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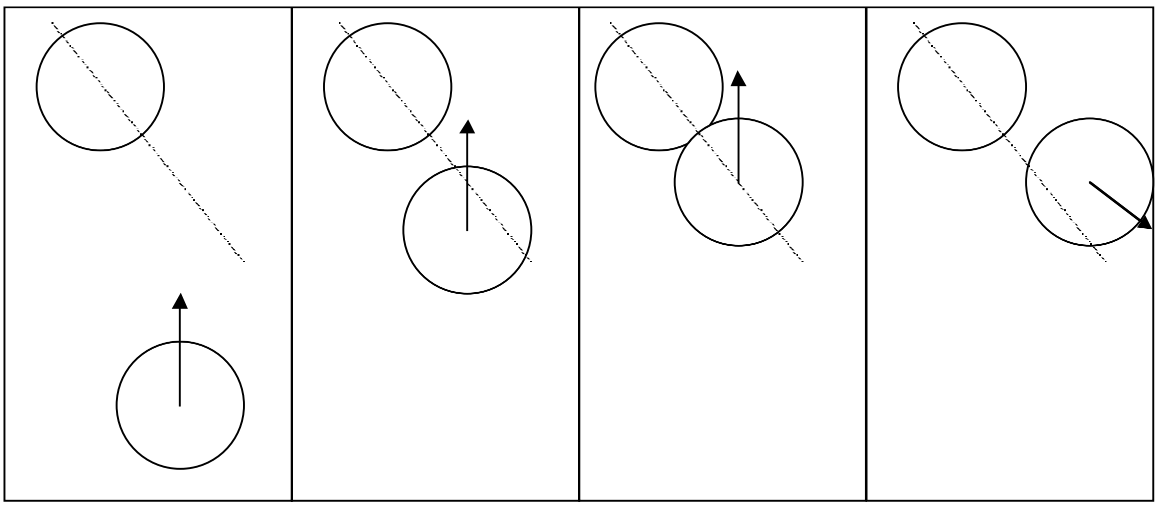
Lecture 5 Scattering

On the first day of class, we looked at the scattering process. We shall now look at it again but this time in more detail. At first, we will continue to use our 'hard sphere' model but shortly we will move on to the much more important coulomb type of collisions. (After all a plasma is a plasma because it has charged particles which will interact at a distance, via coulomb forces.)

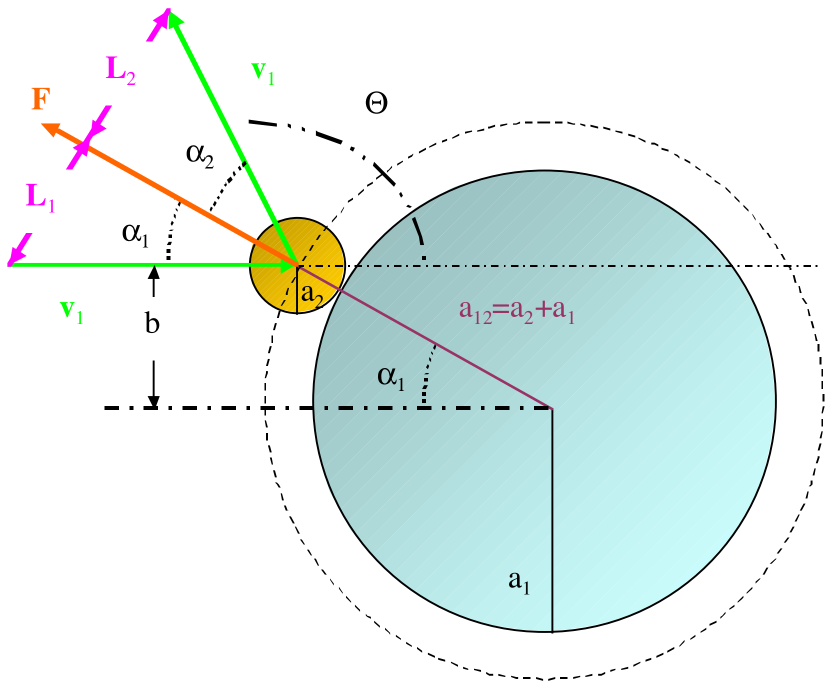
Detailed hard sphere collisions

For now, we consider ourselves 'pool sharks'. ('Pool' is also known as 'billards'. It is a game played on a rectangular surface with 11 hard balls about the size of baseballs. There are pool tables in the Student Union.) We wish to learn how to bank our shots right into the right pocket.



Of particular interest is the point of contact required to scatter into a certain direction. Now the above drawing shows a moving particle colliding with a fixed target particle. The while figure further assumes that the target and moving particle are of the same type and/or size, they do not have to be so. In general, the direction that the particles travel after the collision will depend on the angle at which they collide, the initial/final momentum and the initial/final energy. At first, to simplify things, let us look at a moving particle impacting on a fixed particle. This would require that the 'fixed' particle be much more massive than the 'moving' particle. Determination of the scattering angle is then straight forward.

EXAMPLE: Scattering off of a massive target particle



In this case the scattering will be 'specular', i.e., we will have $\alpha_1 = \alpha_2$. (This is because the large fixed particle does not move. The force acting between the particles occurs along the line marked a_{12}, resulting in a change of sign of the momentum in that direction. We will see below that this also holds in the center of mass frame.) Simple geometry makes

$$\Theta = \pi - 2\alpha$$
$$= \pi - 2\arcsin\left(\frac{b}{a_{12}}\right)$$

Rewriting this, we find (This is also true in general for the CM frame, which we show below. Note: we typically use Θ as the scattering angle in the CM frame. Because we have let $m_2 \to \infty$, the reference that we are using also happens to be the CM frame.)

$$b = a_{12}\sin\left(\frac{\pi - \Theta}{2}\right)$$
$$= a_{12}\cos\left(\frac{\Theta}{2}\right)$$

To get a 90° scatter, one would have to have

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$$b = a_{12}\cos\left(\frac{\pi}{4}\right)$$
$$= a_{12}\frac{1}{\sqrt{2}}$$
$$\Downarrow$$
$$\sigma_{90^\circ} = \frac{1}{2}\sigma = \frac{1}{2}\pi a_{12}^2$$

