

Lecture 8 Plasma Sheaths

New homework problems:

Lieberman 6.1, 6.3 and 6.5 Due March 19th, 2001.

Wednesday's class will be at 1 PM in MP 2.228 - get other make up time for Thursday.

We know from the previous section that the electrons and ions must level a plasma at the same time and position averaged rate. (This implies that we can have the electrons leaving at one location and the ions leaving at another location. It also implies that the ions might leave at a certain time while the electrons leave at a different time – provided the time difference is not great.) We want to examine plasmas in terms of this requirement of charge neutrality.

Our fundamental fluid equations are as before:

$$f|_c = \left(\nabla_r \Sigma \langle n \langle \mathbf{v} \rangle \rangle + \frac{\partial n}{\partial t} \right)$$

$$mn \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \Sigma \nabla_r \langle \mathbf{v} \rangle \right) = \underbrace{\Delta \mathbf{M}|_c}_{\substack{\text{momentum} \\ \text{lost via} \\ \text{collisions}}} - \underbrace{m \langle \mathbf{v} \rangle f|_c}_{\substack{\text{momentum change} \\ \text{via particle gain/loss}}} - \nabla_r \Sigma \mathbf{P} + qn(\mathbf{E} + \langle \mathbf{v} \rangle \wedge \mathbf{B})$$

Let us examine what each of these equations mean.

First integrating over the continuity equation

$$\begin{aligned} \iiint_{Vol} f|_c d\tau - \iiint_{Vol} \frac{\partial n}{\partial t} d\tau &= \iiint_{Vol} \nabla_r \Sigma \langle n \langle \mathbf{v} \rangle \rangle d\tau \\ &= \oint_{\partial Vol} n \langle \mathbf{v} \rangle \Sigma ds \\ &= \oint_{\partial Vol} \Gamma \Sigma ds \end{aligned}$$

where Γ is the flux of the particles.

Likewise we can rewrite of energy equation, here integrating along the \mathbf{E} field to give

$$\int \left(m \frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \frac{1}{2} m \nabla_r \Sigma \mathbf{v} \mathbf{v} \right) \Sigma d\mathbf{l} = \int \left(\frac{1}{n} \Delta \mathbf{M}|_c - \frac{m}{n} \langle \mathbf{v} \rangle f|_c \right) \Sigma d\mathbf{l} - \int \left(\frac{1}{n} \nabla_r \Sigma \mathbf{P} \right) \Sigma d\mathbf{l} + \int q(\mathbf{E} + \langle \mathbf{v} \rangle \wedge \mathbf{B}) \Sigma d\mathbf{l}$$

These are in general very difficult problems to solve. Thus we will make some approximations. First, we will let the collision terms go to zero. Then we will let the pressure differential go to zero. Both of these approximations are reasonable for many plasma systems. Typically they have very thin sheaths compared to the mean-free path and are in low-pressure systems. Thus our equations simplify to:

$$-\iiint_{Vol} \frac{\partial n}{\partial t} d\tau = \oiint_{\partial Vol} \Gamma \Sigma ds$$

$$\int \left(m \frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \frac{1}{2} m \nabla_r \Sigma \mathbf{v} \mathbf{v} \right) \Sigma d\mathbf{l} = \int q (\mathbf{E} + \langle \mathbf{v} \rangle \wedge \mathbf{B}) \Sigma d\mathbf{l}$$

We can further simplify our model by assuming time independence and no magnetic field. This eliminates a lot of real sheaths that we might run across but it makes the model understandable.

Thus, we are dealing with the following set of equations

$$\oiint_{\partial Vol} \Gamma \Sigma ds = 0$$

$$\Rightarrow \Gamma = n\mathbf{v} = const$$

$$\int \left(\frac{1}{2} m \nabla_r \Sigma \mathbf{v} \mathbf{v} \right) \Sigma d\mathbf{l} = \int q \mathbf{E} \Sigma d\mathbf{l} = - \int q \nabla \phi \Sigma d\mathbf{l}$$

$$\Rightarrow \frac{1}{2} m \Delta v^2 = -q \Delta \phi$$

We will use these equations to give us the ion density and velocity at different points in the sheath. (The above equations are general and we had not applied them to any species.) The only thing that we need to know is what is the potential as a function of position. Still, we can determine the velocity and the density as a function of the potential. Thus we have

$$n_i = \frac{v_{is}}{v_i} n_{is}$$

$$\frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i v_{is}^2 + e(\phi_s - \phi)$$

