

Lecture 7 Diffusion

Our fluid equations that we developed before are:

$$f|_c = \left( \nabla_r \Sigma(n\langle \mathbf{v} \rangle) + \frac{\partial n}{\partial t} \right)$$

$$mn \left( \frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \Sigma \nabla_r \langle \mathbf{v} \rangle \right) = \underbrace{\Delta \mathbf{M}|_c}_{\substack{\text{momentum} \\ \text{lost via} \\ \text{collisions}}} - \underbrace{m \langle \mathbf{v} \rangle f|_c}_{\substack{\text{momentum change} \\ \text{via particle gain/loss}}} - \nabla_r \Sigma \mathbf{P} + qn(\mathbf{E} + \langle \mathbf{v} \rangle \wedge \mathbf{B})$$

We have looked collisions and Lorentz force terms in the in the last section two sections. Now we will look at the random and driven flows.

Often, we can approximate the collisional momentum loss as being proportional to the velocity. Here we will assume that

$$-mn\nu_m \mathbf{v} = \underbrace{\Delta \mathbf{M}|_c}_{\substack{\text{momentum} \\ \text{lost via} \\ \text{collisions}}} - \underbrace{m \langle \mathbf{v} \rangle f|_c}_{\substack{\text{momentum change} \\ \text{via particle gain/loss}}}$$

where  $\nu_m$  is the effective momentum transfer collision frequency. (The negative sign is there because we are assuming that we are losing energy to the other species.) Thus,

$$mn \left( \frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \Sigma \nabla_r \langle \mathbf{v} \rangle \right) = -mn\nu_m \mathbf{v} - \nabla_r \Sigma \mathbf{P} + qn(\mathbf{E} + \langle \mathbf{v} \rangle \wedge \mathbf{B})$$

Assuming that all of the accounted for forces balance. Hence, the left hand side of the equation is zero. This gives

$$0 = -mn\nu_m \mathbf{v} - \nabla_r \Sigma \mathbf{P} + qn(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

$$= -mn\nu_m \mathbf{v} - \gamma k T \nabla_r n + qn(\mathbf{E} + \mathbf{v} \wedge \mathbf{B})$$

or

$$\mathbf{v} = -\frac{\gamma k T}{mn\nu_m} \nabla_r n + \frac{qn}{mn\nu_m} (\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) - \text{for now we will let } \mathbf{B} = 0$$

$$= -\frac{\gamma k T}{mn\nu_m} \nabla_r n + \frac{qn}{mn\nu_m} \mathbf{E}$$

$$= -\frac{D}{n} \nabla_r n + \mu \mathbf{E}$$

where

$$D = \frac{\gamma k T}{m\nu_m}$$

$$\mu = \frac{q}{m\nu_m} - \text{note that I have used a slightly different definition for } \mu$$

are the diffusion and the mobility respectively. This is a straight-forward equation to get to but it has a lot of implications. First, we have assumed that the velocity was not a function of position or time. This means that the fluid is flowing from one point to another put that there is also fluid moving the other direction – in some sort of random walk type of fashion. We find – not surprisingly – that this means that if we where to be able to ‘paint’ some of the particles in the species in a certain area, we would see these painted particles drift – or diffuse – away from that area. This diffusion is quite natural. The second thing that we see is that the velocity of charged

particles under the influence is limited. This again is not too surprising – think of terminal velocity for a person who is sky diving (~186 mph). What else can we discover?

First, we can determine the flux of a given species.

$$\Gamma_s = n_s \mathbf{v}_s = -D_s \nabla_r n_s + n_s \mu_s \mathbf{E} \quad \text{for species } s$$

If there is no electric field we get what is known as ‘Fick’s Law’

$$\Gamma_s = n_s \mathbf{v}_s = -D_s \nabla_r n_s$$

This is simply our random walk.

