

Domain Theory I

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Semantics of Loops

B: Boolean-expression

C: Command

...

$C ::= \dots \mid \mathbf{while\ B\ do\ C} \mid \dots$

...

$\mathbf{C}[\mathbf{while\ B\ do\ C}] =$
 $\lambda s. \mathbf{B}[[B]]s \rightarrow \mathbf{C}[\mathbf{while\ B\ do\ C}](\mathbf{C}[[C]]s) [] s$

Problem: meaning of a syntax phrase may be defined only in terms of its proper subparts.

$\mathbf{C}[\mathbf{while\ B\ do\ C}] = w$
 where $w: Store_{\perp} \rightarrow Store_{\perp}$
 $w = \lambda s. \mathbf{B}[[B]]s \rightarrow w(\mathbf{C}[[C]]s) [] s$

Recursion in syntax exchanged for recursion in function notation!

Recursive Function Definitions

- Function Definition

$$q = \lambda n.n \text{ equals zero} \rightarrow \text{one} \sqcup q(n \text{ plus one})$$

- Possible Function Graphs:

- $\{(zero, one)\}$
- $\{(zero, one), (one, \perp), (two, \perp), \dots\}$
- $\{(zero, one), (one, six), (two, six), \dots\}$
- $\{(zero, one), (one, k), (two, k), \dots\}$

Several functions satisfy specification; which one shall we choose?

Least Fixed Point Semantics

Theory that establishes meaning of recursive specifications:

1. Guarantees that every specification has a function satisfying it.
2. Provides means for choosing the “best” function out of the set of possibilities.
3. Ensures that the selected function corresponds to the conventional operational treatment of recursion.

Argument is mapped to defined answer iff simplification of the specification yields a result in a finite number of recursive invocations.

The Factorial Function

$\text{fac}(n) = n$ equals zero \rightarrow one
[] n times ($\text{fac}(n$ minus one))

Only one function satisfies specification:

$\text{graph}(\text{factorial}) =$
 $\{(\text{zero}, \text{one}), (\text{one}, \text{one}),$
 $(\text{two}, \text{two}), (\text{three}, \text{six}),$
 $\dots, (i, i!), \dots\}$

Simplification

fac(three)

→ three equals zero

→ one [] three times fac(three minus one)

= three times fac(three minus one)

= three times fac(two)

→ three times (two equals zero

→ one [] two times fac(two minus one))

= three times (two times fac(one))

→ three times (two times (one equals zero

→ one [] one times fac(one minus one)))

= three times (two times one (times fac(zero)))

→ three times (two times (one times (zero equals zero

→ one [] zero times fac(zero minus one))))

= three times (two times one (times one))

= six

Partial Functions

- Answer is produced in a *finite* number of unfolding steps.
- Idea: place limit on number of unfoldings and investigate resulting graphs
 - zero: $\{\}$
 - one: $\{(\text{zero}, \text{one})\}$
 - two: $\{(\text{zero}, \text{one}), (\text{one}, \text{one})\}$
 - $i + 1$: $\{(\text{zero}, \text{one}), (\text{one}, \text{one}), \dots (i, i!)\}$
- Graph at stage i defines function fac_i .
 - Consistency with each other:

$$\text{graph}(\text{fac}_i) \subseteq \text{graph}(\text{fac}_{i+1})$$
 - Consistency with ultimate solution:

$$\text{graph}(\text{fac}_i) \subseteq \text{graph}(\text{factorial})$$
 - Consequently

$$\bigcup_{i=0}^{\infty} \text{graph}(\text{fac}_i) \subseteq \text{graph}(\text{factorial})$$

Partial Functions

- Any result is computed in a finite number of unfoldings.

$$(a, b) \in \text{graph}(\text{factorial}) \\ \rightarrow (a, b) \in \text{graph}(\text{fac}_i) \text{ (for some } i)$$

- Consequently

$$\text{graph}(\text{factorial}) \subseteq \bigcup_{i=0}^{\infty} \text{graph}(\text{fac}_i)$$

- Thus

$$\text{graph}(\text{factorial}) = \bigcup_{i=0}^{\infty} \text{graph}(\text{fac}_i)$$

Factorial function can be totally understood in terms of the finite subfunctions fac_i !

Partial Functions

- Representations of sub-functions

$$\text{fac}_0 = \lambda n. \perp$$

$$\text{fac}_{i+1} = \lambda n. n \text{ equals zero} \rightarrow \text{one}$$

$$\quad \square n \text{ times } \text{fac}_i(n \text{ minus one})$$

- Each definition is *non-recursive*

Recursive specification can be understood in terms of a family of non-recursive ones.