where 's' stands for the value at the sheath edge. Combining the equations gives

$$\frac{1}{2} m_i \left(\frac{n_{is}}{n_i} v_{is} \right)^2 = \frac{1}{2} m_i v_{is}^2 + e(\phi_s - \phi)$$

$$m_i n_{is}^2 v_{is}^2 = n_i^2 (m_i v_{is}^2 + 2e(\phi_s - \phi))$$

$$n_{is} = n_i \left[1 + \frac{2e}{m_i v_{is}^2} (\phi_s - \phi) \right]^{1/2} \quad or$$

$$n_i = n_{is} \left[1 + \frac{2e}{m_i v_{is}^2} (\phi_s - \phi) \right]^{-1/2}$$

This is slightly different than what is in the book as I have included the fact that the plasma has a potential relative to other items, labeled, ϕ_s .

To get the potential, we can use Poisson's equation.

$$\nabla^2 \phi = -\frac{\rho}{\epsilon} = \frac{e}{\epsilon}(n_e - n_i)$$

Again we are missing a piece of the puzzle, but now it is the electron density. There are a couple of readily useful options. 1) There are no electrons in the sheath. 2) The electron density is set by Boltzmann's equation. The second option is the one used most commonly, while the first is appropriate only for very special situations.

Initially, let us assume that the electrons follow Boltzmann's equation. Thus,

$$n_e = n_0 e^{e\Delta\phi/kT_e}$$

where n_0 is the electron density at the point at which the potential is zero. Or we can write this in terms of the potential at the sheath edge and get

$$n_e = n_{es} e^{e(\phi_s - \phi)/kT_e}$$

This and our equation for the ion density can be plugged into Poisson's equation to give

$$\nabla^2 \phi = \frac{e}{\epsilon} \left(n_{es} e^{-e(\phi_s - \phi)/kT_e} - n_{is} \left[1 + \frac{e}{\frac{1}{2} m_i v_{is}^2} (\phi_s - \phi) \right]^{-1/2} \right)$$

$$\nabla^2 \Phi = \frac{e}{\epsilon} \left(n_{es} e^{e\Phi/kT_e} - n_{is} \left[1 - \frac{e}{E_{is}} (\Phi) \right]^{-1/2} \right)$$

where $\Phi = \phi - \phi_s < 0$ and $E_{is} = \frac{1}{2} m_i v_{is}^2$.

This equation can be integrated once analytically. This is done in the following way. First multiply by $\nabla\Phi$. This gives

$$\nabla\Phi(\nabla^2\Phi) = \nabla\Phi \frac{e}{\epsilon} \left(n_{es} e^{e\Phi/kT_e} - n_{is} \left[1 - \frac{e}{E_{is}} (\Phi) \right]^{-1/2} \right)$$

Now integrate over $d\mathbf{l}$

$$\int_0^{\mathbf{l}} \nabla\Phi(\nabla^2\Phi) \Sigma d\mathbf{l} = \int_0^{\mathbf{l}} \nabla\Phi \frac{e}{\epsilon} \left(n_{es} e^{e\Phi/kT_e} - n_{is} \left[1 - \frac{e\Phi}{E_{is}} \right]^{-1/2} \right) \Sigma d\mathbf{l}$$

$$\int_0^{\mathbf{l}} \frac{1}{2} \nabla(\nabla\Phi)^2 \Sigma d\mathbf{l} = \int_0^{\mathbf{l}} \frac{e}{\epsilon} \nabla \left(n_{es} \frac{kT_e}{e} e^{e\Phi/kT_e} + 2n_{is} \frac{E_{is}}{e} \left[1 - \frac{e\Phi}{E_{is}} \right]^{1/2} \right) \Sigma d\mathbf{l}$$

where 0 is the sheath edge and \mathbf{l} is a position in the sheath. Now, $\nabla = \frac{d}{d\mathbf{l}}$ so that we can use the first principle of calculus and get

$$\int_0^1 \frac{1}{2} \frac{d}{d\mathbf{l}} (\nabla\Phi)^2 \Sigma d\mathbf{l} = \int_0^1 \frac{e}{\varepsilon} \frac{d}{d\mathbf{l}} \left(n_{es} \frac{kT_e}{e} e^{e\Phi/kT_e} + 2n_{is} \frac{E_{is}}{e} \left(1 - \frac{e\Phi}{E_{is}} \right)^{1/2} \right) \Sigma d\mathbf{l}$$

