

Lecture 6 Basic plasma dynamics

Our fluid equations that we developed before are:

$$f|_c = \left(\nabla_r \Sigma(n\langle \mathbf{v} \rangle) + \frac{\partial n}{\partial t} \right)$$

$$mn \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \Sigma \nabla_r \langle \mathbf{v} \rangle \right) = \underbrace{\Delta \mathbf{M}|_c}_{\substack{\text{momentum} \\ \text{lost via} \\ \text{collisions}}} - \underbrace{m \langle \mathbf{v} \rangle f|_c}_{\substack{\text{momentum change} \\ \text{via particle gain/loss}}} - \nabla_r \Sigma \mathbf{P} + qn(\mathbf{E} + \langle \mathbf{v} \rangle \wedge \mathbf{B})$$

We have looked collisions in the last section. Now we will look at the Lorentz force terms. Often, it is best to look at how a single particle reacts to the fields, so we will start there.

Lorentz force

The equation of motion for a single particle of charge q and mass m is given by the Lorentz force

law: $m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}).$

While this equation can be quite complex, for complex fields, it is often easiest to look at the cases when the external fields are simple.

Case 1: $\mathbf{E} = E_z \mathbf{z}'$; $\mathbf{B} = 0$

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

$$= \frac{q}{m} E_z \mathbf{z}'$$

Integrating gives

$$\begin{aligned} x &= x_0 + v_{x0}t \\ y &= y_0 + v_{y0}t \\ z &= z_0 + v_{z0}t + \frac{q}{2m} E_z t^2 \end{aligned} \quad \left\{ \begin{array}{l} \text{Letting } \mathbf{E} = \mathbf{E}_0; \mathbf{B} = 0 \\ \text{Gives } \mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 + \frac{q}{2m} \mathbf{E}_0 t^2 \end{array} \right\}$$

This is quite simple and not terribly informative.

Cases 2: $\mathbf{E} = 0$; $\mathbf{B} = B_0 \mathbf{z}'$

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

$$= \frac{q}{m} \mathbf{v} \wedge B_0 \mathbf{z}'$$

Thus we find

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$$\frac{dv_x}{dt} = \frac{q}{m} v_y B_0$$

$$\frac{dv_y}{dt} = -\frac{q}{m} v_x B_0$$

$$\frac{dv_z}{dt} = 0$$

To solve this set of equations, we must separate the components of the velocity. This is simple to do by differentiating the equations again and substituting to give.

$$\frac{d^2 v_x}{dt^2} = \frac{q}{m} \frac{dv_y}{dt} B_0 = -\left(\frac{q}{m} B_0\right)^2 v_x$$

$$\frac{d^2 v_y}{dt^2} = -\frac{q}{m} \frac{dv_x}{dt} B_0 = -\left(\frac{q}{m} B_0\right)^2 v_y$$

$$\frac{d^2 v_z}{dt^2} = 0$$

These second order equations are of course easily solved as

$$v_x = v_{x0} e^{\pm i\omega_c t}$$

$$v_y = v_{y0} e^{\pm i\omega_c t} \quad \text{where} \quad \omega_c = \frac{q}{m} B_0$$

$$v_z = v_{z0}$$

Now taking into account the original coupled first order equations we find

$$v_x = v_{\perp 0} \cos(\omega_c t + \varphi)$$

$$v_y = -v_{\perp 0} \sin(\omega_c t + \varphi)$$

$$v_z = v_{z0}$$

Integrating a second time we find,

$$x = \frac{v_{\perp 0}}{\omega_c} \sin(\omega_c t + \varphi) + x_0 - \frac{v_{\perp 0}}{\omega_c} \sin(\varphi)$$

$$y = \frac{v_{\perp 0}}{\omega_c} \cos(\omega_c t + \varphi) + y_0 - \frac{v_{\perp 0}}{\omega_c} \cos(\varphi)$$

$$z = v_{z0} t + z_0$$

$$r_c = \frac{v_{\perp 0}}{\omega_c}$$

It is easy to see that $r_c = \frac{v_{\perp 0}}{\omega_c}$ is the radius of a circular orbit around the magnetic field line; it is

known as the Larmor radius or the cyclotron radius. Further we note that the positively charged particles orbit in a left-hand orbit while the negatively charged particles orbit in a right-hand orbit.

Example

Of particular interest is the magnetic field required to give an electron a gyro-frequency of 2.45 GHz. This is of interest because it is required to understand Electron Cyclotron Resonance, ECR, plasma sources. (Given time at the end of the semester, we will discuss these sources in detail.)

$$\begin{aligned}
 f_c &= \frac{\omega_c}{2\pi} \\
 &= \frac{q}{2\pi m} B_0 \\
 &= 2.8E6 B_0 \text{ (Hz / Gauss)} \\
 &\Downarrow \\
 B_0 &\approx 875G
 \end{aligned}$$