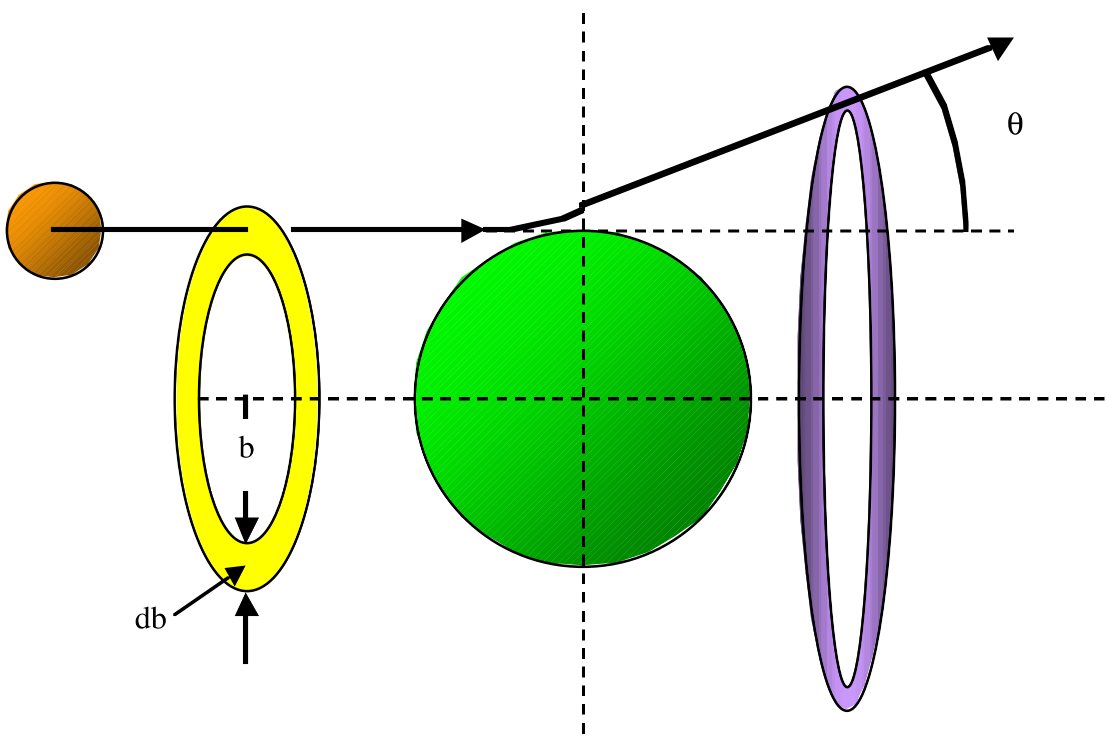
This implies that half of all collisions of this type will scatter at least 90°. This is not true for 'real' collisions.

Differentiating, we find the how the scattering angle changes with respect to incident b for a hard sphere collision,

$$\frac{db}{d\theta} = -\frac{a_{12}}{2}\sin\left(\frac{\theta}{2}\right).$$

This leads us to the general concept of Differential scattering cross section.

Differential Scattering Cross Section



We can generalize scattering to include that the particles are scattered into various angles. This is done in the following manner. Let us assume that a flux of the smaller particles Γ enter the collision through an area given by $bd\phi db$. This gives a total number of particles = $\Gamma bd\phi db$. The

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fraction of these same particles that exit the collision through a given solid angle of $d\phi \sin\theta d\theta$ is $\Gamma I(\mathbf{v},\theta) d\phi \sin\theta d\theta$, then

$$\Gamma b d\phi db = \Gamma I(\mathbf{v},\theta) d\phi \sin\theta d\theta$$

so that

$$I(\mathbf{v},\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

The magnitude $\left|\frac{db}{d\theta}\right|$ is used because typically as b gets smaller, θ gets larger.

To obtain the total cross section, we simply integrate

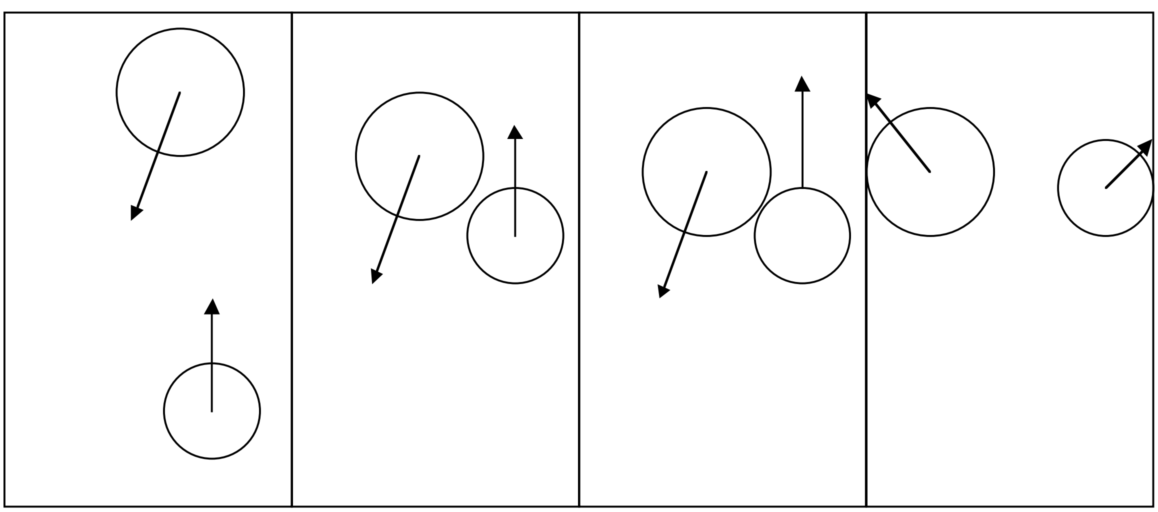
$$\sigma = \int I(\mathbf{v},\theta) d\phi \sin\theta d\theta$$

While we can calculate $I(\mathbf{v},\theta)$ for the hard sphere approximation above, it is truly a quantum mechanical – or coulomb – process and thus is more complicated then what we have derived. To be able to determine the differential scattering cross section, we need to know how a particle will move throughout the collision process.

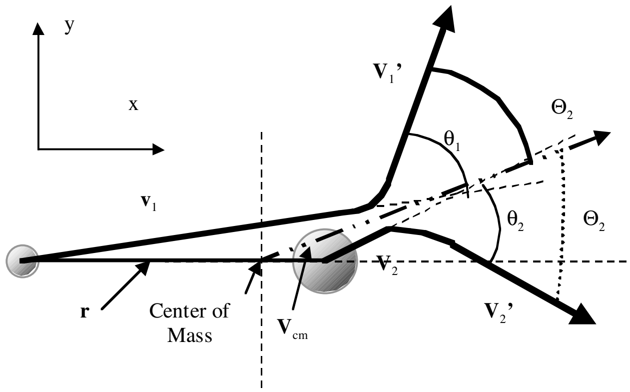
<u>Translation between reference frames</u>

To be able to solve this equation, we will have to learn how to translate between various coordinate systems. This is vital as often the problem is easier to solve in one coordinate system then in another.

To complicate things as much as possible, we will look at a system that consists of two dissimilar particles that are both moving. Sequential pictures of the collision might look like:



For a general reference frame, the collision might look like:

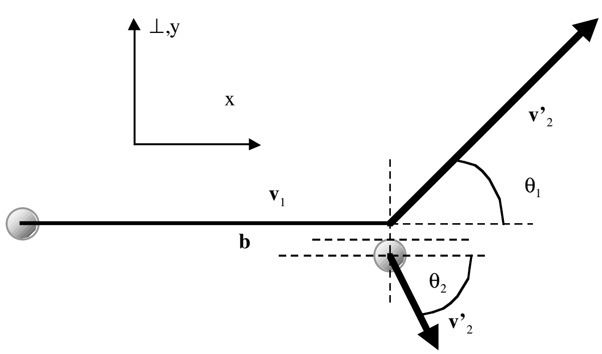


Sketch of system in General Lab frame. Note that both particles are moving. Here, and throughout these notes, θ_1 and θ_2 are the scattering angles in the Lab frame, Θ is the scattering angle in the CM frame, and v_{rel} and v'_{rel} are the relative velocities before and after the collision.