$$\frac{1}{2} (\nabla\Phi)^2 \Big|_{\mathbf{l}=0}^1 = \frac{e}{\varepsilon} \left(n_{es} \frac{kT_e}{e} e^{e\Phi/kT_e} + 2n_{is} \frac{E_{is}}{e} \left(1 - \frac{e\Phi}{E_{is}} \right)^{1/2} \right) \Big|_{\mathbf{l}=0}^1$$

$$\frac{1}{2} \left[(\nabla\Phi(\mathbf{l}))^2 - \left(\nabla\overbrace{\Phi}^{=0}(\mathbf{l}) \right)^2 \right] = \frac{e}{\varepsilon} \left[n_{es} \frac{kT_e}{e} \left[e^{e\Phi(\mathbf{l})/kT_e} - e^{\overbrace{e\Phi(\mathbf{l})}^{=0}/kT_e} \right] + 2n_{is} \frac{E_{is}}{e} \left[\left(1 - \frac{e\Phi(\mathbf{l})}{E_{is}} \right)^{1/2} - \left(1 - \frac{\overbrace{e\Phi(\mathbf{l})}^{=0}}{E_{is}} \right)^{1/2} \right] \right]$$

$$\frac{1}{2} (\nabla\Phi(\mathbf{l}))^2 = \frac{e}{\varepsilon} \left[n_{es} \frac{kT_e}{e} \left[e^{e\Phi(\mathbf{l})/kT_e} - 1 \right] + 2n_{is} \frac{E_{is}}{e} \left[\left(1 - \frac{e\Phi(\mathbf{l})}{E_{is}} \right)^{1/2} - 1 \right] \right]$$

$$\frac{1}{2} \left[(\nabla\Phi(\mathbf{l}))^2 - \left(\nabla\overbrace{\Phi}^{=0}(\mathbf{l}) \right)^2 \right] = \frac{e}{\varepsilon} \left[n_{es} \frac{kT_e}{e} \left[e^{e\Phi(\mathbf{l})/kT_e} - e^{\overbrace{e\Phi(\mathbf{l})}^{=0}/kT_e} \right] + 2n_{is} \frac{E_{is}}{e} \left[\left(1 - \frac{e\Phi(\mathbf{l})}{E_{is}} \right)^{1/2} - \left(1 - \frac{\overbrace{e\Phi(\mathbf{l})}^{=0}}{E_{is}} \right)^{1/2} \right] \right]$$

or

$$\nabla\Phi(\mathbf{l}) = \sqrt{\frac{2e}{\varepsilon} \left[n_{es} \frac{kT_e}{e} \left[e^{e\Phi(\mathbf{l})/kT_e} - 1 \right] + 2n_{is} \frac{E_{is}}{e} \left[\left(1 - \frac{e\Phi(\mathbf{l})}{E_{is}} \right)^{1/2} - 1 \right] \right]}$$

There is no known way to further analytically integrate this equation. However we can still tell that the term inside the square root cannot be negative – or else the potential is imaginary. We can learn something from this requirement.

$$\left[n_{es} \frac{kT_e}{e} \left[e^{e\Phi(\mathbf{l})/kT_e} - 1 \right] + 2n_{is} \frac{E_{is}}{e} \left[\left(1 - \frac{e\Phi(\mathbf{l})}{E_{is}} \right)^{1/2} - 1 \right] \right] \geq 0$$

For small potentials, we can expand the exponential term to get

$$\left[n_{es} \frac{kT_e}{e} \left[\left(1 + \frac{e\Phi(\mathbf{l})}{kT_e} \right) - 1 \right] + 2n_{is} \frac{E_{is}}{e} \left[\left(1 - \frac{e\Phi(\mathbf{l})}{E_{is}} \right)^{1/2} - 1 \right] \right] \geq 0$$

$$\left[n_{es} \Phi(\mathbf{l}) + 2n_{is} \frac{E_{is}}{e} \left[\left(1 - \frac{e\Phi(\mathbf{l})}{E_{is}} \right)^{1/2} - 1 \right] \right] \geq 0$$