Case 3: $\mathbf{E} = E_z \mathbf{z}'$; $\mathbf{B} = B_0 \mathbf{z}'$

$$\begin{aligned}
 \frac{d\mathbf{v}}{dt} &= \frac{q}{m} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \\
 &= \frac{q}{m} \left(E_z \mathbf{z}' + \begin{vmatrix} \mathbf{x}' & \mathbf{y}' & \mathbf{z}' \\ v_x & v_y & v_z \\ 0 & 0 & B_0 \end{vmatrix} \right)
 \end{aligned}$$

or

$$\begin{aligned}
 \frac{dv_x}{dt} &= \frac{q}{m} v_y B_0 \\
 \frac{dv_y}{dt} &= -\frac{q}{m} v_x B_0 \\
 \frac{dv_z}{dt} &= \frac{q}{m} E_z
 \end{aligned}$$

This is easy to solve as we have already done this as parts.

$$x = r_c \sin(\omega_c t + \varphi) + x_0 - r_c \sin(\varphi)$$

$$y = r_c \cos(\omega_c t + \varphi) + y_0 - r_c \cos(\varphi)$$

$$z = z_0 + v_{z0} t + \frac{q}{2m} E_z t^2$$

$$r_c = \frac{v_{\perp 0}}{\omega_c}$$

Case 4: $\mathbf{E} = E_x \hat{\mathbf{x}}$; $\mathbf{B} = B_0 \hat{\mathbf{z}}$. Note that $\mathbf{E} = E_y \hat{\mathbf{y}}$ would also work here

$$\begin{aligned} \frac{d\mathbf{v}}{dt} &= \frac{q}{m} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \\ &= \frac{q}{m} \left(E_x \hat{\mathbf{x}} + \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v_x & v_y & v_z \\ 0 & 0 & B_0 \end{vmatrix} \right) \text{ or} \\ \frac{dv_x}{dt} &= \frac{q}{m} E_x + \omega_c v_y \Rightarrow \quad ? \\ \frac{dv_y}{dt} &= -\omega_c v_x \Rightarrow \quad ? \\ \frac{dv_z}{dt} &= 0 \Rightarrow v_z = v_{z0}; z = v_z t + z_0 \end{aligned}$$

The third equation is easy to solve while the first two are more difficult. Differentiating gives

$$\begin{aligned} \frac{d^2 v_x}{dt^2} &= -\omega_c^2 v_x \\ \Downarrow \\ v_x &= v_{x0} e^{\pm i \omega_c t} \end{aligned}$$

$$\begin{aligned} \frac{d^2 v_y}{dt^2} &= -\omega_c \left(\frac{q}{m} E_x + \omega_c v_y \right) \\ &= -\omega_c \left(\frac{q B_0}{m} \frac{E_x}{B_0} + \omega_c v_y \right) \\ &= -\omega_c^2 \left(\frac{E_x}{B_0} + v_y \right) \text{ but } \frac{d^2 v_y}{dt^2} = \frac{d^2}{dt^2} \left(v_y + \frac{E_x}{B_0} \right) \text{ so} \end{aligned}$$

$$\begin{aligned} v_y + \frac{E_x}{B_0} &= v_{y0} e^{\pm i \omega_c t} \\ v_y &= v_{y0} e^{\pm i \omega_c t} - \frac{E_x}{B_0} \end{aligned}$$

Plugging these into our initial equations gives

$$\begin{aligned} \frac{dv_x}{dt} &= \frac{q}{m} E_x + \omega_c \left(v_{y0} e^{\pm i \omega_c t} - \frac{E_x}{B_0} \right) \\ &= \omega_c v_{y0} e^{\pm i \omega_c t} \\ &= \pm i \omega_c v_{x0} e^{\pm i \omega_c t} \\ \Downarrow \\ v_{y0} &= \pm i v_{x0} \end{aligned}$$

Let $v_{\perp} = \mp i v_{y0} = v_{x0}$ giving

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$$v_x = v_{\perp} e^{\pm i\omega_c t} \Rightarrow v_{\perp} \cos(\omega_c t + \phi)$$

$$v_y = \mp i v_{\perp} e^{\pm i\omega_c t} - \frac{E_x}{B_0} \Rightarrow v_{\perp} \sin(\omega_c t + \phi) - \frac{E_x}{B_0}$$

$$v_z = v_{z0}; \quad z = v_z t + z_0$$

This means that the particle travels along the as it would with just the magnetic field but it also have a drift in the $\mathbf{E} \wedge \mathbf{B}$ direction. We can calculate this in general. First the average force is $\langle \mathbf{F} \rangle = q(\mathbf{E} + \langle \mathbf{v} \rangle \wedge \mathbf{B}) = 0$

Therefore

$$\langle \mathbf{F} \rangle \wedge \mathbf{B} = q(\mathbf{E} \wedge \mathbf{B} + (\langle \mathbf{v} \rangle \wedge \mathbf{B}) \wedge \mathbf{B}) = 0$$

\Downarrow

$$\begin{aligned} \mathbf{E} \wedge \mathbf{B} &= -(\langle \mathbf{v} \rangle \wedge \mathbf{B}) \wedge \mathbf{B} \\ &= \mathbf{B} \wedge (\langle \mathbf{v} \rangle \wedge \mathbf{B}) \\ &= \langle \mathbf{v} \rangle (\mathbf{B} \cdot \mathbf{B}) - \mathbf{B} (\langle \mathbf{v} \rangle \cdot \mathbf{B}) \end{aligned}$$

