution of a gas, and an assortment of different velocity distributions. Further, we related the velocity distribution, f(**v**), to a probability distribution, p(y). Now, we want to return to the concept.

<u>Expectation value</u>

Let us assume that we can randomly pick a y from a set, having a probability $p(y_0)$ of picking y_0. What value would we expect to get – on average – for y? If I were to draw 100 y's from a basket, we would expect $p(y_0)*100$ to be y_0. Assume that I can also pick y_1, y_2, and y_3. Then the average of my 100 picks would be

$$
\begin{aligned}
\langle y \rangle &= \frac{\left(y_0 p(y_0) + y_1 p(y_1) + y_2 p(y_2) + y_3 p(y_3)\right)}{\left(p(y_0) + p(y_1) + p(y_2) + p(y_3)\right)} \\
&= \frac{\sum_i y_i p(y_i)}{\sum_i p(y_i)}
\end{aligned}
$$

If I were to have a continuous set of y's then the equation becomes

$$
\langle y \rangle = \frac{\int y p(y) dy}{\int p(y) dy}
$$

as we have already used. (Often we normalize the probability such that $\int p(y)dy = 1$.) This averaging is important as when we make a measurement, we are typically making a series of measurements – even if we don't know it – and our result is the average of our series of measurements.

f(**r**,**v**,t) is our velocity and density distribution function and it is in a very real sense a probability distribution – with 6 dimensions (**r**,**v**) and one free parameter (t)! (Because of this definition, **r** and **v** are truly independent quantities – **v** does not depend on the location only f. Thus, if we where to desire to know the average velocity then we look at

$$
\langle \mathbf{v} \rangle = \frac{\int \mathbf{v} f(\mathbf{r},\mathbf{v},t) d\mathbf{v}}{\int f(\mathbf{r},\mathbf{v},t) d\mathbf{v}} .
$$

Likewise the average of the $\mathbf{v}^2$ is simply

$$\left\langle \mathbf{v}^2 \right\rangle = \frac{\int \mathbf{v}^2 f(\mathbf{r},\mathbf{v},t)d\mathbf{v}}{\int f(\mathbf{r},\mathbf{v},t)d\mathbf{v}}.$$

We can do this for all sorts of averages

$$\frac{\int \mathbf{r}^0 f(\mathbf{r},\mathbf{v},t)d\mathbf{r}}{\int f(\mathbf{r},\mathbf{v},t)d\mathbf{r}};\ \frac{\int \mathbf{r}^1 f(\mathbf{r},\mathbf{v},t)d\mathbf{r}}{\int f(\mathbf{r},\mathbf{v},t)d\mathbf{r}};\ \frac{\int \mathbf{r}^2 f(\mathbf{r},\mathbf{v},t)d\mathbf{r}}{\int f(\mathbf{r},\mathbf{v},t)d\mathbf{r}};....$$

$$\frac{\int \mathbf{v}^0 f(\mathbf{r},\mathbf{v},t)d\mathbf{v}}{\int f(\mathbf{r},\mathbf{v},t)d\mathbf{v}};\ \frac{\int \mathbf{v}^1 f(\mathbf{r},\mathbf{v},t)d\mathbf{v}}{\int f(\mathbf{r},\mathbf{v},t)d\mathbf{v}};\ \frac{\int \mathbf{v}^2 f(\mathbf{r},\mathbf{v},t)d\mathbf{v}}{\int f(\mathbf{r},\mathbf{v},t)d\mathbf{v}};....$$

However, we are not interested in most of them. In fact we typically only look at