Further we can expand the square root term to get

$$\left(n_{es} \Phi(\mathbf{l}) + 2n_{is} \frac{E_{is}}{e} \left[\left(1 - \frac{1}{2} \frac{e\Phi(\mathbf{l})}{E_{is}} \right) - 1 \right] \right) \geq 0$$

$$\left(n_{es} \Phi(\mathbf{l}) - 2n_{is} \frac{E_{is}}{e} \left[\frac{1}{2} \frac{e\Phi(\mathbf{l})}{E_{is}} \right] \right) \geq 0$$

$$(n_{es} \Phi(\mathbf{l}) - n_{is} \Phi(\mathbf{l})) \geq 0$$

$$n_{es} \leq n_{is} \quad - \text{remember that } \Phi(\mathbf{l}) < 0!$$

This means that the electron density must be less than the ion density but not by much. This is not surprising as we have already said that the sheath occurs because the electrons would otherwise leave faster than the ions. This makes it such that electron density is slightly smaller than the ion density. Now, if we assume that the electron and ion densities are approximately the same, so that $n_{es} \approx n_{is} = n_s$ as we typically assume in the plasma. This is certainly an option.

We can now go back and examine the right hand side of our inequality again.

$$\left(n_s \frac{kT_e}{e} \left[e^{e\Phi(\mathbf{l})/kT_e} - 1 \right] + 2n_s \frac{E_{is}}{e} \left[\left(1 - \frac{e\Phi(\mathbf{l})}{E_{is}} \right)^{1/2} - 1 \right] \right) \geq 0$$

$$\left(\frac{kT_e}{e} \left[e^{e\Phi(\mathbf{l})/kT_e} - 1 \right] + 2 \frac{E_{is}}{e} \left[\left(1 - \frac{e\Phi(\mathbf{l})}{E_{is}} \right)^{1/2} - 1 \right] \right) \geq 0$$

Now we will expand the equation to the second order so that

$$0 \leq \left(\frac{kT_e}{e} \left[\left(1 + \frac{e\Phi(\mathbf{l})}{kT_e} + \frac{e^2\Phi^2(\mathbf{l})}{2k^2T_e^2} \right) - 1 \right] + 2 \frac{E_{is}}{e} \left[\left(1 - \frac{e\Phi(\mathbf{l})}{2E_{is}} - \frac{e^2\Phi^2(\mathbf{l})}{8E_{is}^2} \right) - 1 \right] \right)$$

$$0 \leq \left(\left[\Phi(\mathbf{l}) + \frac{e\Phi^2(\mathbf{l})}{2kT_e} \right] - \left[\Phi(\mathbf{l}) + \frac{e\Phi^2(\mathbf{l})}{4E_{is}} \right] \right)$$

$$0 \leq \left(\left[\frac{e\Phi^2(\mathbf{l})}{2kT_e} \right] - \left[\frac{e\Phi^2(\mathbf{l})}{4E_{is}} \right] \right)$$

$$kT_e \leq 2E_{is}$$

$$v_{is} \geq v_B = \left(\frac{kT_e}{M_i} \right)^{1/2} = \text{ion acoustic / Bohm velocity}$$

This is known as the Bohm Sheath criteria. In effect, it says that the ions have to be traveling at the ion acoustic velocity before they enter the sheath. This makes sense as the if the velocities were zero at the edge of the sheath, then the flux of the ions into the sheath would be zero (unless the density was infinite!). This means that we have to have a region in which the ions are accelerated from 0 to the acoustic velocity. This region is known as the ‘presheath.’

What does this mean? First to accelerate the ions to the Bohm velocity we need to have a potential drop such that energy is conserved. Thus, ignoring collisions we find that

$$\frac{1}{2} m \Delta v^2 = -q \Delta \phi$$

↓

$$\frac{1}{2} M_i \left(v_B^2 - \overbrace{v_{plasma}^2}^{\approx 0} \right) = -q (\phi_B - \phi_{plasma})$$