Now if there is no drift along \mathbf{B} then we get

$$\langle \mathbf{v} \rangle = \frac{\mathbf{E} \wedge \mathbf{B}}{B^2}$$

What we have described above is true in general. Assuming that we have any constant force that is a right angle to the magnetic field.

$$\mathbf{F}_{total} \wedge \mathbf{B} = 0 = \mathbf{F}_{\perp} \wedge \mathbf{B} + q(\mathbf{v} \wedge \mathbf{B}) \wedge \mathbf{B}$$

\Downarrow

$$\mathbf{v}_{drift} = \frac{\mathbf{F}_{\perp} \wedge \mathbf{B}}{qB^2}$$

Case 5: $\mathbf{E} = 0$; $\mathbf{B} = \mathbf{B}_0 + (\mathbf{r} \cdot \nabla) \mathbf{B} + \dots$: Non - uniform magnetic field

Here we look at a magnetic field that is non-uniform in space. The Taylor series expansion of such a field will be of the form

$$\mathbf{B} = \mathbf{B}_0 + (\mathbf{r} \cdot \nabla) \mathbf{B} + \dots$$

This should be straight forward from

$$B_z(y) = B_{z0} + y \partial_y B_z + \dots$$

Now from Lorentz's Force Law we have

$$m \frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) = q(\mathbf{v} \wedge \mathbf{B})$$

or in the y-direction

$$\begin{aligned} m \frac{dv_y}{dt} &= -qv_x B_z \\ &= -qv_x (B_{z0} + y \partial_y B_z + \dots) \end{aligned}$$

(We can do the same thing in the x-direction.)

The force averaged over one gyration is

$$m \left\langle \frac{dv_y}{dt} \right\rangle = -q \left(\langle v_x \rangle B_{z0} + \langle yv_x \rangle \partial_y B_z + \dots \right)$$

The first term is clearly zero. The second term is not as

$$\begin{aligned} \langle yv_x \rangle &= \left\langle \left(r_c \cos(\omega_c t + \varphi) + y_0 - r_c \cos(\varphi) \right) \left(v_{\perp 0} \cos(\omega_c t + \varphi) \right) \right\rangle \\ &= r_c v_{\perp 0} \langle \cos^2(\omega_c t + \varphi) \rangle - \text{all of the other terms are zero} \\ &= \frac{1}{2} r_c v_{\perp 0} \end{aligned}$$

We can now plug this into our average force equation to get

$$\begin{aligned} m \left\langle \frac{dv_y}{dt} \right\rangle &= -q \frac{1}{2} r_c v_{\perp 0} \partial_y B_z \quad \text{– or in general} \\ &= -q \frac{1}{2} r_c v_{\perp 0} \nabla B_z \\ &= \frac{1}{2} m v_{\perp 0}^2 \frac{1}{B} \nabla B_z \end{aligned}$$

In the x-direction this becomes

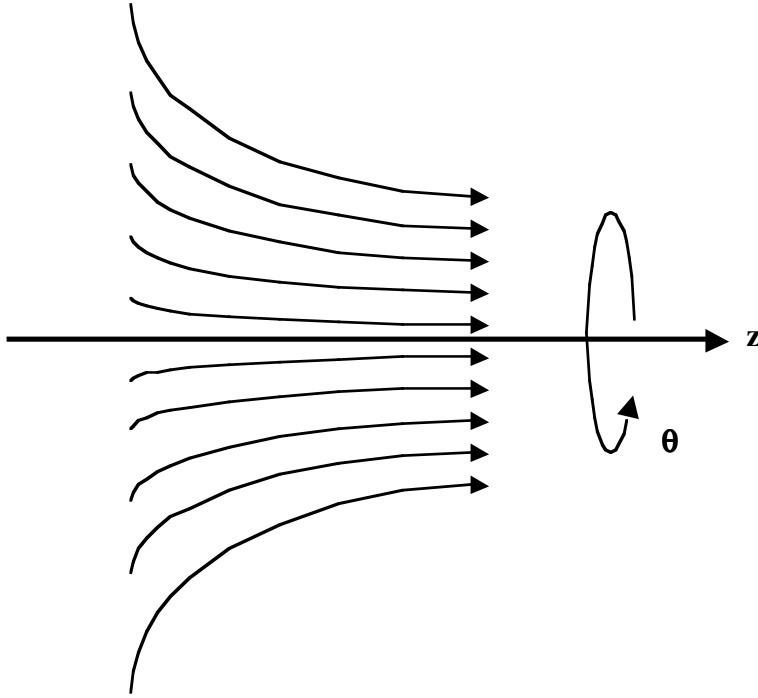
$$\begin{aligned} m \left\langle \frac{dv_x}{dt} \right\rangle &= -q \left(\langle v_y \rangle B_{z0} + \langle yv_y \rangle \partial_y B_z + \dots \right) \\ &\quad \text{but} \\ \langle yv_y \rangle &= \left\langle \left(r_c \cos(\omega_c t + \varphi) + y_0 - r_c \cos(\varphi) \right) \left(v_{\perp 0} \sin(\omega_c t + \varphi) \right) \right\rangle \\ &= r_c v_{\perp 0} \langle \cos(\omega_c t + \varphi) \sin(\omega_c t + \varphi) \rangle \\ &= 0 \end{aligned}$$