$n(\mathbf{r},t) = \int \mathbf{v}^0 f(\mathbf{r},\mathbf{v},t)d\mathbf{v}$	*density*
$\Gamma(\mathbf{r},t) = n\langle \mathbf{v}\rangle = \int \mathbf{v}^1 f(\mathbf{r},\mathbf{v},t)d\mathbf{v}$	*particle flux*
$E_{total} = \frac{1}{2}mn\left\langle \mathbf{v}^2\right\rangle = \frac{1}{2}m\int \mathbf{v}^2 f(\mathbf{r},\mathbf{v},t)d\mathbf{v}$	*total kinetic energy*
$E_{random} = \frac{3}{2}P = \frac{3}{2}nkT = \frac{1}{2}mn\left\langle \left(\mathbf{v}-\langle \mathbf{v}\rangle\right)^2\right\rangle = \frac{1}{2}m\int \left(\mathbf{v}-\langle \mathbf{v}\rangle\right)^2 f(\mathbf{r},\mathbf{v},t)d\mathbf{v}$	*random kinetic energy / pressure* (P)
$E_{directed} = E_{total} - E_{random} = \frac{1}{2}mn\langle \mathbf{v}\rangle^2$	*directed kinetic energy*

(These are known as the zeroth, first, and second moments of the distribution function.)

Why are we bringing this up again? Well we can do the same thing to Boltzmann's Equation.

<u>Zeroth moment of the Boltzmann Equation – The Equation of Continuity (Particle conservation)</u>

$$\int\left(\frac{df}{dt}\right)d\mathbf{v} = Gain - Loss = f\big|_c = \int\left(\mathbf{v}\,\Sigma(\nabla_\mathbf{r} f) + \frac{\mathbf{F}}{m}\Sigma(\nabla_\mathbf{v} f) + \frac{\partial f}{\partial t}\right)d\mathbf{v}$$

$$= \left(\int\left(\mathbf{v}\,\Sigma(\nabla_\mathbf{r} f)\right)d\mathbf{v} + \int\left(\frac{\mathbf{F}}{m}\Sigma(\nabla_\mathbf{v} f)\right)d\mathbf{v} + \int\left(\frac{\partial f}{\partial t}\right)d\mathbf{v}\right)$$

$$= \left(\nabla_\mathbf{r}\,\Sigma\int(\mathbf{v}f)d\mathbf{v} + \int\left(\frac{\mathbf{F}}{m}\Sigma(\nabla_\mathbf{v} f)\right)d\mathbf{v} + \left(\frac{\partial}{\partial t}\right)\int f d\mathbf{v}\right)$$

$$= \left(\nabla_\mathbf{r}\,\Sigma n\langle \mathbf{v}\rangle + \int\left(\frac{\mathbf{F}}{m}\Sigma(\nabla_\mathbf{v} f)\right)d\mathbf{v} + \frac{\partial n}{\partial t}\right)$$

Now what happens to the central term on the right hand side? First let us assume that force, $\mathbf{F}$, is independent of the velocity. (Strictly speaking, this is not true as $\mathbf{F} = e(\mathbf{v}\wedge\mathbf{B})$.) Then the integral becomes

$$\int\left(\frac{\mathbf{F}}{m}\cdot(\nabla_{\mathbf{v}} f)\right)d\mathbf{v} = \frac{\mathbf{F}}{m}\cdot\int\left((\nabla_{\mathbf{v}} f)\right)d\mathbf{v}$$
$$= \frac{\mathbf{F}}{m}(f)\Big|_{-\infty}^{\infty} = \frac{\mathbf{F}}{m}(0-0) = 0 .$$

The distribution goes to zero for physical reasons – we don't want an infinite density. If the force is dependent on velocity, it is typically just to first order as in the magnetic force. Then

$$\int\left(\frac{\mathbf{F}}{m}\cdot(\nabla_{\mathbf{v}} f)\right)d\mathbf{v} = \frac{q}{m}\int\left((\mathbf{v}\wedge\mathbf{B})\cdot\nabla_{\mathbf{v}} f\right)d\mathbf{v}$$

To solve this, we must deal with some vector identities. First,

$$\mathbf{A}\wedge(\mathbf{B}\wedge\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) \quad and$$
$$\mathbf{A}\cdot(\mathbf{B}\wedge\mathbf{C}) = (\mathbf{A}\wedge\mathbf{B})\cdot\mathbf{C} = -(\mathbf{B}\wedge\mathbf{A})\cdot\mathbf{C}.... .$$