To track a particle through the collision process, we assume that momentum and energy are conserved. This gives

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}'_1 + m_2\mathbf{v}'_2$$
$$m_1\mathbf{v}_1^2 + m_2\mathbf{v}_2^2 = m_1\mathbf{v}_1'^2 + m_2\mathbf{v}_2'^2 + \Delta E_{internal}$$

This however, is impossible to solve as is…because we have 7 unknowns, the three $\mathbf{v}'_1$, the three $\mathbf{v}'_2$ and the one $\Delta E_{internal}$, but only four equations, the three momentum equations and the one energy equation. (This is also true for a collision in two dimensions.) Thus we need to make some simplifying assumptions in order to make the problem tractable. With out loss of generality, we can pick a reference frame such that the target particle, particle 2, is initially at rest. Second, we assume that the collision is elastic, e.g. $\Delta E_{internal} = 0$. Third, we will setup the coordinate system such that the test particle, particle 1, is initially moving is a single dimension. Fourth, we will assume that our problem is two-dimensional. (This not really a limitation as once we have gone to the rest frame of the target particle and made the coordinate system choice, the problem becomes two dimensional.) This leaves picture that looks like



and a simplified system of equations of

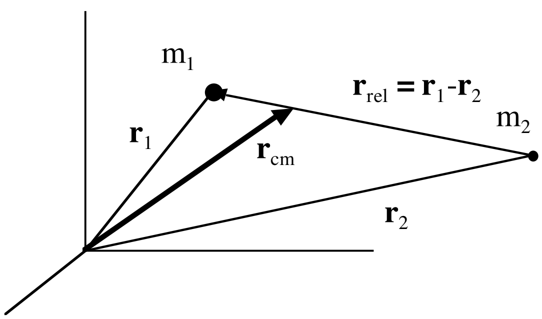
$$x - components \qquad\qquad -or-$$
$$m_1 v_1 = m_1 v_1' \cos\theta_1 + m_2 v_2' \cos\theta_2 \qquad\qquad m_1 v_{1x} = m_1 v_{1x}' + m_2 v_{2x}'$$
$$y - components$$
$$0 = m_1 v_1' \sin\theta_1 - m_2 v_2' \sin\theta_2 \qquad\qquad 0 = m_1 v_{1y}' - m_2 v_{2y}'$$
$$energy$$
$$m_1 v_1^2 = m_1 v_1'^2 + m_2 v_2'^2 \qquad\qquad m_1\left(v_{1x}^2\right) = m_1\left(v_{1x}'^2 + v_{1y}'^2\right) + m_2\left(v_{2x}'^2 + v_{2y}'^2\right)$$

This reference frame is called the Lab Frame.

Sketch of system in Particle 2 initial rest frame, which we will call the Lab frame.

Center of Mass Frame

Of particular interest is the center-of-mass rest frame. The center of mass is given by



$$\mathbf{r}_{cm} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

Likewise, we can define a second vector, $\mathbf{r}_{rel}$, as the relative position vector between the particles. Hence $\mathbf{r}_{rel}$ is simply

$$\mathbf{r}_{rel} = \mathbf{r}_1 - \mathbf{r}_2$$

Each of these vectors will change in time and these are

$$\mathbf{V}_{cm} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}$$

$$\mathbf{v}_{rel} = \mathbf{v}_1 - \mathbf{v}_2$$

When the two particles collide we find that

$$\mathbf{F}_{2\to1} = m_1\dot{\mathbf{v}}_1$$
$$\mathbf{F}_{1\to2} = m_2\dot{\mathbf{v}}_2 = -\mathbf{F}_{2\to1}$$

From this, it is simple to see that

$$\dot{\mathbf{V}}_{cm} = \frac{m_1\dot{\mathbf{v}}_1 + m_2\dot{\mathbf{v}}_2}{m_1 + m_2}$$
$$= \frac{\mathbf{F}_{2\to1} + \mathbf{F}_{1\to2}}{m_1 + m_2}$$
$$= 0$$

which is not massively exciting but it is very important. This implies that the momentum of the system in the CM frame is zero and it remains zero. Also we find that

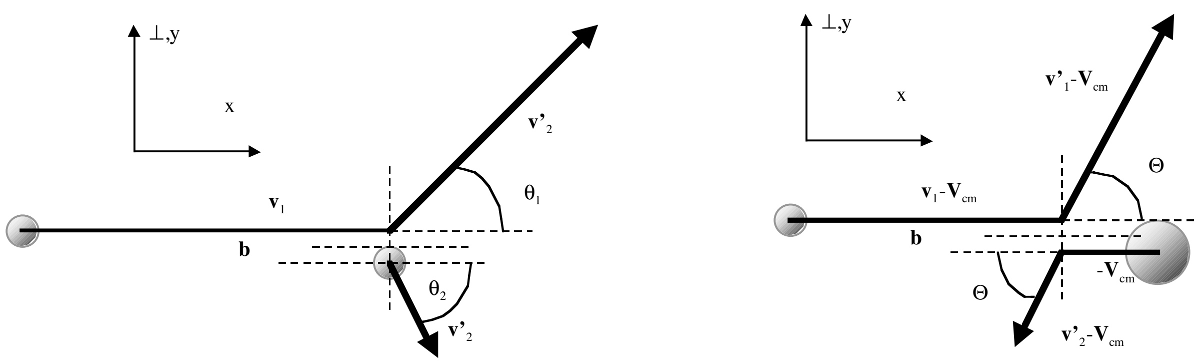
$$\begin{aligned}
\dot{\mathbf{v}}_{rel} &= \dot{\mathbf{v}}_1 - \dot{\mathbf{v}}_2 \\
&= \frac{\mathbf{F}_{2\to 1}}{m_1} - \frac{\mathbf{F}_{1\to 2}}{m_2} \\
&= \mathbf{F}_{2\to 1}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \\
&\Downarrow
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}_{2\to 1} &= \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} \dot{\mathbf{v}}_{rel} \\
&= \frac{m_1 m_2}{m_1 + m_2} \dot{\mathbf{v}}_{rel} \\
&= \mu \dot{\mathbf{v}}_{rel}
\end{aligned}$$

Here $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the 'reduced' mass of the system.

This means that the relative position of the particles acts as if it were a single particle moving about a fixed position, the center of mass. Now this fictitious single particle will approach the center of mass and scatter off of it. (This will hold true for any central force problem.)

Now, we can examine the center of mass, CM, frame in terms of the Lab Frame.



Lab Frame

Center of Mass Frame

CAREFUL, I THINK LIEBERMAN'S FIGURES ARE SOMEWHAT WRONG

To translate from the Lab frame to the CM frame, we simply subtract $\mathbf{V}_{cm}$ from each of the velocities in the Lab frame.

In the CM frame the total momentum is zero. This can be checked easily.

$\mathbf{p}^f = m_1\mathbf{v}_1^f + m_2\mathbf{v}_2^f$; f represents the CM frame

$= m_1(\mathbf{v}_1 - \mathbf{V}_{cm}) + m_2(\mathbf{v}_2 - \mathbf{V}_{cm})$

$= m_1\mathbf{v}_1 + m_2\mathbf{v}_2 - (m_1 + m_2)\mathbf{V}_{cm}$;   but $\mathbf{V}_{cm} = \dfrac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}$

$= 0$

Further the center of mass does not move in the CM frame, which can also be easily checked.

$$\mathbf{V}_{cm}^f = \frac{\mathbf{p}_{cm}^f}{(m_1 + m_2)} = \frac{m_1\mathbf{v}_1^f + m_2\mathbf{v}_2^f}{(m_1 + m_2)}$$
$$= 0$$