Thus, we have from above a drift velocity

$$\begin{aligned} \mathbf{v}_{drift} &= \frac{\mathbf{F}_{\perp} \wedge \mathbf{B}}{qB^2} \\ &= \frac{1}{2} r_c v_{\perp 0} \frac{\mathbf{B} \wedge \nabla B_z}{B^2} \\ &= \frac{1}{2} m v_{\perp 0}^2 \frac{1}{B} \frac{\mathbf{B} \wedge \nabla B_z}{B^2} \end{aligned}$$

This leads into a topic known as magnetic mirrors. Magnetic mirrors are naturally occurring phenomena that happen at the magnetic poles of planets and stars. In laboratory-based plasmas magnetic mirror are used in some process systems to confine the plasma. (They were also used – quite unsuccessfully - as a confinement mechanism for fusion plasmas.)

In a mirror the gradient of the magnetic field is parallel to the direction of the field lines. This sort of arrangement is known as a cusp field and looks like the figure below. (This is the geometry one finds with permanent magnets.)



From Maxwell's Equations we have

$$\nabla \cdot \mathbf{B} = 0$$

↓ - in cylindrical coordinates

$$r^{-1} \partial_r (r B_r) + \partial_z (B_z) = 0 \quad \text{- or}$$

$$r^{-1} \partial_r (r B_r) = -\partial_z (B_z)$$

Integrating over r and assuming $\partial_z (B_z)$ is not a function r gives

$$\int \partial_r (r B_r) dr = -\int r \partial_z (B_z) dr$$

$$r B_r \approx -\frac{r^2}{2} \partial_z (B_z) \Big|_{r=0}$$

$$B_r \approx -\frac{r}{2} \partial_z (B_z) \Big|_{r=0}$$

Now from Lorentz,

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

↓

$$F_r = q \left(v_\theta B_z - v_z \overset{=0}{\overline{B_\theta}} \right) = q v_\theta B_z$$

$$F_\theta = q (v_z B_r - v_r B_z)$$

$$F_z = q \left(v_r \overset{=0}{\overline{B_\theta}} - v_\theta B_r \right) = -q v_\theta B_r = q v_\theta \frac{r}{2} \partial_z (B_z) \Big|_{r=0}$$

Now

$$v_\theta = \sqrt{v_x^2 + v_y^2} \text{ has to do with the direction of } \theta.$$

$$= \mp v_\perp$$

and

$$r = r_c = \frac{v_\perp}{\omega_c} \text{ where } \omega_c = \frac{q}{m} B_0$$

Thus

$$F_z = \mp q v_\perp \frac{r_c}{2} \partial_z (B_z)$$

$$= \mp q \frac{v_\perp^2}{2\omega_c} \partial_z (B_z)$$

$$= -\frac{1}{2} m v_\perp^2 \frac{1}{B} \partial_z (B_z)$$

$$= -\mu \partial_z (B_z) \text{ where } \mu = \frac{1}{2} m v_\perp^2 \frac{1}{B}$$

μ is the magnetic moment of a particle gyrating around a point. This can be easily seen from $\mu = IA$ - where I and A are the current and area

$$= \left(\frac{q\omega_c}{2\pi} \right) (\pi r_c^2)$$

$$= \frac{q v_\perp^2}{2\omega_c}$$

$$= \frac{\frac{1}{2} m v_\perp^2}{B}$$

What does this mean?

As a particle moves into a region of increasing B, the Larmor radius shrinks but the magnetic moment remains constant. (This is shown in a number of books, including Chen.) Since the B field strength is increasing the particles tangential velocity must increase to keep μ constant. The total energy of the particle must also remain constant and thus the particle velocity parallel to the magnetic field must decrease. This causes the particle to bounce off of the ‘magnetic mirror’. (There are still ways for some of the particles to ‘leak’ through the mirror. This is one of the major reasons that magnetic mirrors did not work in the fusion field.)

Additional drift motions

There are a number of additional drift motions that occur for single particles. We unfortunately do not have time to cover these drifts. The other drifts include:

Curved **B**: Curvature drift

Non-uniform **E**

Curved Vacuum fields

Polarization Drift.

Most of these are covered in the main text or in Chen.

Bulk Motions

At this point, we need to deal with some of the bulk motions that occur in plasmas. These are not single particle motions but rather collective motion of all/most of the charge species in the plasma. The first, and most important is the electrostatic plasma oscillation, giving rise to the plasma frequency. [This but just one of a very wide variety of waves in plasmas.] These oscillations occur because one of the species becomes displaced from the other. When it accelerates back toward the other species, it gains too much energy and over shoots. To derive the plasma frequency, we will assume the simplest of geometries and plasmas.