- Common format can be extracted

“Functional” F

$$F: (\text{Nat} \rightarrow \text{Nat}_\perp) \rightarrow (\text{Nat} \rightarrow \text{Nat}_\perp)$$

$$F = \lambda f. \lambda n. n \text{ equals zero} \rightarrow \text{one}$$

$$\quad \square n \text{ times } f(n \text{ minus one})$$

Each subfunction is an instance of the functional!

Functional and Fixed Point

- Partial functions

$$\text{fac}_{i+1} = F(\text{fac}_i) = F^i(\perp)$$

$$\perp := (\lambda n. \perp)$$

- Function graph

$$\text{graph}(\text{factorial}) = \bigcup_{i=0}^{\infty} \text{graph}(F^i(\perp))$$

- Fixed point property

$$\text{graph}(F(\text{factorial})) = \text{graph}(\text{factorial})$$

$$F(\text{factorial}) = \text{factorial}$$

The function factorial is a fixed point of the functional F !

q Function

$$Q = \lambda q. \lambda n. n \text{ equals zero} \\ \rightarrow \text{one} \sqcap q(n \text{ plus one})$$

$$Q^0(\perp) = (\lambda n. \perp) \\ \text{graph}(Q^0(\perp)) = \{\}$$

$$Q^1(\perp) = \lambda n. n \text{ equals zero} \\ \rightarrow \text{one} \sqcap (\lambda n. \perp)(n \text{ plus one}) \\ = \lambda n. n \text{ equals zero} \rightarrow \text{one} \sqcap \perp \\ \text{graph}(Q^1(\perp)) = \{(\text{zero}, \text{one})\}$$

$$Q^2(\perp) = Q(Q^1(\perp)) = \lambda n. n \text{ equals zero} \\ \rightarrow \text{one} \sqcap ((n \text{ plus one}) \text{ equals zero} \rightarrow \text{one} \sqcap \perp) \\ \text{graph}(Q^2(\perp)) = \{(\text{zero}, \text{one})\}$$

q Function

- Convergence has occurred

$$\text{graph}(Q^i(\perp)) = \{(\text{zero}, \text{one})\}, i \geq 1$$

- Resulting graph

$$\bigcup_{i=0}^{\infty} \text{graph}(Q^i(\perp)) = \{(\text{zero}, \text{one})\}$$

- Fix point property

$$Q(\text{qlimit}) = \text{qlimit}$$

- Still many solutions possible

$$\begin{aligned} \text{graph}(q_k) = \\ \{ (\text{zero}, \text{one}), (\text{one}, k), \dots, (i, k), \dots \} \end{aligned}$$

- Least fixed point property

$$\text{graph}(\text{qlimit}) \subseteq \text{graph}(q_k)$$

The function qlimit is the least fixed point of the functional Q!

Recursive Specifications

The meaning of a recursive specification $f = F(f)$ is taken to be $\text{fix}(F)$, the least fixed point of the functional denoted by F .

$$\text{graph}(\text{fix } F) = \bigcup_{i=0}^{\infty} \text{graph}(F^i(\perp))$$

- The domain D of F must be a pointed cpo
 - partial ordering on D ,
 - every chain in D has a least upper bound in D ,
 - D has a least element.
- F must be continuous
 - Preserves limits of chains.
- Semantic domains are cpos and their operations are continuous.
 - Pointed cpos are created from primitive domains and union domains by lifting.

Factorial Function

$$F = \lambda f. \lambda n. n \text{ equals zero} \rightarrow \text{one} \\ \quad [] n \text{ times } (f(n \text{ minus one}))$$

Simplification rule

$$\text{fix } F = F(\text{fix } F)$$

$$\begin{aligned} & (\text{fix } F)(\text{three}) \\ &= (F (\text{fix } F))(\text{three}) \\ &= (\lambda f. \lambda n. n \text{ equals zero} \rightarrow \text{one} \\ & \quad [] n \text{ times } (f(n \text{ minus one})))(\text{fix } F)(\text{three}) \\ &= (\lambda n. n \text{ equals zero} \rightarrow \text{one} \\ & \quad [] n \text{ times } (\text{fix } F)(n \text{ minus one}))(\text{three}) \\ &= \text{three equals zero} \rightarrow \text{one} \\ & \quad [] \text{three times } (\text{fix } F)(\text{three minus one}) \\ &= \text{three times } (\text{fix } F)(\text{two}) \\ &= \text{three times } (F (\text{fix } F))(\text{two}) \\ &= \dots \\ &= \text{three times } (\text{two times } (\text{fix } F)(\text{two})) \end{aligned}$$

Fixed point property justifies rec. unfolding!