Thus,

$$(\mathbf{v}\wedge\mathbf{B})\cdot\nabla_{\mathbf{v}} f = (\mathbf{v}\wedge\mathbf{B})\cdot\nabla_{\mathbf{v}}[f]$$
$$= \nabla_{\mathbf{v}}[f]\cdot(\mathbf{v}\wedge\mathbf{B})$$
$$= \left(\nabla_{\mathbf{v}}[f]\wedge\mathbf{v}\right)\cdot\mathbf{B}$$
$$= \left(\nabla_{\mathbf{v}}\wedge[f\mathbf{v}]\right)\cdot\mathbf{B} - \left(f\nabla_{\mathbf{v}}\wedge[\mathbf{v}]\right)\cdot\mathbf{B}$$
$$= \nabla_{\mathbf{v}}\cdot\left([f\mathbf{v}]\wedge\mathbf{B}\right) - f\nabla_{\mathbf{v}}\cdot(\mathbf{v}\wedge\mathbf{B})$$

First we will look at the second term on the right.

$$\nabla_{\mathbf{v}}\cdot(\mathbf{v}\wedge\mathbf{B}) = \partial_{V_x}\left(V_yB_z - V_zB_y\right) + \partial_{V_y}\left(V_zB_x - V_xB_z\right) + \partial_{V_z}\left(V_xB_y - V_yB_x\right)$$
$$= \mathbf{0}$$

Now we can go back to the integral and we find

$$\int\left(\frac{\mathbf{F}}{m}\cdot(\nabla_{\mathbf{v}} f)\right)d\mathbf{v} = \frac{-q}{m}\int\left((\mathbf{v}\wedge\mathbf{B})\cdot\nabla_{\mathbf{v}} f\right)d\mathbf{v}$$
$$= \frac{-q}{m}\int\left(\nabla_{\mathbf{v}}\cdot([f\mathbf{v}]\wedge\mathbf{B})\right)d\mathbf{v}$$
$$= \frac{-q}{m}\oint_{\substack{surface\\ at\ \infty}}\left(([f\mathbf{v}]\wedge\mathbf{B})\right)d\mathbf{v}$$
$$= 0$$

Here $f\mathbf{v}$ goes to zero on the surface for physical reasons – we don't want an infinite energy so f must go to zero faster than $\mathbf{v}$.

This leaves the continuity equation

$$\boxed{f\big|_c = \left(\nabla_{\mathbf{r}}\cdot(n\langle\mathbf{v}\rangle) + \frac{\partial n}{\partial t}\right)}$$

<u>First moment of the Boltzmann equation – Momentum Conservation</u>

Now we multiple Boltzmann's equation

$$\frac{df}{dt} = \mathbf{v}\cdot(\nabla_{\mathbf{r}} f) + \frac{\mathbf{F}}{m}\cdot(\nabla_{\mathbf{v}} f) + \frac{\partial f}{\partial t}$$

by $m\mathbf{v}$ and integrate over velocity to get

$$m\int \mathbf{v}\left(\frac{df}{dt}\right)d\mathbf{v} = \Delta Momentum = m\int \mathbf{v}\left(\mathbf{v}\,\Sigma(\nabla_{\mathbf{r}} f) + \frac{\mathbf{F}}{m}\Sigma(\nabla_{\mathbf{v}} f) + \frac{\partial f}{\partial t}\right)d\mathbf{v}$$

$$= m\left(\int \mathbf{v}\left(\mathbf{v}\,\Sigma(\nabla_{\mathbf{r}} f)\right)d\mathbf{v} + \int \mathbf{v}\left(\frac{\mathbf{F}}{m}\Sigma(\nabla_{\mathbf{v}} f)\right)d\mathbf{v} + \int \mathbf{v}\left(\frac{\partial f}{\partial t}\right)d\mathbf{v}\right).$$

