

Sets, Functions, Domains

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Sets

Collections of elements.

- Enumeration of elements

$\{1, \{1, 4, 7\}, 4\}, \{\text{red, yellow}\}, \{\}$

- Defining property

$\{x | P(x)\}$

$\{x | x \text{ is an even integer}\}$

Examples

- Natural numbers **N**

$\{0, 1, 2, \dots\}$

- Truth values (Booleans) **B**

$\{\text{true, false}\}$

- Rational numbers **Q**

$\{x | x = p/q \text{ for some } p, q \in \mathbf{N}, q \neq 0\}$

Set Predicates

Based on concept of membership

- Membership $x \in S$

Only basic predicate
(sets are black boxes otherwise)

- Equivalence $R = S$

$x \in R \Leftrightarrow x \in S$ (for all x)
(extensionality principle)

- Subset $R \subseteq S$

$x \in R \Rightarrow x \in S$ (for all x)
($\{\} \subseteq S, S \subseteq S$)

Set Constructions

Composition of sets

- Union $R \cup S$

$$\{x \mid x \in R \text{ or } x \in S\}$$
$$\cup_i S_i = S_{j_1} \cup S_{j_2} \cup \dots \cup S_{j_n}$$

- Intersection $R \cap S$

$$\{x \mid x \in R \text{ and } x \in S\}$$
$$\cap_i S_i = S_{j_1} \cap S_{j_2} \cap \dots \cap S_{j_n}$$

- Powerset $\mathbf{P}(R)$

$$\{x \mid x \subseteq R\}$$
$$(\{\} \in \mathbf{P}(R), R \in \mathbf{P}(R))$$

Product

Concept of ordered pair

- Constructor (x, y)

- Selectors

$$\text{fst}(x,y) = x$$

$$\text{snd}(x,y) = y$$

- Product $R \times S$

$$\{(x, y) | x \in R \text{ and } x \in S\}$$

Sum

Concept of disjoint union

- Sum $R + S$

$$\{(\text{zero}, x) \mid x \in R\} \cup \{(\text{one}, y) \mid y \in S\}$$

“tags” to preserve origin of element

- Constructors

$$\text{inR}(x) = (\text{zero}, x) \text{ (for } x \in R)$$

$$\text{inS}(y) = (\text{one}, y) \text{ (for } y \in S)$$

- Selector

cases m of

$$\text{isR}(x) \Rightarrow \boxed{\dots x \dots}$$

$$\text{isS}(y) \Rightarrow \boxed{\dots y \dots}$$

end

$$m = (\text{zero}, x) \Rightarrow \boxed{\dots x \dots}$$

$$m = (\text{one}, y) \Rightarrow \boxed{\dots y \dots}$$

Functions

Black box that accepts object as input and produces another object as output

Definition in terms of sets

- $f : R \Rightarrow S$

f is function from R to S

R domain of f , S codomain of f

$R \Rightarrow S$ arity (functionality) of f

- Application $f(a)$

$$a \in R, f(a) \in S$$

- Equality $f = g$

$$f, g : R \Rightarrow S \quad f(x) = g(x) \text{ (for all } x\text{)}$$

(extensionality principle)

- Composition $f \circ g$

$$f : R \Rightarrow S, g : S \Rightarrow T$$

$$f \circ g : R \Rightarrow T$$

$$(f \circ g)(x) = g(f(x))$$

Classification of Mappings

- **Injective (one-one, 1-1)**

$$f(x) = f(y) \Rightarrow x = y \text{ (for all } x, y \in R)$$

- **Surjective (onto)**

for every $y \in S$ there is some $x \in R$ such that
 $f(x) = y$

- **Identity function**

$$f : R \Rightarrow R$$

$$f(x) = x \text{ (for all } x \in R)$$

- **Inverse function**

$f : R \Rightarrow S$ injective and surjective

$$g : S \Rightarrow R, g(y) = x \Leftrightarrow f(x) = y$$

$$g = f^{-1}$$

Isomorphism

Relationship between sets defined by functions

R and S are isomorphic if there is a pair of functions

$$f : R \Rightarrow S$$

$$g : S \Rightarrow R$$

$$f \circ g \text{ is identity on } R$$

$$g \circ f \text{ is identity on } S$$

f and g are then called *isomorphisms*.