- 1) No external fields. (This can be relaxed and the same result can be obtained.)
- 2) No random motion of the particles. (Hence, all particles of a species move at the same velocity at the same point in space. This can be relaxed and one can get the same result – it is just harder to do.)
- 3) Only the electrons move. (This is not a bad assumption for many aspects of plasmas.)
- 4) The plasma is one-dimensional and of such a length that the walls do not influence the result. (This implies that we are considering just regions that are at least several λ_{debye} from the walls.)

From Maxwell’s equations we have,

$$\nabla \cdot \mathbf{E} = \rho/\epsilon$$

$$\nabla \wedge \mathbf{E} = 0$$

(We will ignore the induced magnetic field.) Then our equation of motion (momentum conservation) and continuity (particle conservation) become

Continuity Equation

$$f|_c = \left(\nabla_r \cdot \Sigma(n\langle \mathbf{v} \rangle) + \frac{\partial n}{\partial t} \right)$$

⇓

$$0 = \nabla_r \cdot \Sigma(n\langle \mathbf{v} \rangle) + \frac{\partial n}{\partial t}$$

Energy Equation

$$mn \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \cdot \nabla_r \langle \mathbf{v} \rangle \right) = \underbrace{\Delta \mathbf{M}|_c}_{\substack{\text{momentum} \\ \text{lost via} \\ \text{collisions}}} - \underbrace{m \langle \mathbf{v} \rangle f|_c}_{\substack{\text{momentum change} \\ \text{via particle gain/loss}}} - \nabla_r \cdot \Sigma \mathbf{P} + qn(\mathbf{E} + \langle \mathbf{v} \rangle \wedge \mathbf{B})$$

⇓

$$mn \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \cdot \nabla_r \langle \mathbf{v} \rangle \right) = qn(\mathbf{E})$$

For this particular wave, we are considering deviations from charge neutrality. Thus we will have an induced electric field given by

$$\epsilon \nabla \cdot \mathbf{E} = \rho = e(n_i - n_e)$$

We have three items that are changing with time, \mathbf{E} , n_e and $\langle \mathbf{v}_e \rangle$. We will expand each of these items to produce a time average term, denoted with a ‘0’ and an oscillating term denoted with a ‘1’. Thus

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$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 \quad \text{- but } \mathbf{E}_0 \approx 0!$$

$$n_e = n_0 + n_1$$

$$\mathbf{v}_e = \mathbf{v}_0 + \mathbf{v}_1 \quad \text{- but } \mathbf{v}_0 \approx 0!$$

Then our conservation of momentum (energy) equation becomes

$$\left(\frac{\partial(\mathbf{v}_0 + \mathbf{v}_1)}{\partial t} + (\mathbf{v}_0 + \mathbf{v}_1) \cdot \nabla_r (\mathbf{v}_0 + \mathbf{v}_1) \right) = \frac{q}{m} (\mathbf{E}_0 + \mathbf{E}_1)$$

$$\left(\frac{\partial(\mathbf{v}_1)}{\partial t} + (\mathbf{v}_1) \cdot \nabla_r (\mathbf{v}_1) \right) = \frac{q}{m} (\mathbf{E}_1)$$

Now the second term on the left-hand side is small compared to the other two. (Two oscillating terms as opposed to one.) Thus we are left with

$$\frac{\partial(\mathbf{v}_1)}{\partial t} = \frac{q}{m} (\mathbf{E}_1) \quad \text{- let } \mathbf{v}_1 \propto e^{i(\omega t - \beta x)} \quad \text{- (implying we have a travelling wave)}$$

↓

$$i\omega \mathbf{v}_1 = -\frac{e}{m} \mathbf{E}_1$$

Now we can do the same thing to the continuity equation

$$\nabla_r \cdot \mathbf{\Sigma}(n\langle\mathbf{v}\rangle) + \frac{\partial n}{\partial t} = 0$$

$$\nabla_r \cdot \mathbf{\Sigma}((n_0 + n_1)\langle\mathbf{v}_1\rangle) + \frac{\partial(n_0 + n_1)}{\partial t} = 0$$

$$\nabla_r \cdot \mathbf{\Sigma}((n_1)\langle\mathbf{v}_1\rangle) + \frac{\partial(n_1)}{\partial t} = 0$$

Where again we have dropped the higher order terms. Thus

$$n_1 \nabla_r \cdot \mathbf{\Sigma}(\mathbf{v}_1) = -\partial_t(n_1) \quad \text{- letting } n_1 \propto e^{i(\omega t - \beta x)}$$

↓

$$in_1 \beta \mathbf{v}_1 = -i\omega n_1$$

Finally, we solve Poisson's Equation

$$\epsilon \nabla \cdot \mathbf{\Sigma} \mathbf{E} = e(n_i - n_e)$$

$$\epsilon \nabla \cdot \mathbf{\Sigma} \mathbf{E}_1 = e(n_0 - (n_0 + n_1))$$

$$-i\beta \epsilon E_1 = -en_1$$

This gives us three equations and three unknowns

$$in_1 \beta \mathbf{v}_1 = -i\omega n_1, \quad i\beta \epsilon E_1 = en_1, \quad i\omega \mathbf{v}_1 = -\frac{e}{m} \mathbf{E}_1.$$

Combining the last two to eliminate E_1 gives

$$\mathbf{v}_1 = -\frac{e^2 n_1}{\omega m \beta \epsilon}$$

Placing this into the first to eliminate n_1 gives

$$\omega_{pe}^2 = \frac{n_0 e^2}{m \epsilon}; \quad \omega_{pe} = f_{pe} / 2\pi,$$

the angular electron frequency of the plasma. This is also known as the dispersion relation.