Double Recursion

$g = \lambda n. n \text{ equals zero} \rightarrow \text{one}$

$[\] (g(n \text{ minus one}) \text{ plus } g(n \text{ minus one})) \text{ minus one}$

$\text{graph}(F^0(\perp)) = \{\}$

$\text{graph}(F^1(\perp)) = \{(\text{zero}, \text{one})\}$

$\text{graph}(F^2(\perp)) = \{(\text{zero}, \text{one}), (\text{one}, \text{one})\}$

$\text{graph}(F^3(\perp)) = \{(\text{zero}, \text{one}), (\text{one}, \text{one}), (\text{two}, \text{one})\}$

...

$\text{graph}(F^{i+1}(\perp)) = \{(\text{zero}, \text{one}), \dots, (i, \text{one})\}$

$\text{fix } F = \lambda n. \text{one}$

Stepwise construction of graph yields insight!

Simultaneous Definitions

$f, g: \text{Nat} \rightarrow \text{Nat}_\perp$

$f = \lambda x. x \text{ equals zero} \rightarrow g(\text{zero})$

$\square f(g(x \text{ minus one})) \text{ plus two}$

$g = \lambda y. y \text{ equals zero} \rightarrow \text{zero}$

$\square y \text{ times } f(y \text{ minus one})$

$T = \text{Nat} \rightarrow \text{Nat}_\perp$

$F: (T \times T) \rightarrow (T \times T)$

$F = \lambda(f,g).(\dots, \dots)$

$F^0(\perp) = (\{\}, \{\})$

$F^1(\perp) = (\{\}, \{(\text{zero}, \text{zero})\})$

$F^2(\perp) = (\{(\text{zero}, \text{zero})\}, \{(\text{zero}, \text{zero})\})$

...

$F^5(\perp) = (\{(\text{zero}, \text{zero}), (\text{one}, \text{two}), (\text{two}, \text{two})\},$

$\{(\text{zero}, \text{zero}), (\text{one}, \text{zero}),$

$(\text{two}, \text{four}), (\text{three}, \text{six})\})$

$F^i(\perp) = F^5(\perp), i > 5$

$\text{fix}(F) = (f,g)$

While Loops

$$\mathbf{C}[\text{while } B \text{ do } C] = \text{fix}(\lambda f. \lambda s. \mathbf{B}[[B]]s \rightarrow f(\mathbf{C}[[C]]s) [] s)$$

Function: $\text{Store}_{\perp} \rightarrow \text{Store}_{\perp}$

Example:

$$\begin{aligned} \mathbf{C}[\text{while } A > 0 \text{ do } (A := A - 1; B := B + 1)] \\ = \text{fix } F \text{ where} \\ F = \lambda f. \lambda s. \text{test } s \rightarrow f(\text{adjust } s) [] s \\ \text{test} = \mathbf{B}[[A > 0]] \\ \text{adjust} = \mathbf{C}[[A := A - 1; B := B + 1]] \end{aligned}$$

Partial function graphs:

- Each pair in graph shows store prior to loop entry and after loop exit.
- Each graph $F^{i+1}(\perp)$ contains those pairs whose input stores finish processing in at most i iterations.

Example

$$\begin{aligned}
 \text{graph}(F^0(\perp)) &= \{\} \\
 \text{graph}(F^1(\perp)) &= \{ \\
 &\quad (\{ ([[A]], \text{zero}), ([[B]], \text{zero}), \dots \}, \\
 &\quad \{ ([[A]], \text{zero}), ([[B]], \text{zero}), \dots \}), \\
 &\quad \dots \\
 &\quad (\{ ([[A]], \text{zero}), ([[B]], \text{four}), \dots \}, \\
 &\quad \{ ([[A]], \text{zero}), ([[B]], \text{four}), \dots \}), \dots \} \\
 \text{graph}(F^2(\perp)) &= \{ \\
 &\quad (\{ ([[A]], \text{zero}), ([[B]], \text{zero}), \dots \}, \\
 &\quad \{ ([[A]], \text{zero}), ([[B]], \text{zero}), \dots \}), \\
 &\quad \dots \\
 &\quad (\{ ([[A]], \text{zero}), ([[B]], \text{four}), \dots \}, \\
 &\quad \{ ([[A]], \text{zero}), ([[B]], \text{four}), \dots \}), \\
 &\quad \dots \\
 &\quad (\{ ([[A]], \text{one}), ([[B]], \text{zero}), \dots \}, \\
 &\quad \{ ([[A]], \text{zero}), ([[B]], \text{one}), \dots \}), \\
 &\quad \dots \\
 &\quad (\{ ([[A]], \text{one}), ([[B]], \text{four}), \dots \}, \\
 &\quad \{ ([[A]], \text{zero}), ([[B]], \text{five}), \dots \}), \dots \}
 \end{aligned}$$