$$\frac{1}{2} M_i \frac{kT_e}{M_i} = -q \Phi_B$$

$$q \Phi_B = -\frac{1}{2} kT_e$$

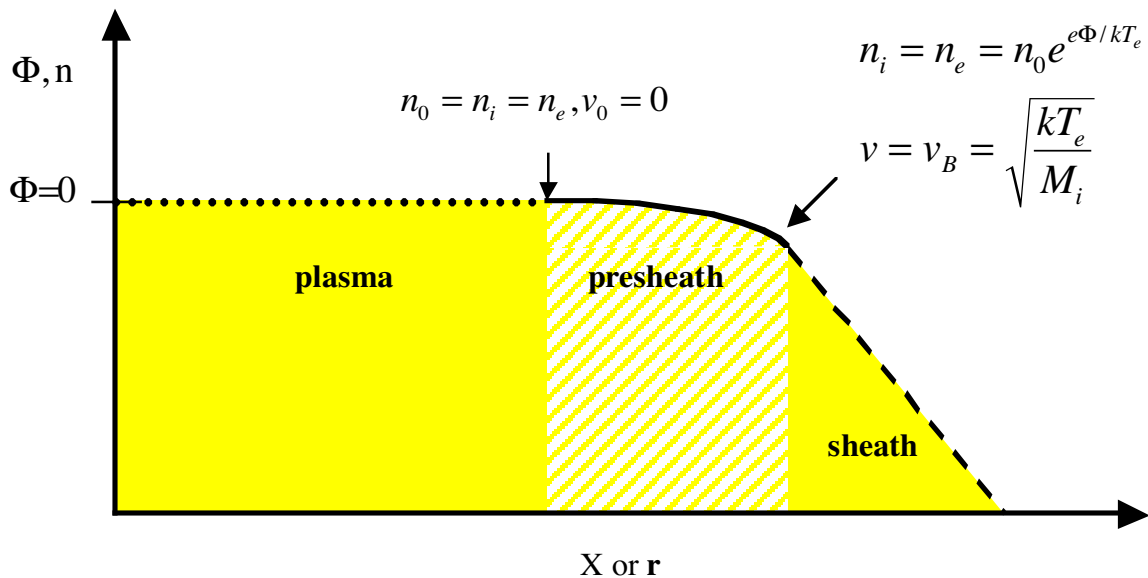
Likewise we know that we can determine the electron density at the edge of the sheath from Boltzmann's equation.

$$n_s = n_{plasma} e^{-q\Phi_B/kT_e}$$

$$= n_{plasma} e^{-1/2}$$

$$= 0.61 n_{plasma}$$

The above discussion implies that there has to be some sort of presheath region which accelerates the ions from zero to the Bohm velocity. We can get some handle on the subject if we assume a few simple things. First that the ion and electron densities are equal throughout the presheath. Second, that the electron density follows Boltzmann's equation. Third and final, that the ions enter the presheath at zero velocity and leave on the other side at the Bohm velocity. Thus we have the following picture.



To understand the presheath, we need to look at the ion flux through the presheath. This is simply

$$\Gamma_i = n_i v_i.$$

Taking the derivative of both sides gives

$$\begin{aligned}\nabla \Sigma \Gamma_i &= \nabla \Sigma (n_i \mathbf{v}_i) \\ &= n_i \nabla \Sigma (\mathbf{v}_i) + \mathbf{v}_i \Sigma \nabla (n_i)\end{aligned}$$

now dividing through by the flux, we get

$$\frac{1}{\Gamma_i} \nabla \Sigma \Gamma_i = \frac{1}{\mathbf{v}_i} \nabla \Sigma (\mathbf{v}_i) + \frac{1}{n_i} \nabla (n_i).$$

Plugging in Boltzmann's equation gives

$$\begin{aligned}\frac{1}{\Gamma_i} \nabla \Sigma \Gamma_i &= \frac{1}{\mathbf{v}_i} \nabla \Sigma (\mathbf{v}_i) + \frac{1}{n_i} \nabla (n_i) \\ &= \frac{1}{\mathbf{v}_i} \nabla \Sigma (\mathbf{v}_i) + \frac{1}{n_0 e^{e\Phi/kT_e}} \nabla (n_0 e^{e\Phi/kT_e}) \\ &= \frac{1}{\mathbf{v}_i} \nabla \Sigma (\mathbf{v}_i) + \frac{1}{n_0 e^{e\Phi/kT_e}} n_0 e^{e\Phi/kT_e} \frac{e}{kT_e} \nabla (\Phi) \\ &= \frac{1}{\mathbf{v}_i} \nabla \Sigma (\mathbf{v}_i) + \frac{e}{kT_e} \nabla (\Phi)\end{aligned}$$

What does this mean?

Well we can get some handle on the subject if we examine what happens to the ion energy as it passes through the pre sheath. The only simple example is conserved energy. (Obviously there are ways in which energy can also be added or subtracted!)