First we will look at the third part of the right-hand side

$$\int \mathbf{v}\left(\frac{\partial f}{\partial t}\right)d\mathbf{v} = \frac{\partial}{\partial t}\left[\int \mathbf{v}(f)d\mathbf{v}\right] - \int \frac{\partial \mathbf{v}}{\partial t}(f)d\mathbf{v}$$

$$= \frac{\partial}{\partial t}\left[\int \mathbf{v}(f)d\mathbf{v}\right] - \underbrace{\int \mathbf{a}(f)d\mathbf{v}}_{\substack{Which\ is\ zero\\ from\ before}}$$

$$= \frac{\partial}{\partial t}\left[\int \mathbf{v}(f)d\mathbf{v}\right]$$

$$= \frac{\partial}{\partial t}\left[n\langle \mathbf{v}\rangle\right]$$

Now for the second term

$$\int \mathbf{v}\left(\frac{\mathbf{F}}{m}\Sigma(\nabla_{\mathbf{v}} f)\right)d\mathbf{v} = \frac{q}{m}\int \mathbf{v}\left[(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})\,\Sigma(\nabla_{\mathbf{v}} f)\right]d\mathbf{v}\ \ using\ \ the\ \ chain\ \ rule$$

$$= \frac{q}{m}\int \nabla_{\mathbf{v}} \underbrace{\Sigma}_{\substack{This\ operates\\ on\ the\ force}} \underbrace{\left[f\mathbf{v}(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})\right]}_{\substack{Notice\ that\ this\\ notation\ is\ correct\\ -This\ is\ a\ tensor(matrix)}} d\mathbf{v}$$

$$-\frac{q}{m}\int f\mathbf{v}\nabla_{\mathbf{v}}\,\Sigma\left[(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})\right]d\mathbf{v} - \frac{q}{m}\int\left[f(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})\right]\Sigma\nabla_{\mathbf{v}}\mathbf{v}\,d\mathbf{v}$$

$$= \frac{q}{m}\overbrace{\oint_{surface} f\mathbf{v}(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})d\mathbf{v}}^{=0\ \ from\ \ before}$$

$$-\frac{q}{m}\int f\mathbf{v}\overbrace{\nabla_{\mathbf{v}}\,\Sigma\left[(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})\right]}^{=0\ \ from\ \ before}d\mathbf{v} - \frac{q}{m}\int\left[f(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})\right]\Sigma\overbrace{\nabla_{\mathbf{v}}\mathbf{v}}^{\mathbf{I}=identity\ \ tensor}d\mathbf{v}$$

$$= -\frac{q}{m}\int\left[f(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})\right]\Sigma\mathbf{I}\,d\mathbf{v}$$

$$= -\frac{q}{m}\int\left[f(\mathbf{E}+\mathbf{v}\wedge\mathbf{B})\right]d\mathbf{v}$$

$$= -\frac{q}{m}n\left(\mathbf{E}+\langle\mathbf{v}\rangle\wedge\mathbf{B}\right)$$

Finally, we can deal with the first term…

(HOLD ON THIS GETS EASIER!)

$$\int \mathbf{v}\left(\mathbf{v}\,\Sigma(\nabla_{\mathbf{r}} f)\right)d\mathbf{v} = \nabla_{\mathbf{r}}\,\Sigma\int f\mathbf{v}\mathbf{v}d\mathbf{v}$$
$$= \nabla_{\mathbf{r}}\,\Sigma\left(n\langle\mathbf{v}\mathbf{v}\rangle\right)$$