Thus we find

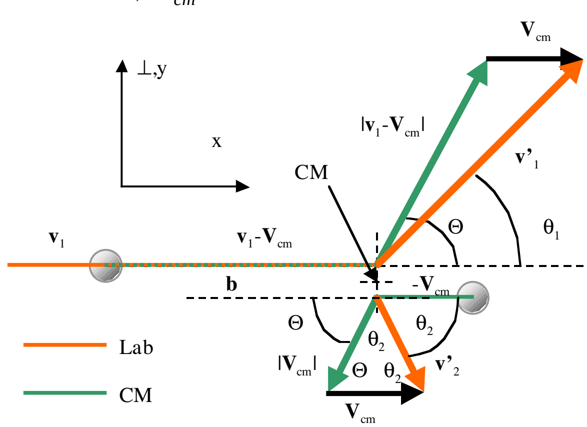
$$\frac{-\mathbf{v}_2^f}{\mathbf{v}_1^f} = \frac{-\mathbf{v}_2'^f}{\mathbf{v}_1'^f} = \frac{m_1}{m_2}$$

Example:

If we have conservation of energy we get

$$|\mathbf{v}_2^f| = |\mathbf{v}_2'^f|$$
$$|\mathbf{v}_1^f| = |\mathbf{v}_1'^f|$$

We can now look at the transition from lab frame to CM frame. This requires simply subtracting the vector, $\mathbf{V}_{cm}$



In the lab frame we have

$$\tan\theta_1 = \left.\frac{\mathbf{v}_{1y}'}{\mathbf{v}_{1x}'}\right|_{Lab\ \ Frame} = \left.\frac{|\mathbf{v}_1 - \mathbf{V}_{cm}|\sin\Theta}{|\mathbf{V}_{cm}| + |\mathbf{v}_1 - \mathbf{V}_{cm}|\cos\Theta}\right|_{CM\ \ Frame}$$
$$= \frac{\sin\Theta}{\dfrac{|\mathbf{V}_{cm}|}{|\mathbf{v}_1 - \mathbf{V}_{cm}|} + \cos\Theta}$$

But from above,

$$\frac{-\mathbf{v}_2}{\mathbf{v}_1} = \left|\frac{\mathbf{v}_2}{\mathbf{v}_1}\right| = \frac{-\mathbf{v}'_2}{\mathbf{v}'_1} = \left|\frac{\mathbf{v}'_2}{\mathbf{v}'_1}\right| = \frac{m_1}{m_2}\text{, so}$$

$$\tan\theta_1 = \frac{\sin\Theta}{\frac{m_1}{m_2} + \cos\Theta}$$

We can do the same thing for θ_2.

$$\tan\theta_2 = \left.\frac{\mathbf{v}'_{2y}}{\mathbf{v}'_{2x}}\right|_{Lab\ \ Frame} = \left.\frac{|\mathbf{V}_{cm}|\sin\Theta}{\mathbf{V}_{cm} - |\mathbf{V}_{cm}|\cos\Theta}\right|_{CM\ \ Frame}$$

$$= \frac{\sin\Theta}{1 - \cos\Theta}$$

If energy is not conserved than we get the equations in the book.

$$\tan\theta_1 = \frac{\sin\Theta}{\frac{m_1 v_{rel}}{m_2 v'_{rel}} + \cos\Theta}$$

$$\tan\theta_2 = \frac{\sin\Theta}{\frac{v_{rel}}{v'_{rel}} - \cos\Theta} \text{ and finally}$$

$$|\mathbf{v}'_1|\sin\theta_1 = |\mathbf{v}_1|\frac{m_2}{m_1 + m_2}\sin\Theta$$

<u>Energy Transfer</u>

We see in our Lab frame picture above that the second particle, which is originally at rest, is moving after the collision process. This is energy transfer. Energy transfer occurs in all collision types. Most, however, involve significant amounts of energy deposited into the internal structure of the particles. (For a solid ball, this would result in heating or permanent deformation of the ball. For an atom/molecule this would involve exciting an electron to a higher energy state or ionizing the atom/molecule. For molecules, you can also break or excite bonds.) If energy is not deposited into the internal structure of the particles, we have what is known as 'elastic' collisions.

In the Lab frame, we have a simplified system of equations of

$$x - components \qquad\qquad -or-$$

$$m_1 v_1 = m_1 v'_1 \cos\theta_1 + m_2 v'_2 \cos\theta_2 \qquad\qquad m_1 v_{1x} = m_1 v'_{1x} + m_2 v'_{2x}$$

$$y - components$$

$$0 = m_1 v'_1 \sin\theta_1 - m_2 v'_2 \sin\theta_2 \qquad\qquad 0 = m_1 v'_{1y} - m_2 v'_{2y}$$

$$energy$$

$$m_1 v_1^2 = m_1 v_1'^2 + m_2 v_2'^2 \qquad\qquad m_1\left(v_{1x}^2\right) = m_1\left(v_{1x}'^2 + v_{1y}'^2\right) + m_2\left(v_{2x}'^2 + v_{2y}'^2\right)$$

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While this is still not a tractable problem, we have three equations and four unknowns - assuming we know all of the initial conditions, we can determine the energy transfer. First we solve the first two equations for v'_{1x} and v'_{1y} respectively, giving,

$$v'_{1x} = v_{1x} - \frac{m_2}{m_1} v'_{2x}$$

$$v'_{1y} = \frac{m_2}{m_1} v'_{2y}$$

which in turn can be placed in the energy equation to give

$$\begin{aligned}(v_{1x}^2) &= \left(\left(v_{1x} - \frac{m_2}{m_1} v'_{2x} \right)^2 + \left(\frac{m_2}{m_1} v'_{2y} \right)^2 \right) + \frac{m_2}{m_1} \left(v'^2_{2x} + v'^2_{2y} \right) \\ &= \left(v_{1x}^2 - 2\frac{m_2}{m_1} v_{1x} v'_{2x} + \frac{m_2^2}{m_1^2} v'^2_{2x} \right) + \left(\frac{m_2^2}{m_1^2} v'^2_{2y} \right) + \frac{m_2}{m_1} \left(v'^2_{2x} + v'^2_{2y} \right)\end{aligned}$$

Combining like terms gives

$$2\frac{m_2}{m_1} v_{1x} v'_{2x} = \left(v'^2_{2x} + v'^2_{2y} \right) \left(\frac{m_2^2}{m_1^2} + \frac{m_2}{m_1} \right); \; but \;\; v'_{2x} = v'_2 \cos\theta_2, \; v_{1x} = v_1 \;\; and \;\; v'^2_2 = v'^2_{2x} + v'^2_{2y}$$

$$2 v_1 \cos\theta_2 = v'_2 \left(\frac{m_2 + m_1}{m_1} \right) - \; now \;\; squaring \;\; both \;\; sides$$

$$v'^2_2 = v_1^2 \cos^2\theta_2 \left(\frac{4 m_1^2}{(m_1 + m_2)^2} \right)$$

$$\frac{1}{2} m_2 v'^2_2 = \frac{1}{2} m_1 v_1^2 \cos^2\theta_2 \left(\frac{4 m_1^2}{(m_1 + m_2)^2} \right)$$

This means that the incident particle gives up energy to the target particle. For electron collision with anything else, this energy transfer is very small, on the order of the mass ratio. For particles of similar mass the energy transfer can be quite large, on the order of the initial energy. The upshot of this is that particles of similar mass tend to have similar energy distributions, because of the efficient energy transfer, while particles of distinctly different mass say an electron and just about anything else, can have very dissimilar energy distributions.