Typically for process plasmas the density is $\sim 10^{10-12} \text{ cm}^{-3}$. Thus, $f_{pe} \sim 1$ to 10 GHz.

This is, in some sense, the simplest oscillation that can exist in a plasma. Note that this is not a wave in the typical sense! (Energy does not move in this oscillation, the group velocity, $\frac{d\omega}{d\beta}$, is zero. Here ‘ β ’ is the wave vector.)

There are numerous other oscillations that are waves that that can transfer energy. They can be divided into electrostatic waves and electromagnetic waves. We will deal first with the electrostatic waves.

Electrostatic waves in plasmas

Let us go back to our fundamental fluid equations, the continuity equation, the energy equation, and Poisson’s equation. As before we will ignore the collision terms and assume that the magnetic field is zero. Here however, we will include the pressure variations of the species.

Poisson’s equation

$$\epsilon \nabla \cdot \mathbf{E} = \rho = e(n_i - n_e)$$

Continuity Equation

$$f|_c = \left(\nabla_r \cdot \Sigma(n\langle \mathbf{v} \rangle) + \frac{\partial n}{\partial t} \right)$$

↓

$$0 = \nabla_r \cdot \Sigma(n\langle \mathbf{v} \rangle) + \frac{\partial n}{\partial t}$$

Energy Equation

$$mn \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \cdot \Sigma \nabla_r \langle \mathbf{v} \rangle \right) = \underbrace{\Delta \mathbf{M}|_c}_{\substack{\text{momentum} \\ \text{lost via} \\ \text{collisions}}} - \underbrace{m \langle \mathbf{v} \rangle f|_c}_{\substack{\text{momentum change} \\ \text{via particle gain/loss}}} - \nabla_r \cdot \Sigma \mathbf{P} + qn(\mathbf{E} + \langle \mathbf{v} \rangle \wedge \mathbf{B})$$

↓

$$mn \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \cdot \Sigma \nabla_r \langle \mathbf{v} \rangle \right) = qn(\mathbf{E}) - \nabla_r \cdot \Sigma \mathbf{P}$$

This would be identical to the plasma oscillations except for the pressure term. We will deal with that term first.

We know from the ideal gas law that $p = nkT$. Then, assuming an isotropic, or one dimensional, plasma

$$\begin{aligned}\nabla_r \Sigma \mathbf{P} &= \nabla_r p \\ &= kT \nabla_r n\end{aligned}$$

This is true provided that the compression is ‘isothermal’. In other words the temperature stays the same during the compression. Often, this is not true. Rather, we have ‘adiabatic’ compression, where the temperature changes. In this case, it can be shown using thermodynamics that

$$p = Cn^\gamma.$$

Here C is a constant and $\gamma = C_p/C_v$ is the ratio of the specific heats. We can see from the above equation that

$$\begin{aligned}\nabla p &= \nabla(Cn^\gamma) \\ &= C \nabla(n^\gamma) \\ &= C \gamma n^{\gamma-1} \nabla n \\ &= \gamma C n^\gamma \frac{\nabla n}{n} \\ &= \gamma p \frac{\nabla n}{n}\end{aligned}$$

Further, it can be shown that $\gamma = \frac{N+2}{N}$ where N is the number of degrees of freedom. Thus for N=1, $\gamma = 3$. (This is a crude approximation but it works for our needs.)

Thus, our equation of motion becomes

$$mn \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \Sigma \nabla_r \langle \mathbf{v} \rangle \right) = qn(\mathbf{E}) - \gamma kT \nabla_r n.$$

As before, we will assume that the density, velocity and electric field consists of a time-averaged term and an oscillating term. Thus,

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 \quad \text{but } \mathbf{E}_0 \approx 0!$$

$$n_e = n_0 + n_1$$

$$\mathbf{v}_e = \mathbf{v}_0 + \mathbf{v}_1 \quad \text{but } \mathbf{v}_0 \approx 0!$$

letting

$$\mathbf{v}_1, \mathbf{E}_1, n_1 \propto e^{i(\omega t - \beta x)}$$

We can now follow the derivation that we made before but this time we will add our additional term.