What precisely is this last term? Well let us consider what **v** is…

$\mathbf{v} = \mathbf{v}_{directed} + \mathbf{v}_{random}$

Thus

$$\nabla_{\mathbf{r}}\,\Sigma\left[n\langle\mathbf{v}\mathbf{v}\rangle\right] = \nabla_{\mathbf{r}}\,\Sigma\left[n\left\langle(\mathbf{v}_{dir}+\mathbf{v}_{rnd})(\mathbf{v}_{dir}+\mathbf{v}_{rnd})\right\rangle\right]$$
$$= \nabla_{\mathbf{r}}\,\Sigma\left[n\left\langle(\mathbf{v}_{dir}\mathbf{v}_{dir}+\mathbf{v}_{dir}\mathbf{v}_{rnd}+\mathbf{v}_{rnd}\mathbf{v}_{dir}+\mathbf{v}_{rnd}\mathbf{v}_{rnd})\right\rangle\right] \quad but$$
$$\langle\mathbf{v}_{dir}\rangle = \mathbf{v}_{dir} \quad so$$
$$= \nabla_{\mathbf{r}}\,\Sigma\left(n\mathbf{v}_{dir}\mathbf{v}_{dir}\right) + \nabla_{\mathbf{r}}\,\Sigma\left(2n\mathbf{v}_{dir}\underbrace{\langle\mathbf{v}_{rnd}\rangle}_{=0\ -\ it\ is\ random!}\right) + \nabla_{\mathbf{r}}\,\Sigma\left(n\langle\mathbf{v}_{rnd}\mathbf{v}_{rnd}\rangle\right)$$

The last term we have dealt with before. We know for a Maxwellian that

$$\langle E\rangle = \tfrac{1}{2}m\langle\mathbf{v}^2\rangle = \tfrac{3}{2}kT = \tfrac{3}{2}\frac{\mathbf{P}}{n} \quad but \quad for \quad our \quad calculations$$

$\mathbf{v} = \mathbf{v}_{rnd}!$

Thus we find

$\mathbf{P} = mn\langle\mathbf{v}_{rnd}\mathbf{v}_{rnd}\rangle$

(Notice that **P** is a tensor – as might make sense – Pressure is direction dependent)

Now going back to our momentum conservation equation we find

$$\Delta Momentum \equiv \Delta\mathbf{M}\big|_c = m\left(\int\mathbf{v}\left(\mathbf{v}\,\Sigma(\nabla_{\mathbf{r}} f)\right)d\mathbf{v} + \int\mathbf{v}\left(\frac{\mathbf{F}}{m}\Sigma(\nabla_{\mathbf{v}} f)\right)d\mathbf{v} + \int\mathbf{v}\left(\frac{\partial f}{\partial t}\right)d\mathbf{v}\right)$$
$$= m\left(\nabla_{\mathbf{r}}\,\Sigma\left(n\langle\mathbf{v}\rangle\langle\mathbf{v}\rangle\right) + \nabla_{\mathbf{r}}\,\Sigma\frac{\mathbf{P}}{m} + -\frac{q}{m}n(\mathbf{E}+\langle\mathbf{v}\rangle\wedge\mathbf{B}) + \frac{\partial}{\partial t}\left[n\langle\mathbf{v}\rangle\right]\right)$$
$$= \left(m\langle\mathbf{v}\rangle\nabla_{\mathbf{r}}\,\Sigma\left(n\langle\mathbf{v}\rangle\right) + mn\langle\mathbf{v}\rangle\,\Sigma\nabla_{\mathbf{r}}\left(\langle\mathbf{v}\rangle\right) + m\nabla_{\mathbf{r}}\,\Sigma\mathbf{P} - qn(\mathbf{E}+\langle\mathbf{v}\rangle\wedge\mathbf{B}) + m\frac{\partial}{\partial t}\left[n\langle\mathbf{v}\rangle\right]\right)$$

using the continuity equation

$$f\big|_c = \left(\nabla_{\mathbf{r}}\,\Sigma\left(n\langle\mathbf{v}\rangle\right) + \frac{\partial n}{\partial t}\right)$$

and rearranging…

$$\Delta\mathbf{M}\big|_c = \left(m\langle\mathbf{v}\rangle\nabla_{\mathbf{r}}\,\Sigma\big(n\langle\mathbf{v}\rangle\big) + mn\langle\mathbf{v}\rangle\,\Sigma\nabla_{\mathbf{r}}\big(\langle\mathbf{v}\rangle\big) + \nabla_{\mathbf{r}}\,\Sigma\mathbf{P} - qn\big(\mathbf{E} + \langle\mathbf{v}\rangle \wedge \mathbf{B}\big) + m\frac{\partial}{\partial t}\big[n\langle\mathbf{v}\rangle\big] \right)$$