### Ambipolar Diffusion

If one were to turn off a plasma, most of them will decay via a process known as ambipolar diffusion. ‘Ambipolar’ comes from two words. ‘Ambhi’ which is Latin for ‘on both sides’ or ‘round’ and polar (Duh?). In essence it means that both positive and negative charged diffuse at the same rate. Such a requirement is not particularly surprising as we have required and expect that the density of the positive and negative species must be about the same.

Without an applied electric field we clearly find that

$$\Gamma_e = -D_e \nabla_r n_e \gg \Gamma_i = -D_i \nabla_r n_i$$

This implies that we must have an electric field which pushes the ions out and serves to retain the electron. Thus,

$$\Gamma_e = -D_e \nabla_r n_e + n_e \mu_e \mathbf{E} = \Gamma_i = -D_i \nabla_r n_i + n_i \mu_i \mathbf{E}$$

We can now solve for the  $\mathbf{E}$  field.

$$(D_i - D_e) \nabla_r \tilde{n}_e = (\mu_i - \mu_e) \tilde{n}_e \mathbf{E}$$

$$\Rightarrow \mathbf{E} = \frac{(D_i - D_e) \nabla_r n_0}{(\mu_i - \mu_e) n_0}$$

Placing this into our common diffusion equation gives

$$\Gamma_a = -D_i \nabla_r n_0 + \mu_i \frac{(D_i - D_e)}{(\mu_i - \mu_e)} \nabla_r n_0$$

$$= \frac{(\mu_e D_i - \mu_i D_e)}{(\mu_i - \mu_e)} \nabla_r n_0$$

$$= D_a \nabla_r n_0$$

where

$$D_a = \frac{(\mu_e D_i - \mu_i D_e)}{(\mu_i - \mu_e)}$$

is the ambipolar diffusion coefficient.

We can determine approximately what the ambipolar diffusion coefficient is by making some simple estimates.

First  $\mu_{i,e} = \frac{q_{i,e}}{m_{i,e} v_m}$ . Now  $v_m \approx n \sigma v$ . Assuming that both the ions and the electrons have the

same energy, then the velocity of the electron will be on the order of  $\frac{v_e}{v_i} \approx \sqrt{\frac{M_i}{m_e}}$ . (This is

actually an under estimate as often the electron energy is higher than the ion energy.) Further, we will assume that the collision cross sections are on the same order of magnitude. This is not a bad assumption as coulomb collisions will be very similar as well as collisions with neutrals.

Thus, we expect

$$\frac{\mu_e}{\mu_i} = -\sqrt{\frac{M_i}{m_e}}$$

Then

$$D_a = D_i - \frac{\mu_i}{\mu_e} D_e$$

It just so happens that the ratio of

$$\frac{\mu}{D} = \frac{q}{\gamma k T}$$

So that

$$\begin{aligned} \frac{\mu_i}{\mu_e} D_e &= \frac{e}{\gamma_i k T_i} \frac{\gamma_e k T_e}{-e} D_i \\ &= -\frac{\gamma_e T_e}{\gamma_i T_i} D_i \quad \text{- often } \gamma_e = 1 \\ &= -\frac{T_e}{\gamma_i T_i} D_i \end{aligned}$$

Plugging this into the above equation for  $D_a$  gives

$$\begin{aligned} D_a &= D_i - \frac{\mu_i}{\mu_e} D_e \\ &= \left(1 + \frac{T_e}{\gamma_i T_i}\right) D_i \\ &\approx (2 \text{ to } 10) D_i \end{aligned}$$

Thus we see that the ambipolar diffusion is tied to the slower diffusion rate but which often greatly exceeds that slower rate.