Assuming that we are examining electrons, our conservation of momentum (energy) equation becomes

$$m(n_0 + n_1) \left(\frac{\partial \left(\frac{\bar{\mathbf{v}}_0}{\omega} + \mathbf{v}_1 \right)}{\partial t} + \left(\frac{\bar{\mathbf{v}}_0}{\omega} + \mathbf{v}_1 \right) \cdot \nabla_r \left(\frac{\bar{\mathbf{v}}_0}{\omega} + \mathbf{v}_1 \right) \right) = -e(n_0 + n_1) \left(\frac{\bar{\mathbf{E}}_0}{\omega} + \mathbf{E}_1 \right) - \gamma k T_e \nabla(n_0 + n_1)$$

$$mn_0 \frac{\partial(\mathbf{v}_1)}{\partial t} = -en_0 \mathbf{E}_1 - \gamma k T_e \nabla n_1$$

$$\Downarrow$$

$$imn_0 \omega \mathbf{v}_1 = -en_0 \mathbf{E}_1 + i \gamma k T_e \beta n_1$$

Now we can do the same thing to the continuity equation

$$\nabla_r \cdot \Sigma(n\mathbf{v}) + \frac{\partial n}{\partial t} = 0$$

$$\nabla_r \cdot \Sigma((n_0 + n_1)(\mathbf{v}_1)) + \frac{\partial(n_0 + n_1)}{\partial t} = 0$$

$$\nabla_r \cdot \Sigma((n_0)(\mathbf{v}_1)) + \frac{\partial(n_1)}{\partial t} = 0$$

$$n_0 \nabla_r \cdot \Sigma(\mathbf{v}_1) = -\partial_t(n_1)$$

$$\Downarrow$$

$$in_0 \beta \mathbf{v}_1 = i \omega n_1$$

Finally, we solve Poisson's Equation

$$\epsilon \nabla \cdot \Sigma \mathbf{E} = e(n_i - n_e)$$

$$\epsilon \nabla \cdot \Sigma \mathbf{E}_1 = e(n_0 - (n_0 + n_1))$$

$$-i\beta \epsilon \mathbf{E}_1 = -en_1$$

This gives us three equations and three unknowns

$$in_0 \beta \mathbf{v}_1 = i \omega n_1, \quad -i\beta \epsilon \mathbf{E}_1 = -en_1, \quad imn_0 \omega \mathbf{v}_1 = -en_0 \mathbf{E}_1 + i \gamma k T_e \beta n_1.$$

Combining the last two to eliminate \mathbf{E}_1 gives

$$\mathbf{v}_1 = \left(\frac{e^2}{m\beta\epsilon\omega} + \frac{\gamma k T_e \beta}{mn_0\omega} \right) n_1$$

which is very similar to what we got before except we now have a second term.

Placing this into the first to eliminate n_1 gives

$$\omega^2 = \frac{e^2 n_0}{m\epsilon} + \frac{\gamma k T_e}{m} \beta^2$$

$$= \omega_{pe}^2 + c_\gamma^2 \beta^2 \quad - \text{where } c_\gamma^2 = \frac{\gamma k T_e}{m} \text{ is the electron sound speed}$$

Note that here the group velocity is non-zero, meaning that energy can be carried by the wave.

Now, let us assume that we are dealing with (positive) ions. Here, however, the electric field is determined by the electrons, not the ions. Thus, we need to replace the electric field with the gradient of the potential and use Boltzmann's relation on the electron density. Thus, our equations become

$$\nabla_r \Sigma(n\mathbf{v}) + \frac{\partial n}{\partial t} = 0$$

↓

$$n_0 \beta \mathbf{v}_1 = \omega n_1$$

(as before)

$$\begin{aligned} m_i n \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \Sigma \nabla_r \langle \mathbf{v} \rangle \right) &= qn(\mathbf{E}) - \gamma k T \nabla_r n \\ &= -en(\nabla \phi) - \gamma k T_i \nabla_r n \end{aligned}$$

$$m_i (n_0 + n_1) \left(\frac{\partial \left(\overset{=0}{\mathbf{v}_0} + \mathbf{v}_1 \right)}{\partial t} + \left(\overset{=0}{\mathbf{v}_0} + \mathbf{v}_1 \right) \Sigma \nabla_r \left(\overset{=0}{\mathbf{v}_0} + \mathbf{v}_1 \right) \right) = -e(n_0 + n_1) \nabla \left(\overset{=0}{\phi_0} + \phi_1 \right) - \gamma k T_i \nabla (n_0 + n_1)$$

$$m_i n_0 \frac{\partial (\mathbf{v}_1)}{\partial t} = -e n_0 \nabla \phi_1 - \gamma k T_i \nabla n_1$$

↓

$$m_i n_0 \omega \mathbf{v}_1 = e n_0 \beta \phi_1 + \gamma k T_i \beta n_1$$

Now, we can't use Poisson's equation but rather we assume that the change in local density can be modeled with Boltzmann's equation. E.g. that the local ion density is the same as the local electron density and that the electron density is given by

$$n_i \approx n_e = n_0 e^{e\phi/kT_e} \text{ (This approximation causes some small error)}$$

$$= n_0 e^{\left(\overset{=0}{\phi_0} + \phi_1 \right) / kT_e} = n_0 e^{e\phi_1 / kT_e}$$

$$\approx n_0 \left(1 + \frac{e\phi_1}{kT_e} \right)$$

↓

$$n_1 = n_0 \frac{e\phi_1}{kT_e}$$

This gives us our three equations and three unknowns

$$n_1 = n_0 \frac{e\phi_1}{kT_e}, \quad n_0 \beta \mathbf{v}_1 = \omega n_1, \quad m_i n_0 \omega \mathbf{v}_1 = e n_0 \beta \phi_1 + \gamma k T_i \beta n_1$$

Putting together the first two gives

$$\mathbf{v}_1 = \frac{\omega e \phi_1}{\beta k T_e}$$

Plugging this and the first into the third gives

$$\frac{\omega^2}{\beta^2} = \frac{kT_e + \gamma kT_i}{m_i}$$

This is the ion-acoustic or ion-sound waves.