While Loops

Representation by finite subfunctions

$$\begin{aligned}
 \mathbf{C}[\mathbf{while\ B\ do\ C}] &= \sqcup\{ \\
 &\quad \lambda s. \perp, \\
 &\quad \lambda s. \mathbf{B}[\mathbf{B}]s \rightarrow \perp \quad \square \quad s, \\
 &\quad \lambda s. \mathbf{B}[\mathbf{B}]s \rightarrow (\mathbf{B}[\mathbf{B}](\mathbf{C}[\mathbf{C}]s) \rightarrow \perp \quad \square \quad \mathbf{C}[\mathbf{C}]s) \\
 &\quad \quad \square \quad s, \\
 &\quad \lambda s. \mathbf{B}[\mathbf{B}]s \rightarrow (\mathbf{B}[\mathbf{B}](\mathbf{C}[\mathbf{C}]s) \rightarrow \\
 &\quad \quad (\mathbf{B}[\mathbf{B}](\mathbf{C}[\mathbf{C}](\mathbf{C}[\mathbf{C}]s)) \rightarrow \perp \\
 &\quad \quad \square \quad \mathbf{C}[\mathbf{C}](\mathbf{C}[\mathbf{C}]s)) \quad \square \quad \mathbf{C}[\mathbf{C}]s \quad \square \quad s, \dots \} \\
 &= \sqcup\{ \\
 &\quad \mathbf{C}[\mathbf{diverge}], \\
 &\quad \mathbf{C}[\mathbf{if\ B\ then\ diverge\ else\ skip}], \\
 &\quad \mathbf{C}[\mathbf{if\ B\ then\ (C;\ if\ B\ then\ diverge\ else\ skip)} \\
 &\quad \quad \mathbf{else\ skip}], \\
 &\quad \mathbf{C}[\mathbf{if\ B\ then\ (C;\ if\ B\ then} \\
 &\quad \quad (\mathbf{C;\ if\ B\ then\ diverge\ else\ skip)} \\
 &\quad \quad \mathbf{else\ skip)\ else\ skip}], \dots \}
 \end{aligned}$$

Loop iteration can be understood by sequence of non-iterating programs.

Reasoning about Least Fixed Points

- Fixed Point Induction Principle:

To prove $P(\text{fix } F)$, it suffices to prove

1. $P(\perp)$
2. $P(d) \rightarrow P(F(d))$, for arbitrary $d \in D$

for pointed cpo D , continuous functional $F : D \rightarrow D$, and inclusive predicate $P : D \rightarrow \mathbf{B}$.

- Inclusiveness of predicates

If predicate holds for every element of chain, it also holds for its least upper bound.

- All universally quantified combinations of conjunctions/disjunctions that use only \sqsubseteq over functional expressions are inclusive.

Mainly useful for showing equivalences of program constructs.

Reasoning about Least Fixed Points

$$\mathbf{C}[\text{repeat C until B}] = \text{fix}(\lambda f. \lambda s. \\ \text{let } s' = \mathbf{C}[[\mathbf{C}]]s \text{ in } \mathbf{B}[[\mathbf{B}]]s' \rightarrow s' [] (f s'))$$

$$\mathbf{C}[[\mathbf{C}; \text{while } \neg B \text{ do C}]] \stackrel{?}{=} \mathbf{C}[\text{repeat C until B}]$$

Proof: $P(f, g) = \forall s. f(\mathbf{C}[[\mathbf{C}]]s) = (g s)$

1. $P(\perp, \perp)$ holds obviously.

2. Prove $P(F(f), G(g))$

$$F = (\lambda f. \lambda s. \mathbf{B}[[\neg B]]s \rightarrow f(\mathbf{C}[[\mathbf{C}]]s) [] s) \\ G = (\lambda f. \lambda s. \text{let } s' = \mathbf{C}[[\mathbf{C}]]s \text{ in } \mathbf{B}[[\mathbf{B}]]s' \\ \rightarrow s' [] (f s'))$$

(a) $F(f)(\mathbf{C}[[\mathbf{C}]]\perp) = \perp = G(g)(\perp)$.

(b) $s \neq \perp$:

- i. $\mathbf{C}[[\mathbf{C}]]s = \perp$: $F(f)(\perp) = \perp = (\text{let } s' = \perp \text{ in } \mathbf{B}[[\mathbf{B}]]s' \rightarrow s' [] (g s')) = G(g)(\perp)$.
- ii. $\mathbf{C}[[\mathbf{C}]]s = s_0 \neq \perp$: $F(f)(s_0) = \mathbf{B}[[\neg B]]s_0 \rightarrow f(\mathbf{C}[[\mathbf{C}]]s_0) [] s_0 = \mathbf{B}[[\mathbf{B}]]s_0 \rightarrow s_0 [] f(\mathbf{C}[[\mathbf{C}]]s_0) = \mathbf{B}[[\mathbf{B}]]s_0 \rightarrow s_0 [] f(g s_0) = G(g)(s)$