$$= \begin{pmatrix} m\langle\mathbf{v}\rangle\left(f\big|_c - \dfrac{\partial n}{\partial t}\right) + mn\langle\mathbf{v}\rangle\,\Sigma\nabla_{\mathbf{r}}\big(\langle\mathbf{v}\rangle\big) + \nabla_{\mathbf{r}}\,\Sigma\mathbf{P} \\ -qn\big(\mathbf{E} + \langle\mathbf{v}\rangle \wedge \mathbf{B}\big) + mn\dfrac{\partial}{\partial t}\big[\langle\mathbf{v}\rangle\big] + m\langle\mathbf{v}\rangle\dfrac{\partial n}{\partial t} \end{pmatrix}$$

$$\Downarrow$$

$$\boxed{mn\left(\frac{\partial\langle\mathbf{v}\rangle}{\partial t} + \langle\mathbf{v}\rangle\,\Sigma\nabla_{\mathbf{r}}\langle\mathbf{v}\rangle\right) = \underbrace{\Delta\mathbf{M}\big|_c}_{\substack{\text{momentum}\\ \text{lost via}\\ \text{collisions}}} - \underbrace{m\langle\mathbf{v}\rangle f\big|_c}_{\substack{\text{momentum change}\\ \text{via particle gain/loss}}} - \nabla_{\mathbf{r}}\,\Sigma\mathbf{P} + qn\big(\mathbf{E} + \langle\mathbf{v}\rangle \wedge \mathbf{B}\big)}$$

This is slightly different then Lieberman but it is correct.
This is also known as the <u>fluid equation of motion</u>

<u>Energy Conservation (Heat flow equation)</u>
Now we multiply Boltzmann's equation

$$\frac{df}{dt} = \mathbf{v}\,\Sigma\big(\nabla_{\mathbf{r}} f\big) + \frac{\mathbf{F}}{m}\,\Sigma\big(\nabla_{\mathbf{v}} f\big) + \frac{\partial f}{\partial t}$$

by $\frac{1}{2}m\mathbf{v}^2$ and integrate over velocity to get

$$\tfrac{1}{2}m\int \mathbf{v}^2\left(\frac{df}{dt}\right)d\mathbf{v} = \Delta Energy = \tfrac{1}{2}m\left(\int \mathbf{v}^2\big(\mathbf{v}\,\Sigma(\nabla_{\mathbf{r}} f)\big)d\mathbf{v} + \int \mathbf{v}^2\left(\frac{\mathbf{F}}{m}\,\Sigma(\nabla_{\mathbf{v}} f)\right)d\mathbf{v} + \int \mathbf{v}^2\left(\frac{\partial f}{\partial t}\right)d\mathbf{v}\right).$$

This is trivial and is left to the reader (yeah right!)

Rather than go through this derivation, we will accept Lieberman's formula. (We don't have time to do this in class now anyway and the first two formula are more important.)

<u>SIDE NOTE ON VECTORS</u>

We know that in matrix notation the inner product is

$$\mathbf{A}\,\Sigma\mathbf{B} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \Sigma \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

$$= \overbrace{\begin{pmatrix} A_x & A_y & A_z \end{pmatrix}}^{\Rightarrow} \left.\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}\right\}^{\Downarrow}$$

$$= A_x B_x + A_y B_y + A_z B_z$$

What we have above is the form

$$\mathbf{AB} = \overbrace{\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}}^{\Rightarrow}\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}\Bigg\}^{\Downarrow} \quad \begin{matrix} \textit{multiply squential} \\ \textit{row elements of } \mathbf{A} \textit{ with} \\ \textit{column elements of } \mathbf{B} \end{matrix}$$

$$= \begin{pmatrix} A_x B_x & A_y B_x & A_z B_x \\ A_x B_y & A_y B_y & A_z B_y \\ A_x B_z & A_y B_z & A_z B_z \end{pmatrix}$$