Return to the Differential Scattering Cross Section

Now that we can move from one reference frame to another, we wish to look at the concept of the differential scattering cross section. The first thing that we note is that our original definition is independent of our reference frame. Thus we have,

$$\Gamma b d\phi_1 db = \Gamma I(\mathbf{v}_1, \theta_1) d\phi_1 \sin\theta_1 d\theta_1 = \Gamma I(\mathbf{v}_r, \Theta) d\Phi \sin\Theta d\Theta$$

or intergrating around ϕ_1 and Φ (to get 2π) so that

$$bdb = I(\mathbf{v}_1, \theta_1) \sin\theta_1 d\theta_1 = I(\mathbf{v}_{rel}, \Theta) \sin\Theta d\Theta$$

<u>Example: The differential scattering cross section off of a hard sphere collision in the CM frame.</u>

We know from above that in the center of mass frame that

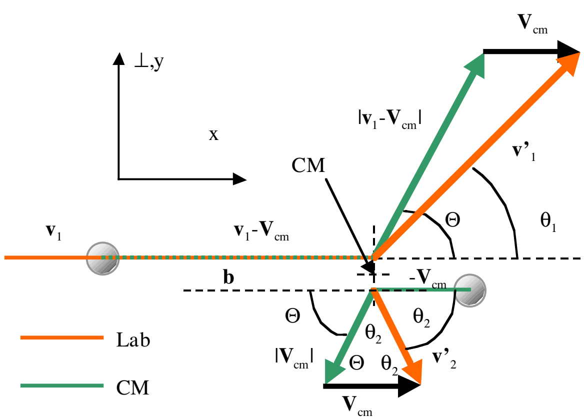
$$\frac{-\mathbf{v}_2}{\mathbf{v}_1} = \left|\frac{\mathbf{v}_2}{\mathbf{v}_1}\right| = \frac{-\mathbf{v}'_2}{\mathbf{v}'_1} = \left|\frac{\mathbf{v}'_2}{\mathbf{v}'_1}\right| = \frac{m_1}{m_2},$$

which can be found from zero total momentum.
Likewise, we know that for a hard sphere, the energy lost to the internal structure is assumed to be zero so that

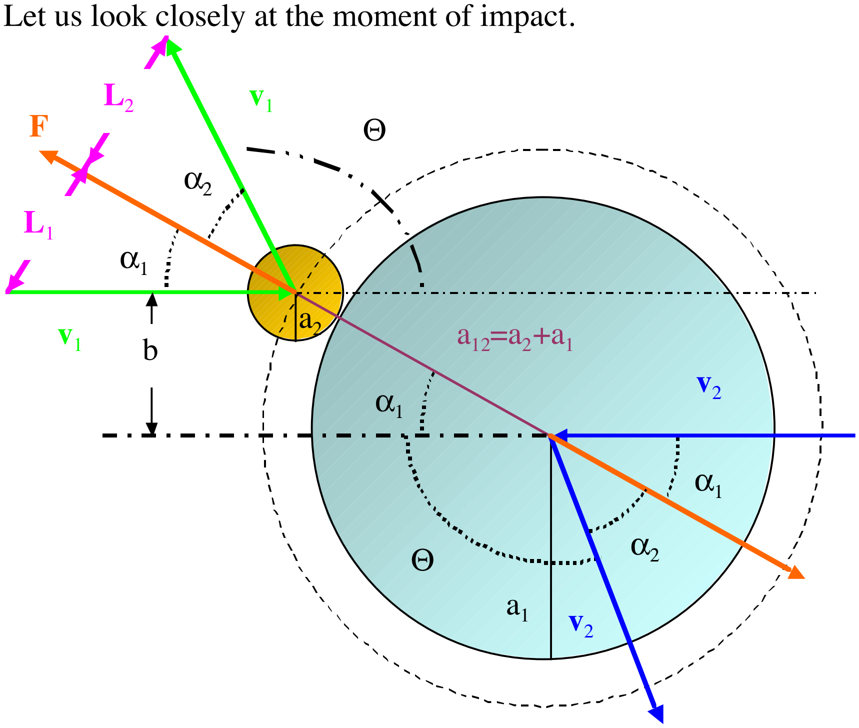
$$|\mathbf{v}_2| = |\mathbf{v}'_2|$$

$$|\mathbf{v}_1| = |\mathbf{v}'_1|$$

Now in the CM frame, this implies that we have a collision that looks like



Let us look closely at the moment of impact.



Because the force is only along a_{12}, and $|\mathbf{v}_1| = |\mathbf{v}'_1|$ in the CM frame, this forces $L_1 = L_2$. Thus,

$\alpha_1 = \alpha_2 = \alpha$ and

$$\Theta = \pi - 2\alpha$$
$$= \pi - 2\arcsin\left(\frac{b}{a_{12}}\right)$$

Thus we again find

$$b = a_{12}\sin(\alpha) = a_{12}\sin\left(\frac{\pi - \Theta}{2}\right) = a_{12}\cos\left(\frac{\Theta}{2}\right);\ \text{and}$$

$$\text{db} = a_{12}\cos(\alpha)d\alpha$$

From above we have

$$bdb = I(\mathbf{v}_1, \theta_1)\sin\theta_1 d\theta_1 = I(\mathbf{v}_{rel}, \Theta)\sin\Theta d\Theta$$

so

$$bdb = a_{12}\sin(\alpha)\sum a_{12}\cos(\alpha)d\alpha$$
$$= \frac{a_{12}^2}{2}\sin(2\alpha)d\alpha$$
$$= \frac{a_{12}^2}{2}\sin(\Theta - \pi)d\left(\frac{\Theta}{2}\right)$$
$$= -\frac{a_{12}^2}{4}\sin(\Theta)d\Theta$$
$$= I(\mathbf{v}_{rel}, \Theta)\sin\Theta d\Theta$$

rearranging and normalizing

$$I(\mathbf{v}_{rel}, \Theta) = \frac{a_{12}^2}{4}$$

Small angle scattering

As we will show below, the most important type of elastic collisions are the small angle collisions. They are more numerous than large angle collisions and thus account for most of the scattering. We will begin by looking at Coulomb collisions and then look at collisions in general.

Example: Small vs. Large angle scattering for charged particles

We already know that the electric field in a plasma decays as

$$\Phi \approx \Phi_0 \exp\left(\frac{-|\mathbf{r}|}{\lambda_{Debye}}\right)\ \textit{where}$$

$$\lambda_{Debye} = \left(\frac{\varepsilon kT}{e^2 n_0}\right)^{\frac{1}{2}}$$

This implies that a charged particle in the plasma will only interact will other charged particles that are in the same area. This number of particles is given by

$$\Lambda_s = n_0 \lambda_s^3$$

where

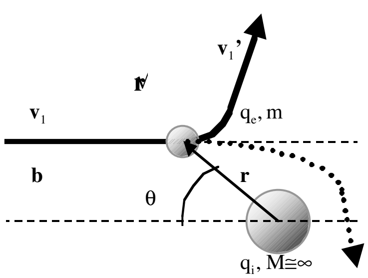
$$\lambda_s = \left(\frac{\varepsilon k T_s}{e^2 n_0} \right)^{\frac{1}{2}}$$

This number of particles is known as the plasma parameter of species s. Typically we define the plasma parameter as

$$\Lambda = n_0 \lambda_{Debye}^3 .$$

Often Λ is on the order of 10^6.

This number is important, as it tells us how many particles (only ions and electrons!) that our test particle is interacting with at a given time. Now let us consider a collision between our test particle, an electron, and our target particle, an ion. (Note that we could also look at ion (species 1)-ion (species 2) or ion (moving) – electron (fixed) collisions. While the first of these might be important, the second is rare – for hopefully obvious reasons. However, electron (fast) – ion (slow) is the dominant collision type.)



Note either path might occur – depending on the relative sign of the charges.