Finally, we have to this point ignored the continuity equation.

$$f|_c = \left( \nabla_r \Sigma(n \langle \mathbf{v} \rangle) + \frac{\partial n}{\partial t} \right)$$

Ignoring the collisional term, and using the definition of the flux, it is easy to see that

$$\frac{\partial n_s}{\partial t} = -\nabla_{\mathbf{r}} \cdot \Sigma(n_s \langle \mathbf{v}_s \rangle) \quad \text{but} \quad \Gamma_s = n_s \mathbf{v}_s = -D_s \nabla_{\mathbf{r}} n_s + n_s \mu_s \mathbf{E}$$

↓

$$\begin{aligned} \frac{\partial n_s}{\partial t} &= -\nabla_{\mathbf{r}} \cdot \Sigma(-D_s \nabla_{\mathbf{r}} n_s + n_s \mu_s \mathbf{E}) \\ &= D_s \nabla_{\mathbf{r}}^2 n_s - n_s \mu_s \nabla_{\mathbf{r}} \cdot \Sigma \mathbf{E} \quad - \text{letting } \mathbf{E} = 0 \\ &= D_s \nabla_{\mathbf{r}}^2 n_s \end{aligned}$$

### Diffusion in a Slab

This last equation allows us to look closer at the decay of a plasma

First let's let

$$n_s(\mathbf{r}, t) = T(t)S(\mathbf{r})$$

Then using the standard separation of variable technique for solving PDEs we get

$$\frac{\partial n_s}{\partial t} = D_s \nabla_{\mathbf{r}}^2 n_s$$

$$\frac{\partial TS}{\partial t} = D_s \nabla_{\mathbf{r}}^2 TS$$

$$S \frac{\partial T}{\partial t} = T D_s \nabla_{\mathbf{r}}^2 S$$

$$\frac{1}{T} \frac{\partial T}{\partial t} = \frac{1}{S} D_s \nabla_{\mathbf{r}}^2 S = \text{Const} \quad \text{because we have to vary } t \text{ and } \mathbf{r} \text{ independently}$$

$$\text{We will call that } \text{Const} \equiv \frac{1}{\tau}$$

Then

$$\frac{\partial T}{\partial t} = -\frac{T}{\tau} \Rightarrow T = T_0 e^{-t/\tau}$$

and

$$\nabla_{\mathbf{r}}^2 S = -\frac{S}{D_s \tau} = -\frac{S}{\Lambda^2}$$

↓

$$S = S_{0+} e^{+i|\mathbf{r}|/\Lambda} + S_{0-} e^{-i|\mathbf{r}|/\Lambda} \quad \text{or}$$

$$S = S_{0A} \cos(|\mathbf{r}|/\Lambda) + S_{0B} \sin(|\mathbf{r}|/\Lambda)$$

Assuming that the plasma has a length of L, with a density of 0 at  $\pm L/2$ , we find

Class notes for EE6318/Phys 6383 – Spring 2001

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$$S|_{r=\pm L/2} = 0 = S_A \cos(|\mathbf{r}| / \Lambda) + S_B \sin(|\mathbf{r}| / \Lambda)$$

⇓

$$\begin{aligned} S_A \cos(L / 2\Lambda) &= S_B \sin(L / 2\Lambda) \quad (\text{from } -L / 2) \\ &= -S_B \sin(L / 2\Lambda) \quad (\text{from } +L / 2) \end{aligned}$$

⇓

$$S_B = 0 \text{ and } \frac{L}{2\Lambda} = \frac{\pi}{2} \pm j\pi$$

Thus,

$$\Lambda = \frac{L}{\pi} (1 \pm 2j)^{-1} = \sqrt{D_s \tau} \Rightarrow \tau_j = \frac{L^2}{D_s \pi^2} (1 \pm 2j)^{-2}$$

and

$$S = \sum_j S_{Aj} \cos\left(\frac{|\mathbf{r}|\pi}{L} (1 \pm 2j)\right)$$

Therefore

$$n_s(\mathbf{r}, t) = T(t)S(\mathbf{r})$$

$$= \sum_{j=-\infty}^{\infty} n_{0j} \exp\left(-t \frac{D_s \pi^2}{L^2} (1 + 2j)^2\right) \cos\left(\frac{|\mathbf{r}|\pi}{L} (1 + 2j)\right)$$

From  $\tau_j = \frac{L^2}{D_s \pi^2} (1 \pm 2j)^{-2}$  it is readily apparent that the higher order modes will decay faster than lower order modes. Thus as a plasma shuts off, we would expect to see a ‘cos’ profile to the density distribution.