Equilibrium Properties

Boltzmann's Relation

Boltzmann's relation makes use of the momentum conservation equation

$$mn\left(\frac{\partial\langle\mathbf{v}\rangle}{\partial t} + \langle\mathbf{v}\rangle\Sigma\nabla_{\mathbf{r}}\langle\mathbf{v}\rangle\right) = \Delta\mathbf{M}\big|_c - m\langle\mathbf{v}\rangle f\big|_c - \nabla_{\mathbf{r}}\Sigma\mathbf{P} + qn(\mathbf{E} + \langle\mathbf{v}\rangle\wedge\mathbf{B}).$$

Under many conditions, most of these terms are small, particularly for the electron. This is particularly true for the electrons. If we assume that the electron is massless, $m_e \approx 0$, then our equation reduces to

$$0 = -\nabla_{\mathbf{r}}\Sigma\mathbf{P} + qn(\mathbf{E} + \langle\mathbf{v}\rangle\wedge\mathbf{B}).$$

If we further assume that the electron is either traveling along the **B** field or that **B**=0 then

$$\nabla_{\mathbf{r}}\Sigma\mathbf{P} = qn\mathbf{E}$$

but

$$\nabla_{\mathbf{r}} \Sigma \mathbf{P} = \nabla \Sigma \begin{pmatrix} P_x & 0 & 0 \\ 0 & P_y & 0 \\ 0 & 0 & P_z \end{pmatrix}$$

$$\begin{aligned} \partial_x P_x + \partial_y P_y + \partial_z P_z &= \partial_{\mathbf{r}} \Sigma \mathbf{P} = \partial_{\mathbf{r}} (nkT_e) \\ &= kT_e \partial_{\mathbf{r}} (n) \\ &= qn\mathbf{E} = qn(-\nabla \Phi) \end{aligned}$$

$$\Downarrow$$

$$\partial_{\mathbf{r}} \Phi = \frac{kT_e}{e} \frac{1}{n} \partial_{\mathbf{r}} (n)$$

$$\Downarrow$$

$$\Phi = \frac{kT_e}{e} \ln(n) + \underbrace{const}_{\equiv \frac{kT}{e} \ln(n_0)}$$

$$\Downarrow$$

$$n = n_0 \exp\left(\frac{e\Phi}{kT_e}\right)$$

<u>Boltzmann Relation</u>

$$\boxed{n = n_0 \exp\left(\frac{e\Phi}{kT_e}\right)}$$

<u>Debye Length</u>

We can now calculate the Debye length – an effective length over which a plasma will shield an electric field. (The length is the 1/e distance for reducing a potential.)

First, we have Poisson's equation

$$\begin{aligned} \nabla^2 \Phi &= -\frac{\rho}{\varepsilon} \\ &= \frac{e}{\varepsilon} (n_e - n_i) \end{aligned}$$

We make the further assumption that the density of the electrons in absence of the potential is the same as the ion density. (We make this assumption because if either the electrons or the ions were to leave an area, a significant electric field would be setup to try to pull them back together.) Thus,

$$n_i = n_0$$

Plugging this and the Boltzmann relation into Poisson's equation gives

$$\nabla^2 \Phi = \frac{en_0}{\varepsilon} \left(\exp\left(\frac{e\Phi}{kT}\right) - 1 \right)$$

Now, using the Taylor series expansion of the exponential, which we assume is approximately 1, gives

$$\nabla^2\Phi = \frac{en_0}{\varepsilon}\left(1 + \frac{e\Phi}{kT} + O\left(\left(\frac{e\Phi}{kT}\right)^2\right) - 1\right)$$

$$\approx \frac{e^2 n_0 \Phi}{\varepsilon kT}$$

Solving the differential equation leaves

$$\Phi \approx \Phi_0 \exp\left(\frac{-|\mathbf{r}|}{\lambda_{Debye}}\right) \text{ where}$$

$$\lambda_{Debye} = \left(\frac{e^2 n_0}{\varepsilon kT}\right)^{-\frac{1}{2}}$$