We know from electromagnetism the force on the electron from the ion is:

$$\mathbf{F} = m_e \ddot{\mathbf{r}} = \frac{1}{4\pi\varepsilon} \frac{q_i q_e}{r^2} \hat{\mathbf{r}}$$

Now the change in velocity perpendicular to the initial path is given by

$$mv_\perp = \int_{-\infty}^{\infty} F_\perp dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{4\pi\varepsilon} \frac{q_i q_e}{r^2} \sin\theta dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{4\pi\varepsilon} \frac{q_i q_e}{(b/\sin\theta)^2} \sin\theta dt; \quad from \;\; b = r\sin\theta$$

$$= \int_{-\infty}^{\infty} \frac{1}{4\pi\varepsilon} \frac{q_i q_e}{b^2} \sin^3\theta dt$$

Now, provided that we know the initial velocity of the electron, then we can determine the angle, θ, as a function of time.

$$x = v_o t = -r\cos\theta = -b\frac{\cos\theta}{\sin\theta}$$

(This makes the assumption that the velocity in the x direction does not change much during the process.) Differentiating the above equation gives

$$dt = -\frac{b}{v_o} d\left(\frac{\cos\theta}{\sin\theta}\right) = -\frac{b}{v_o}\left(-d\theta - \frac{\cos^2\theta}{\sin^2\theta}d\theta\right)$$

$$= \frac{b}{v_o}\frac{1}{\sin^2\theta}d\theta$$

so that

$$mv_\perp = \int_{-\infty}^{\infty}\frac{1}{4\pi\varepsilon}\frac{q_i q_e}{b^2}\sin^3\theta\frac{b}{v_o}\frac{1}{\sin^2\theta}d\theta$$

$$= \int_0^{\pi}\frac{1}{4\pi\varepsilon}\frac{q_i q_e}{v_o b}\sin\theta d\theta$$

$$= \frac{1}{2\pi\varepsilon}\frac{q_i q_e}{v_o b}$$

As is common, we will rearrange the above equation to give

$$\frac{v_\perp}{v_o} = \frac{q_i q_e}{4\pi\varepsilon\left(\frac{1}{2}mv_o^2\right)}\frac{1}{b};\ \ defining\ \ b_o = \frac{q_i q_e}{4\pi\varepsilon\left(\frac{1}{2}mv_o^2\right)}$$

$$= \frac{b_o}{b}$$

b_o is known as the Landau length. (This is a slightly different definition than what is given in Lieberman. It is, however, closely related.)

The scattering angle can be determined directly from the above equation. Assuming that the angle is small, which was already necessary for $\Delta v_o \approx 0$, lets approximate

$$\theta \approx \tan\theta = \frac{v_\perp}{v_o} = \frac{b_o}{b}$$

Now, how do we 'find' the impact parameter required to scatter into large angles. Well we can get an approximation by using the above formula – noting that the formula is really just for small angle impact. Let's assume

$$v_\perp \approx v_o \quad so\ \ that$$

$$b_o \approx b$$

Now the collision frequency of such a collision is

$$\nu_{large} = v_o/\lambda_{mfp}$$

$$= v_o n_o \sigma_{large\ \ angle}$$

$$= v_o n_o \pi b_o^2$$

$$= n_o\frac{q_i^2 q_e^2}{4\pi\varepsilon^2\left(m^2 v_o^3\right)}$$

Now, let us consider small angle collisions. Let the test particle travel through the plasma in the x-direction. Then the scattering of the velocity into the z- and y-directions is given by

$$\Delta \mathbf{v}_\perp = \Delta v_y \hat{\mathbf{y}} + \Delta v_z \hat{\mathbf{z}} = v_o \frac{b_o}{b}$$

so that

$$\Delta v_\perp^2 = \Delta v_y^2 + \Delta v_z^2 = v_o^2 \frac{b_o^2}{b^2}$$

Now, the change in $v_\perp$ for any given collision will be independent of previous collisions. Thus,

$$\Delta v_{\perp\ total}^2 = \sum_{i-th\ \ collision} \Delta v_{\perp\ i}^2$$

On average,

$$\left\langle \Delta v_{\perp\ i}^2 \right\rangle = \left\langle \Delta v_{\perp\ j}^2 \right\rangle;\ \ j \neq i\ \ so\ \ that$$

$$\begin{aligned}\left\langle \Delta v_{\perp\ total}^2 \right\rangle &= \left\langle \sum_{i-th\ \ collision} \Delta v_{\perp\ i}^2 \right\rangle \\ &= \sum_{i-th\ \ collision} \left\langle \Delta v_{\perp\ i}^2 \right\rangle \\ &= N \left\langle \Delta v_{\perp\ i}^2 \right\rangle\end{aligned}$$

Likewise,

$$\left\langle \Delta v_{\perp\ total}^2 \right\rangle = \left\langle \Delta v_{x\ total}^2 \right\rangle + \left\langle \Delta v_{y\ total}^2 \right\rangle\ \ but$$

$$\left\langle \Delta v_x^2 \right\rangle = \left\langle \Delta v_y^2 \right\rangle\ \ so$$

$$\left\langle \Delta v_{\perp\ total}^2 \right\rangle = 2N \left\langle \Delta v_{x\ i}^2 \right\rangle$$

giving (finally!)

$$\left\langle \Delta v_{x\ total}^2 \right\rangle = \frac{N}{2} v_o^2 \frac{b_o^2}{b^2}$$

for a particular b – noting that there are other possible impact parameters.

The rate of change of $\left\langle \Delta v_{x\ total}^2 \right\rangle$ is

$$\begin{aligned}\frac{d}{dt}\left\langle \Delta v_{x\ total}^2 \right\rangle &= \frac{1}{2}\frac{b_o^2}{b^2} v_o^2 \frac{d}{dt}(N) \\ &= \frac{1}{2}\frac{b_o^2}{b^2} v_o^2 \left(2\pi b db\ \ n_o v_o\right)\end{aligned}$$

where $n_o v_o$ is the number of scattering centers past in time t and $2\pi b db$ is the area of each center between b and b +db. Now integrating over all b gives (remember we have only looked at a single b!)

$$\begin{aligned}\frac{d}{dt}\left\langle \Delta v_{x\ total}^2 \right\rangle &= \int_{b\ \min}^{b\ \max} \pi b_o^2 n_o v_o^3 b^{-1} db \\ &= \pi b_o^2 n_o v_o^3 \ln(b)\Big|_{b\ \min}^{b\ \max}\end{aligned}$$

At this point we need to decide what we should use for b_{max} and b_{min}. Clearly the maximum distance over which an electric field can act in a plasma is $\lambda_{Debye} \Rightarrow b_{\max} \approx \lambda_{Debye}$. Likewise, our model for small angle collisions works only for $b_o \Rightarrow b_{\min} \approx b_o$. Plugging these in we find

$$\frac{\lambda_{Debye}}{b_o} = \frac{4\pi\varepsilon\left(\frac{1}{2}mv_o^2\right)\lambda_D}{q_i q_e};\ let\ \ v_o^2 \approx v_{th}^2 = \frac{kT}{m}$$

$$\approx \frac{4\pi\varepsilon\left(\frac{1}{2}kT\right)\lambda_D}{q_i q_e};\ noting\ \ \lambda_s = \left(\frac{\varepsilon k T_s}{e^2 n_0}\right)^{\frac{1}{2}}$$

$$= 2\pi n_0 \lambda_D^3$$

$$= 2\pi\Lambda \approx \Lambda \approx 10^6$$

Thus,

$$\frac{d}{dt}\left\langle \Delta v_{x\ total}^2 \right\rangle = \frac{n_o q_i^2 q_e^2}{8\pi\varepsilon^2 m^2 v_o}\ln(\Lambda)$$