If energy is conserved then $\frac{1}{2} m \Delta v^2 = -e \Delta \phi$. This implies for our case that

$$\frac{1}{2} M_i v^2 = -e \Phi \quad \text{or} \quad v = \sqrt{\frac{-2e\Phi}{M_i}}$$

thus

$$\begin{aligned}\nabla v_i &= \nabla \sqrt{\frac{-2e\Phi}{M_i}} \\ &= \sqrt{\frac{-2e}{M_i}} \nabla \sqrt{\Phi} \\ &= \frac{1}{2} \sqrt{\frac{-2e}{M_i}} \frac{1}{\sqrt{\Phi}} \nabla \Phi \\ &= \frac{1}{2} \sqrt{\frac{-2e\Phi}{M_i}} \Phi^{-1} \nabla \Phi \\ &= \frac{1}{2} v \Phi^{-1} \nabla \Phi \\ &\Downarrow\end{aligned}$$

$$\frac{1}{v_i} \nabla v_i = \frac{1}{2} \Phi^{-1} \nabla \Phi$$

and at the sheath edge

$$\frac{1}{2} M_i \frac{kT_e}{M_i} = -e\Phi$$

$$\Phi = -\frac{1}{2} \frac{kT_e}{e}$$

Plugging this into our flux equation we find

$$\begin{aligned} \frac{1}{\Gamma_i} \nabla \Sigma \Gamma_i &= \frac{1}{\mathbf{v}_i} \nabla \Sigma(\mathbf{v}_i) + \frac{e}{kT_e} \nabla \Phi \\ &= \frac{1}{2} \Phi^{-1} \nabla \Phi + \frac{e}{kT_e} \nabla \Phi \\ &= \frac{1}{2} \frac{2e}{kT_e} \nabla \Phi + \frac{e}{kT_e} \nabla \Phi - \text{at the sheath edge} \\ &= 0 \end{aligned}$$

This implies that

$$\nabla \Sigma \Gamma_i = 0$$

in the presheath if the energy is conserved. Other solutions typically have to be found via computer simulations.

Classified Sheaths

There are a few important standard sheaths. These are sheaths to a floating wall, sheaths to a grounded wall and sheaths to a driven wall. We will start with the sheath potential at a floating wall.

If a wall or other object is inserted into a plasma and there is no way for the charge to drain from the surface, we have a situation in which the wall is electrically floating. This gives rise to a ‘floating sheath’. In this case, the object will charge up until the flux of electrons and ions match. Thus to find floating potential of the object we simply need to determine the potential at which the fluxes match.

The flux of ions to the wall is given by

$$\Gamma_i = \text{const}$$

$$= n_s v_B$$

$$= n_s \sqrt{\frac{kT_e}{M_i}}$$

This flux is matched by the flux of electrons to the surface. The electron flux is those electron headed toward the surface which also have enough energy to overcome the potential. Thus,

$$\Gamma_e = \int_{v_{\min}}^{\infty} v_x f(v) dv_x dv_y dv_z$$

where

$$v_{\min} = \sqrt{\frac{2e\Phi_w}{m_e}}$$

is set by energy requirements

Thus

$$\begin{aligned}
 \Gamma_e &= n_e \left(\frac{m_e}{2\pi kT_e} \right)^{3/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{\frac{2e\Phi_w}{m_e}} v_x \exp\left[\frac{-m_e(v_x^2 + v_y^2 + v_z^2)}{2kT_e} \right] dv_x dv_y dv_z \\
 &= n_e \left(\frac{m_e}{2\pi kT_e} \right)^{1/2} \int_{-\infty}^{\infty} \sqrt{\frac{2e\Phi_w}{m_e}} v_x \exp\left[\frac{-m_e(v_x^2)}{2kT_e} \right] dv_x \\
 &= n_e \frac{1}{2} \left(\frac{2kT_e}{\pi m_e} \right)^{1/2} \int_{-\infty}^{\infty} \sqrt{\frac{2e\Phi_w}{m_e}} \exp\left[\frac{-m_e(v_x^2)}{2kT_e} \right] d\left(\frac{m_e v_x^2}{2kT_e} \right) \\
 &= n_e \frac{1}{2} \left(\frac{2kT_e}{\pi m_e} \right)^{1/2} \int_{-\infty}^{\infty} \sqrt{\frac{2e\Phi_w}{m_e}} \exp[-U] d(U) \\
 &= n_e \left(\frac{kT_e}{2\pi m_e} \right)^{1/2} \exp\left[\frac{-e\Phi_w}{kT_e} \right] \left\{ = n_e \frac{1}{4} \overbrace{\langle v \rangle}^{\text{average speed}} \exp\left[\frac{-e\Phi_w}{kT_e} \right] \right\}
 \end{aligned}$$