Examples

- $R = \{1, 4, 7\}$ is isomorphic to $S = \{2, 4, 6\}$
- $A \times B$ is isomorphic to $B \times A$
- \mathbf{N} is isomorphic to \mathbf{Z}

Functions as Sets

Every function $f : R \Rightarrow S$ can be represented by its *graph*:

$$\begin{aligned} \text{graph}(f) &= \{(x, f(x)) \mid x \in R\} \\ &\subseteq R \times S \end{aligned}$$

Successor function on \mathbf{Z}

$$\{\dots, (-2, -1), (-1, 0), (0, 1), (1, 2), \dots\}$$

- Function application

$$\begin{aligned} f(a) = b &\Leftrightarrow (a, b) \in \text{graph}(f) \\ f(a) &:= \text{apply}(\text{graph}(f), a) \end{aligned}$$

- Function composition

$$\begin{aligned} \text{graph}(f \circ g) &= \\ &\{(x, z) \mid x \in R \text{ and, for some } y \in S, \\ &\quad (x, y) \in \text{graph}(f) \text{ and} \\ &\quad (y, z) \in \text{graph}(g)\} \end{aligned}$$

Examples

- $add : (\mathbf{N} \times \mathbf{N}) \Rightarrow \mathbf{N}$

$$\{((0, 0), 0), ((1, 0), 1), ((0, 1), 1), \\ ((1, 1), 2), ((2, 0), 2), ((2, 1), 3), \\ ((2, 2), 4), \dots\}$$

- $duplicate : R \Rightarrow R \times R, = \{1, 4, 7\}$

$$\{(1, (1, 1)), (4, (4, 4)), (7, (7, 7))\}$$

- $which : (\mathbf{B} + \mathbf{N}) \Rightarrow \{\text{isbool}, \text{isnum}\}$

$$\{((\text{zero}, \text{true}), \text{isbool}), \\ ((\text{zero}, \text{false}), \text{isbool}), \\ ((\text{one}, 0), \text{isnum}), ((\text{one}, 1), \text{isnum}), \\ ((\text{one}, 2), \text{isnum}), \dots\}$$

- $singleton : \mathbf{N} \Rightarrow \mathbf{P}(\mathbf{N})$

$$\{(0, \{0\}), (1, \{1\}), \dots, (n, \{n\}), \dots\}$$

- $\text{nothing} : \mathbf{B} \cap \mathbf{N} \Rightarrow \mathbf{B}$

$$\{\}$$

Examples

- *split-add* : $\mathbf{N} \Rightarrow (\mathbf{N} \Rightarrow \mathbf{N})$

$$\{ (0, \underline{\{(0, 0), (1, 1), (2, 2), \dots\}}), \\ (1, \underline{\{(0, 1), (1, 2), (2, 3), \dots\}}), \\ (2, \underline{\{(0, 2), (1, 3), (2, 4), \dots\}}), \\ \dots, \\ (n, \underline{\{(0, n), (1, n + 1), (2, n + 2), \dots\}}), \\ \dots \}$$

- *make-succ* : $(\mathbf{N} \Rightarrow \mathbf{N}) \Rightarrow (\mathbf{N} \Rightarrow \mathbf{N})$

$$\{ \dots, \\ (\underline{\{(0, 1), (1, 1), (2, 1), (3, 6), \dots\}}), \\ (\underline{\{(0, 2), (1, 2), (2, 2), (3, 7), \dots\}}), \dots \}$$