All of the above are just a few examples of electrostatic waves. There are many more electrostatic waves.

We will now deal briefly with electromagnetic waves. **It is important to note that we will only look at the very simplest cases. There are a wide variety of electromagnetic waves that are sustained in plasmas.**

First, standard light waves exist. This comes directly from Maxwell's equation.

$$\nabla^2 \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \sigma \mu \partial_t \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} + \epsilon \mu \partial_t^2 \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

$$|\mathbf{E}| = \pm \frac{|\mathbf{H}|}{\eta}; \quad \eta = \frac{i\mu\omega}{\gamma} = \sqrt{\frac{i\mu\omega}{\sigma + i\epsilon\omega}} \left(= \sqrt{\frac{\mu}{\epsilon}} \text{ if } \sigma = 0 \right)$$

sign determined by growth / decay

(growth => -, decay => +)

$$\beta = \frac{2\pi}{\lambda} = \frac{c}{\omega}; c = \frac{1}{\sqrt{\epsilon\mu}}$$

In plasmas, this is not quite correct. What happens if the waves are interacting with the plasma?

From Maxwell's equations we have

$$\nabla \wedge \mathbf{E} = -\partial_t \mathbf{B}$$

$$\nabla \wedge \mathbf{H} = \mathbf{J}_{free} + \partial_t \mathbf{D}$$

$$\nabla \cdot \mathbf{D} = \rho_{free} = e(n_i - n_e) \approx 0$$

$$\nabla \cdot \mathbf{B} = 0$$

However the current is not zero. This changes the fields. To solve the problem, we will consider only the time varying components. Taking the curl of the first equation and the time derivative of the second equation gives

$$\nabla \wedge (\nabla \wedge \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\partial_t \nabla \wedge \mathbf{B}$$

$$\mu^{-1} \partial_t \nabla \wedge \mathbf{B} = \partial_t \mathbf{J}_{free} + \partial_t^2 \epsilon \mathbf{E}$$

We can combine these to give

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu \partial_t \mathbf{J}_{free} - \epsilon \mu \partial_t^2 \mathbf{E}$$

$$\beta \left(\overbrace{\nabla \cdot \mathbf{E}}^{\rho=0} \right) + \beta^2 \mathbf{E} = -i\omega \mu \mathbf{J}_{free} + \epsilon \mu \omega^2 \mathbf{E} \quad - \quad c^{-2} = \epsilon \mu$$

⇓

$$(\beta^2 - c^{-2} \omega^2) \mathbf{E} = -i\omega \mu \mathbf{J}_{free}$$

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If the light is at a high frequency, then the ions are effectively fixed. Thus the current is almost entirely due to the motion of the electrons. Then the current can be given as

$$\mathbf{J}_{free} = -en_0\mathbf{v}_{e1}$$

from the equation of motion

$$\mathbf{F} = m \frac{d\mathbf{v}_{e1}}{dt} = -e\mathbf{E}$$

so that

$$\mathbf{J}_{free} = -\frac{in_0e^2\mathbf{E}}{\omega m}$$

plugging this in to Maxwell's equation gives

$$(\beta^2 - c^{-2}\omega^2)\mathbf{E} = -\frac{n_0\mu e^2\mathbf{E}}{m}$$

⇓

$$0 = \left(\omega^2 - c^2 \frac{n_0\mu e^2}{m} - c^2\beta^2 \right) \mathbf{E}$$

$$= \left(\omega^2 - \frac{n_0e^2}{\epsilon m} - c^2\beta^2 \right) \mathbf{E}$$

$$= (\omega^2 - \omega_{pe}^2 - c^2\beta^2)\mathbf{E}$$

⇓

$$\omega^2 = \omega_{pe}^2 + c^2\beta^2$$

This is the dispersion relation of electromagnetic waves in a plasma.

There is a very useful application of this dispersion relation.

Then is a very interesting phenomenon that occurs, known as cutoff.

Lecture 7 Diffusion

Our fluid equations that we developed before are:

$$f|_c = \left(\nabla_r \Sigma(n\langle \mathbf{v} \rangle) + \frac{\partial n}{\partial t} \right)$$

$$mn \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \Sigma \nabla_r \langle \mathbf{v} \rangle \right) = \underbrace{\Delta \mathbf{M}|_c}_{\substack{\text{momentum} \\ \text{lost via} \\ \text{collisions}}} - \underbrace{m \langle \mathbf{v} \rangle f|_c}_{\substack{\text{momentum change} \\ \text{via particle gain/loss}}} - \nabla_r \Sigma \mathbf{P} + qn(\mathbf{E} + \langle \mathbf{v} \rangle \wedge \mathbf{B})$$

We have looked collisions in the last section. Now we will look at the Lorentz force terms. Often, it is best to look at how a single particle reacts to the fields, so we will start there.