Now the flux of the ions and the electrons must come into balance so that

$$\begin{aligned}
 n_e \left(\frac{kT_e}{2\pi m_e} \right)^{1/2} \exp\left[\frac{-e\Phi_w}{kT_e} \right] &= n_s \sqrt{\frac{kT_e}{M_i}} \\
 &= n_e \sqrt{\frac{kT_e}{M_i}} \\
 \Downarrow \\
 \Phi_w &= -\frac{kT_e}{e} \ln\left(\sqrt{\frac{2\pi m_e}{M_i}} \right)
 \end{aligned}$$

Grounded Sheath

In this case the electrode is electrically connected such that we can add or subtract any necessary charge to provide balance. As before the flux of ions to the surface is

$$\Gamma_i = n_i(x)v_i(x) = \text{const}$$

(Note that while I am doing this for flux, you really want to do this for current. The distinction is that the ions might be doubly – or greater – charged. Thus the flux and the current are not equivalent. You really need to know how much charge you are carrying to the surface.)

To keep things as simple as possible, we will assume that the energy is conserved. Thus

$$\frac{1}{2} M_i \Delta v^2 = -e\Delta\Phi.$$

If we assume further that the initial kinetic energy and potential is zero – hence ignore the presheath – we find

$$\frac{1}{2} M_i (v^2 - v_B^2) = -e(\Phi - \Phi_s) \quad \text{or}$$

$$v = \sqrt{v_B^2 - \frac{2e(\Phi - \Phi_s)}{M_i}}$$

but

$$v_B^2 = \frac{kT_e}{M_i}$$

$$\Phi_s = -\frac{1}{2} \frac{kT_e}{e}$$

thus

$$\begin{aligned} v &= \sqrt{\frac{kT_e}{M_i} - \frac{2e(\Phi)}{M_i} - \frac{kT_e}{M_i}} \\ &= \sqrt{-\frac{2e(\Phi)}{M_i}} \end{aligned}$$

and

$$\Gamma_i = n_i(x)v_i(x) = \text{const}$$

$$= n_i(x) \sqrt{-\frac{2e\Phi}{M_i}}$$

↓

$$n_i(x) = \Gamma_i \sqrt{\frac{-M_i}{2e\Phi(x)}}$$

This means that in the sheath, the ion density can be determined if we know the potential. We can find the potential from Poisson's equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon} = \frac{e}{\epsilon} (n_e - n_i)$$

Plugging in our ion density variation electron density from the Boltzmann's equation

$$n_e = n_0 e^{e\Phi/kT_e}$$

we find

$$\nabla^2 \phi = -\frac{\rho}{\epsilon} = \frac{e}{\epsilon} \left(n_0 e^{e\Phi/kT_e} - \Gamma_i \sqrt{\frac{-M_i}{2e\Phi(x)}} \right)$$

As we did before, we can integrate this by multiplying by $\nabla \phi$ and then integrating. Thus

$$\begin{aligned} \nabla^2 \Phi \nabla \Phi &= \frac{e}{\epsilon} \left(n_0 e^{e\Phi/kT_e} \nabla \Phi + \Gamma_i \sqrt{\frac{M_i}{2e(-\Phi)}} \nabla(-\Phi) \right) \\ &\Downarrow \\ \int_0^r \nabla^2 \Phi \nabla \Phi \Sigma d\mathbf{r} &= \frac{e}{\epsilon} \int_0^r \left(n_0 e^{e\Phi/kT_e} \nabla \Phi + \Gamma_i \sqrt{\frac{M_i}{2e(-\Phi)}} \nabla(-\Phi) \right) \Sigma d\mathbf{r} \\ \int_0^r \frac{1}{2} \nabla(\nabla \Phi)^2 \Sigma d\mathbf{r} &= \frac{e}{\epsilon} \int_0^r \left(n_0 \frac{kT_e}{e} \nabla(e^{e\Phi/kT_e}) + \Gamma_i \nabla \left(2 \sqrt{\frac{M_i(-\Phi)}{2e}} \right) \right) \Sigma d\mathbf{r} \\ &\Downarrow \\ \frac{1}{2} (\nabla \Phi)^2 \Big|_0^r &= \frac{e}{\epsilon} \left[n_0 \frac{kT_e}{e} (e^{e\Phi/kT_e}) + \Gamma_i \left(2 \sqrt{\frac{-M_i \Phi}{2e}} \right) \right] \Big|_0^r \\ &= \frac{e}{\epsilon} \left[\frac{kT_e}{e} n_e(\mathbf{r}) + \Gamma_i \left(2 \sqrt{\frac{-M_i \Phi}{2e}} \right) \right] \Big|_0^r \end{aligned}$$