- *apply* : $((\mathbf{N} \Rightarrow \mathbf{N}) \times \mathbf{N}) \Rightarrow \mathbf{N}$

$$\{ \dots, \\ ((\underline{\{(0, 1), (1, 1), (2, 1), (3, 6), \dots\}}, 0), 1), \\ ((\underline{\{(0, 1), (1, 1), (2, 1), (3, 6), \dots\}}, 1), 1), \\ ((\underline{\{(0, 1), (1, 1), (2, 1), (3, 6), \dots\}}, 2), 1), \\ ((\underline{\{(0, 1), (1, 1), (2, 1), (3, 6), \dots\}}, 3), 6), \\ \dots \}$$

Functions as Equations

More convenient form of specification

- $add : (\mathbf{N} \times \mathbf{N}) \Rightarrow \mathbf{N}$

$$add(m, n) = m + n$$

- $duplicate : R \Rightarrow R \times R, = \{1, 4, 7\}$

$$duplicate(r) = (r, r)$$

- $which : (\mathbf{B} + \mathbf{N}) \Rightarrow \{\text{isbool}, \text{isnum}\}$

$$which(m) = \text{cases } m \text{ of}$$

$$\text{isB}(b) \Rightarrow \text{isbool}$$

$$\text{isN}(n) \Rightarrow \text{isnum}$$

end

- $singleton : \mathbf{N} \Rightarrow \mathbf{P}(\mathbf{N})$

$$singleton(n) = \{n\}$$

- $\text{nothing} : \mathbf{B} \cap \mathbf{N} \Rightarrow \mathbf{B}$

no equational definition (domain empty)!

Equations are just function representations!

Evaluation of Equations

- Definition $f : A \Rightarrow B, f(x) = \alpha$
- Application $f(a_0)$
 1. Substitution $[a_0/x]\alpha$
 2. Simplification to underlying value

Lambda Notation $f = \lambda x.\alpha$

$$\text{split-add}(x) = \lambda y.x + y$$

$$\text{slit-add} = \lambda x.\lambda y.x + y$$

Updating Functions $[a_0 \mapsto b_0]f$

$$([a_0 \mapsto b_0]f)(a_0) = b_0$$

$$([a_0 \mapsto b_0]f)(a) = f(a), \text{ for all } a \neq a_0$$

Semantic Domains

Those sets that are used as value spaces in denotational semantics.

- Primitive domains

N, Z, B, ...

- Compound domains

- Product domains $A \times B$

- Sum domains $A + B$

- Function domains $A \Rightarrow B$

- Lifted domains $A_{\perp} = A \cup \{\perp\}$

- * \perp = “bottom”

- * Non-termination, no value at all

- * Strict functions $f : A_{\perp} \Rightarrow B_{\perp}, f = \underline{\lambda}x.\alpha$

$$f(\perp) = \perp$$

$$f(a) = [a/x]\alpha, \text{ for } a \in A$$

Semantic Algebras

Format for presenting semantic domains

Rational Numbers

Domain $\text{Rat} = (\mathbf{Z} \times \mathbf{Z})_{\perp}$

Operations

$\text{makerat} : \mathbf{Z} \Rightarrow (\mathbf{Z} \Rightarrow \text{Rat})$

$\text{makerat} = \lambda p. \lambda q. (q = 0) \Rightarrow \perp \square (p, q)$

$\text{addrat} : \text{Rat} \Rightarrow (\text{Rat} \Rightarrow \text{Rat})$

$\text{addrat} = \underline{\lambda}(p_1, q_1). \underline{\lambda}(p_2, q_2). ((p_1 * q_2) + (p_2 * q_1), q_1 * q_2)$

$\text{multrat} : \text{Rat} \Rightarrow (\text{Rat} \Rightarrow \text{Rat})$

$\text{multrat} = \underline{\lambda}(p_1, q_1). \underline{\lambda}(p_2, q_2). (p_1 * p_2, q_1 * q_2)$

Choice function $e_1 \Rightarrow e_2 \square e_3$

$\Rightarrow e_2$, if $e_1 = \text{true}$

$\Rightarrow e_3$, if $e_1 = \text{false}$