To compare this to large angle collisions, we want to know the time it takes to have

$$\left\langle \Delta v_{x\ total}^2 \right\rangle \approx v_o^2$$

which is the same energy change for a large angle collision. The time required for this to happen is simply

$$\frac{d}{dt}\left\langle \Delta v_{x\ total}^2 \right\rangle = \frac{v_o^2}{\tau} = \frac{n_o q_i^2 q_e^2}{8\pi\varepsilon^2 m^2 v_o}\ln(\Lambda)$$

$$\Downarrow$$

$$\nu_{small} = \tau^{-1} = \frac{n_o q_i^2 q_e^2}{8\pi\varepsilon^2 m^2 v_o^3}\ln(\Lambda)$$

but

$$\boxed{\begin{aligned} \nu_{large} &= n_o \frac{q_i^2 q_e^2}{4\pi\varepsilon^2\left(m^2 v_o^3\right)}\ \ so\ \ that \\ \nu_{small} &= 2\ln(\Lambda)\nu_{large} \\ &\approx 30\nu_{large} \end{aligned}}$$

<u>This shows that for coulomb collisions, the small numerous angle collisions are more important than less frequent large angle collisions.</u>

Now, we can go back and redo this the way that Lieberman does it, i.e. in a much more general form.

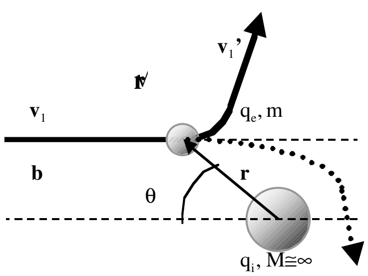
We know that there are more potential forces that one particle can assert on another particle than a monopole electric field. For example, many particles exhibit dipole, permanent and induced, structures. (These particles have a net charge of zero.) The general form of the potential from a multipole structure is given by

$$V_{monopole} = \frac{q}{4\pi\varepsilon r}$$
$$V_{dipole} = \frac{q\mathbf{s}\,\Sigma\mathbf{r}}{4\pi\varepsilon r^2};\ \mathbf{s}\ \text{is the charge separation}$$
$$V_{quadrapole} = \frac{qs^2}{4\pi\varepsilon r^3}\left(3\cos^2\theta - 1\right)$$
$$V_{order\ l\ pole} \propto \frac{q}{4\pi\varepsilon r^{l+1}}$$

See for example: Paul Lorrain and Dale Corson, Electromagnetic Fields and Waves 2nd Ed., (Freeman, San Francisco, 1970) secs 2.9 – 2.11.

Hard Spheres have delta function like forces and hence will have potentials that are like

$$V_{hard\ sphere} \propto \frac{1}{(r-a)^\infty};\ \text{where } a \text{ is the particle radius}$$



The distance from the target (fixed) particle to the test (moving) particle is simply
$r = \left(b^2 + v_o^2 t^2\right)^{1/2} \approx \left(b^2 + v_{rel}^2 t^2\right)^{1/2};\ \mathbf{v}_o \neq \mathbf{v}_{rel}!\ (close\ \ but\ \ \neq)$
As we see from above, the central force has a potential of
$$V = \frac{C}{r^i},$$
giving the force of the form
$$\mathbf{F} = -\nabla V = -\partial_r\left(\frac{C}{r^i}\right)\hat{\mathbf{r}}$$
The part of the force to the velocity is simply
$$\begin{aligned}\mathbf{F}_\perp &= -\sin\theta\ \ \nabla V\\ &= -\frac{b}{r}\ \ \nabla V\\ &= -\frac{b}{r}\ \ \partial_r\left(\frac{C}{r^i}\right)\end{aligned}$$
Then the impulse is simply

$$m\mathbf{v}_\perp = \int_{-\infty}^{\infty} \mathbf{F}_\perp dt$$
$$= \int_{-\infty}^{\infty} -\frac{b}{r}\ \partial_r\left(\frac{C}{r^i}\right) dt$$
$$= \int_{-\infty}^{\infty} \frac{b}{r}\ \left|\partial_r\left(\frac{C}{r^i}\right)\right| dt$$

As before,

$$dt = \frac{b}{v_o}\frac{1}{\sin^2\theta}d\theta$$
$$= \left(\frac{1}{r^2 - b^2}\right)^{1/2} \frac{r}{v_o} dr$$

so that

$$m\mathbf{v}_\perp = \int_{b_o}^{\infty} \frac{b}{r}\ \left|\partial_r\left(\frac{C}{r^i}\right)\right| \left(\frac{1}{r^2 - b^2}\right)^{1/2} \frac{r}{v_o} dr$$
$$= \frac{b}{v_o}\int_{b_o}^{\infty} \left|\partial_r\left(\frac{C}{r^i}\right)\right| \left(\frac{1}{r^2 - b^2}\right)^{1/2} dr$$

Lieberman claims that this can be solved to give (I will try to look up the paper that he quotes. I also think that energy needs to be a the relative energy…)

$$\Theta \approx \frac{\Delta\mathbf{v}_\perp}{\mathbf{v}_\parallel} = \frac{A}{\left(\frac{1}{2}mv_o^2\right)b^i}$$

where

$$A = \frac{C\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)};\ \Gamma(l) = (l-1)!,\ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\ .$$

From this, we can determine the differential cross section,

$$I(\mathbf{v},\Theta) = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right|$$
$$\approx \frac{b}{\Theta}\left|\frac{db}{d\Theta}\right|$$
$$b \approx \left(\frac{A}{\left(\frac{1}{2}mv_o^2\right)\Theta}\right)^{1/i}$$
$$\Downarrow$$
$$\frac{db}{d\Theta} = -\frac{1}{i}\left(\frac{A}{\left(\frac{1}{2}mv_o^2\right)}\right)^{1/i}\Theta^{-\frac{1}{i}-1}$$

$$
\begin{aligned}
I(\mathbf{v},\Theta) &\approx \frac{b}{\Theta}\frac{1}{i}\left(\frac{A}{\left(\frac{1}{2}mv_o^2\right)}\right)^{1/i}\Theta^{-\frac{1}{i}-1} \\
&= \frac{1}{i}\left(\frac{A}{\left(\frac{1}{2}mv_o^2\right)}\right)^{2/i}\Theta^{-\frac{2}{i}-2}
\end{aligned}
$$

<u>Inelastic Collisions</u>

Inelastic collisions are at the very least, immensely more complicated than elastic collisions. This is because kinetic energy is no longer conserved. The lost kinetic energy can be spent in a lot of ways.

Excitation of an electron in an atomic orbital
Ionization of an atom
Ionization of a molecule
Excitation of a bond structures in a molecule
Dissociation of a molecule
Etc.

Each of these processes could almost be a semester-long class in and by themselves. In fact some people spend their whole life studying just one of these processes. All of the processes involve some form of quantum mechanics. Thus, we will now have a brief introduction to Quantum Mechanics.

<u>Introductory Quantum Mechanics</u>

The quantum mechanical nature of the universe became apparent when in the late 1800's what was then the standard model, now called classical physics, failed. For example according to classical electrodynamics, the electrons orbiting an atomic nucleus should loss energy and decay into the core. Obviously, this was not the case. (Other problems included blackbody radiation etc.) The first break through was the work of Bohr. He made two postulates. 1) He postulated that the electron in a hydrogen atom exists in discrete states given by the relation

$$nh = \oint p_\theta d\theta \Rightarrow p_\theta = \frac{nh}{2\pi} = n\hbar$$

where p_θ is the angular momentum, n in an integer, and h = 6.6E-34 Js, is a constant, now known as Planck's constant.