Now assume the following. 1) $\nabla \Phi(\mathbf{r} = 0) = 0$, 2) $\Phi(\mathbf{r} = 0) = 0$, AND 3) $n_e(\mathbf{r}) = 0$. The first two assumptions are reasonable, while the third assumption is an important approximation that allows us to obtain an analytic solution. With these assumptions we find

$$\begin{aligned} \frac{1}{2} (\nabla \Phi)^2 &= \Gamma_i \frac{e}{\epsilon} \left(2 \sqrt{\frac{-M_i \Phi}{2e}} \right) \\ &\Downarrow \\ -\nabla \Phi &= 2 \left(\frac{e\Gamma_i}{\epsilon} \right)^{1/2} \left(\frac{-M_i(-\Phi)}{2e} \right)^{1/4} \\ &\Downarrow \\ \frac{\nabla(-\Phi)}{(-\Phi)^{1/4}} &= 2 \left(\frac{e\Gamma_i}{\epsilon} \right)^{1/2} \left(\frac{M_i}{2e} \right)^{1/4} \end{aligned}$$

Integrating again

$$\int_0^r \frac{\nabla(-\Phi)}{(-\Phi)^{1/4}} d\mathbf{r} = \int_0^r 2 \left(\frac{e\Gamma_i}{\epsilon} \right)^{1/2} \left(\frac{M_i}{2e} \right)^{1/4} d\mathbf{r}$$

$$\Downarrow$$

$$\int_0^r \frac{4}{3} \nabla(-\Phi)^{3/4}(\mathbf{r}) d\mathbf{r} = 2 \left(\frac{e\Gamma_i}{\epsilon} \right)^{1/2} \left(\frac{M_i}{2e} \right)^{1/4} \int_0^r d\mathbf{r}$$

$$\Downarrow$$

$$\frac{4}{3} (-\Phi)^{3/4}(\mathbf{r}) = 2 \left(\frac{e\Gamma_i}{\epsilon} \right)^{1/2} \left(\frac{M_i}{2e} \right)^{1/4} \mathbf{r}$$

$$\Downarrow$$

$$(-\Phi)^{3/4} = \frac{3}{2} \left(\frac{e\Gamma_i}{\epsilon} \right)^{1/2} \left(\frac{M_i}{2e} \right)^{1/4} \mathbf{r}$$

This is known as the Child's Law sheath. It is the only known analytic description, albeit an approximate solution, to the sheath equation.

Ion Matrix Sheath

The ion matrix sheath occurs when the potential on an object changes very rapidly. In this case, the electric field change is such that the electrons leave the immediate region while the ions remain fixed for a small instance. (The heavy ions can not respond as fast as the light electrons.) Then there is an electric field that occurs through a uniform distribution of ions. How thick is such a sheath and what is the potential structure? From Poisson's equation we know:

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon} = \frac{e}{\epsilon} \left(\overset{=0}{\vec{n}_e} - \overset{=const=n_0}{\vec{n}_i} \right)$$

$$= \frac{-e}{\epsilon} n_0 = const$$

\Downarrow integrating once gives

$$\nabla \Phi = \frac{-e}{\epsilon} n_0 \mathbf{r} \quad (= -\mathbf{E})$$

\Downarrow integrating again gives

$$\Phi = \frac{-e}{2\epsilon} n_0 r^2$$

\Downarrow inverting

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$$r = \sqrt{-\Phi \frac{2\varepsilon}{en_0}} \quad \text{- Thus at the wall}$$

$$\begin{aligned} S_{wall} &= \sqrt{-\Phi_{wall} \frac{2\varepsilon}{en_0}} \\ &= \lambda_{Debye} \sqrt{\frac{-2\Phi_{wall}}{kT_e}}, \quad \lambda_{Debye} = \sqrt{\frac{\varepsilon kT_e}{en_0}} \end{aligned}$$

Now we can start to add back parts to our model of the sheath. One of the important things that we left out was collisions. How do we add these back into our model?