2) He postulated that when an electron changes state the energy lost/gained through a photon is simply the difference in the energy levels of the two states. Hence

$$h\nu = E_n - E_m;\ n \neq m$$

While Bohr was not quite right, his PhD dissertation was only ~2 pages long, he was not far from the truth.

We can use Bohr's model to get a handle on our inelastic collision processes.

First the energy that an electron has when it orbits a proton is simply

$$\begin{aligned}E &= \tfrac{1}{2}mv^2 + \frac{q_e q_p}{4\pi\varepsilon r} \\ &= \tfrac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon r};\ \ p_\theta = mrv \\ &= \frac{p_\theta^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon r} \\ &= \frac{n^2\hbar^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon r}\end{aligned}$$

Now the centripetal force equation requires that

$$\frac{q_e q_p}{4\pi\varepsilon r^2} = \frac{mv^2}{r} = \frac{p_\theta^2}{mr^3} = \frac{n^2\hbar^2}{mr^3}$$

$$\Downarrow$$

$$\frac{q_e q_p}{4\pi\varepsilon r} = \frac{n^2\hbar^2}{mr^2}$$

Plugging this into the above equation gives

$$\begin{aligned}E &= \frac{n^2\hbar^2}{2mr^2} - \frac{n^2\hbar^2}{mr^2} \\ &= -\frac{n^2\hbar^2}{2mr^2};\ \ \textit{further}\end{aligned}$$

$$\frac{e^2}{4\pi\varepsilon r} = \frac{n^2\hbar^2}{mr^2}$$

$$\Downarrow$$

$$r_n = \frac{4\pi\varepsilon n^2\hbar^2}{me^2}$$

Thus,

$$\begin{aligned}E &= -\frac{n^2\hbar^2}{2m}\left(\frac{me^2}{4\pi\varepsilon n^2\hbar^2}\right)^2 \\ &= -\frac{me^4}{32\pi^2\varepsilon^2 n^2\hbar^2};\ \ \textit{letting}\ \ \Re = \frac{me^4}{32\pi^2\varepsilon^2\hbar^2} = 13.6\ \ eV\end{aligned}$$

$$E = -\frac{\Re}{n^2}$$

The negative sign implies the electron is bound and the lowest energy state has a radius of

$$r_1(\equiv a_0) = \frac{4\pi\varepsilon\hbar^2}{me^2} \approx 0.529\mathring{A}$$

which is known as the Bohr radius and

$$r_n = n^2 a_0 .$$

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Unfortunately, Bohr did not know about internal spin and a small host of other physical phenomenon, which make this model incorrect. Fortunately for those of you have not yet had Quantum, this model will suffice for our purposes.

Ionization

At this point, the only thing that we are interested in is the differential ionization cross section. The differential cross section for small angle collisions between electrons is given by

$$I(\mathbf{v}_{rel},\Theta) \approx \frac{b}{\Theta}\frac{1}{i}\left(\frac{A}{\left(\frac{1}{2}\mu v_{rel}^2\right)}\right)^{1/i}\Theta^{-\frac{1}{i}-1}$$

$$= \frac{1}{i}\left(\frac{A}{\left(\frac{1}{2}\mu v_{rel}^2\right)}\right)^{2/i}\Theta^{-\frac{2}{i}-2}$$

where i = 1 and

$$A = \frac{C\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i+2}{2}\right)};\ \Gamma(l) = (l-1)!,\ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{e^2}{4\pi\varepsilon}\frac{\sqrt{\pi}}{2}$$

so that

$$I(\mathbf{v}_{rel},\Theta) \approx \left(\frac{e^2}{4\pi\varepsilon\left(\frac{1}{2}\mu v_{rel}^2\right)}\right)^2\Theta^{-4}$$

This cross section is important as the process requires the test electron to collide with the bound target electron. Note that this electron-electron collision is elastic!, but not the electron-neutral. Now transforming to the lab frame, using

$$\tan\theta_1 = \frac{\sin\Theta}{(m_1/m_2) + \cos\Theta}$$

$$\tan\theta_2 = \frac{\sin\Theta}{1-\cos\Theta}$$

Now $m_1/m_2 = 1$

so

$$\tan\theta_1 = \frac{\sin\Theta}{1+\cos\Theta}$$

$$\Downarrow$$

$$\tan\theta_1 = \sin\Theta - \tan\theta_1\cos\Theta$$

$$= \frac{\sin\Theta\cos\theta_1 - \sin\theta_1\cos\Theta}{\cos\theta_1}$$

$$\frac{\sin\theta_1}{\cos\theta_1} = \frac{\sin(\Theta-\theta_1)}{\cos\theta_1}$$

$$\Downarrow$$

$$\sin\theta_1 = \sin(\Theta-\theta_1)$$

$$\Downarrow$$

$$\Theta = 2\theta_1$$

We can now determine the differential scattering cross section in the lab frame.

$$d\sigma = I(v_{rel},\Theta)2\pi\sin\Theta d\Theta \text{ but } \Theta = 2\theta$$

$$= 2\pi\left(\frac{e^2}{4\pi\varepsilon\left(\frac{1}{2}\mu v_{rel}^2\right)}\right)^2 8\sin 2\theta \ \ \theta^{-4}d\theta$$

$$\approx 2\pi\left(\frac{e^2}{4\pi\varepsilon\left(\frac{1}{2}\mu v_{rel}^2\right)}\right)^2 4\theta^{-3}d\theta$$

$$= 2\pi\left(\frac{e^2}{4\pi\varepsilon\left(\frac{1}{2}m v_1^2\right)}\right)^2 4\theta^{-3}d\theta$$

Likewise we have energy lost from the test electron to the target electron can be given by
$W_{lost} = \zeta_L W$

in the center of mass frame

$$\zeta_L = \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\Theta)$$

$$= \tfrac{1}{2}(1-\cos\Theta)$$

Transforming to the lab frame and expanding the cos term to first order

$$\zeta_L = \tfrac{1}{2}\left(1-\left(1-\frac{\Theta^2}{2}\right)\right)$$

$$= \frac{\Theta^2}{4} = \theta^2$$

so finally

$$W_{lost} = \theta^2 W \ \ and$$

$$dW_{lost} = 2\theta d\theta W$$

$$d\sigma \approx 2\pi\left(\frac{e^2}{4\pi\varepsilon\left(\frac{1}{2}mv_o^2\right)}\right)^2 4\theta^{-3}d\theta$$

$$= 2\pi\left(\frac{e^2}{4\pi\varepsilon}\right)^2 2W^{-2}\theta^{-3}2d\theta$$

$$= 4\pi\left(\frac{e^2}{4\pi\varepsilon}\right)^2 W^{-1}\underbrace{W^{-2}\theta^{-4}}_{W_L^{-2}}\underbrace{2W\theta d\theta}_{dW_L}$$

$$= 4\pi\left(\frac{e^2}{4\pi\varepsilon}\right)^2 W^{-1}W_L^{-2}dW_L$$

To get the cross section for ionization, we need to integrate

$$\sigma = \int d\sigma \approx \int_{U_{ion}}^{W} 4\pi\left(\frac{e^2}{4\pi\varepsilon}\right)^2 W^{-1}W_L^{-2}dW_L$$

$$= 4\pi\left(\frac{e^2}{4\pi\varepsilon}\right)^2 W^{-1}\left(\frac{1}{U_{ion}} - \frac{1}{W}\right)$$

This is known as the Thomson ionization cross section. It is an inexact approximation of the ionization process.