

Problem 1. Simplify the following  $\lambda$ -expression.

$$a. \left( \left( \left( \lambda x. \lambda y. \lambda z. xz(yz) \right) (\lambda x. \lambda y. x) \right) (\lambda x. \lambda y. x) \right) (\lambda x. \lambda y. x)$$

$\quad \quad \quad =K1 \quad \quad \quad =K2 \quad \quad \quad =K3$

$$\begin{aligned} &\Rightarrow \left( \left( \left( \lambda x. \lambda y. \lambda z. xz(yz) \right) K1 \right) K2 \right) K3 \\ &\Rightarrow \left( \left( \left( \lambda x. \lambda y. \lambda z. xz(yz) \right) K1 \right) K2 \right) K3 \\ &\Rightarrow \left( \left( \left( \lambda x. \lambda y1. \lambda z. xz(yz) \right) K1 \right) K2 \right) K3 \\ &\Rightarrow \left( \left( \left[ K1 / x \right] \lambda y. \lambda z. xz(yz) \right) K2 \right) K3 \\ &\Rightarrow \left( \left( \lambda y. \lambda z. K1 z(yz) \right) K2 \right) K3 \\ &\Rightarrow \left( \left( \lambda y. \lambda z. K1 z(yz) \right) K2 \right) K3 \\ &\Rightarrow \left( \left[ K2 / y \right] \lambda z. K1 z(yz) \right) K3 \\ &\Rightarrow \left( \lambda z. K1 z(K2 z) K3 \right) \\ &\Rightarrow \left( \lambda z. K1 z(K2 z) K3 \right) \\ &\Rightarrow \left( \left[ K3 / z \right] K1 z(K2 z) \right) \\ &\Rightarrow \left( K1 K3 (K2 K3) \right) \\ &\Rightarrow \left( \lambda x. \lambda y. x \right) K3 (K2 K3) \\ &\Rightarrow K3 \\ &\Rightarrow (\lambda x. \lambda y. x) \end{aligned}$$

$$b. (\lambda x.(xy)) (\lambda z.z)$$

$$\begin{aligned} &\Rightarrow (\lambda x.(x y)) (\lambda z.z) \\ &\Rightarrow \left[ (\lambda z.z) / x \right] (x y) \\ &\Rightarrow ((\lambda z.z) y) \\ &\Rightarrow [z/y] z \\ &\Rightarrow y \end{aligned}$$

$$c. \left( (\lambda f. \left( (\lambda g. ((f f)g)) (\lambda h.(k h)) \right) \right) (\lambda x. (\lambda y.y)) \right)$$

$$\begin{aligned} &\Rightarrow \left( \left[ (\lambda x. (\lambda y.y)) / f \right] \left( (\lambda g. ((f f)g)) (\lambda h.(k h)) \right) \right) \\ &\Rightarrow \left( (\lambda g. (((\lambda x. (\lambda y.y)) (\lambda x. (\lambda y.y))) g)) (\lambda h.(k h)) \right) \\ &\Rightarrow \left( \left[ (\lambda h.(k h)) / \lambda g \right] (((\lambda x. (\lambda y.y)) (\lambda x. (\lambda y.y))) g) \right) \\ &\Rightarrow \left( \left( (\lambda x. (\lambda y.y)) (\lambda x. (\lambda y.y)) \right) (\lambda h.(k h)) \right) \\ &\Rightarrow \left( \left( (\lambda x1. (\lambda y1.y1)) (\lambda x. (\lambda y.y)) \right) (\lambda h.(k h)) \right) \\ &\Rightarrow \left( \left[ (\lambda x. (\lambda y.y)) / x1 \right] (\lambda y1.y1) \right) (\lambda h.(k h)) \\ &\Rightarrow \left( (\lambda y1.y1) (\lambda h.(k h)) \right) \\ &\Rightarrow \left( \left[ (\lambda h.(k h)) / y1 \right] y1 \right) \\ &\Rightarrow (\lambda h.(k h)) \end{aligned}$$

d.  $(\lambda g.((\lambda f.((\lambda x.(f(x x)))(\lambda x.(f(x x))))))g))$

$$\begin{aligned} &\Rightarrow (\lambda g.(\lambda f.(\lambda x.(f(x x)))(\lambda x.(f(x x)))) g) \\ &\Rightarrow (\lambda g.(\lambda f.(\lambda x.(f(x x)))(\lambda x.(f(x x)))) g) \end{aligned}$$

$$\begin{aligned} &\Rightarrow (\lambda g.(\lambda f.[g/f]\lambda x.(\lambda x.(g(x x)))(\lambda x.(g(x x)))) g) \\ &\Rightarrow (\lambda g.(\lambda x.(\lambda x.(g(x x)))(\lambda x.(g(x x)))) g) \\ &\Rightarrow (\lambda g.(\lambda x.(\lambda x.(g(x x)))/x]g(x x)) \\ &\Rightarrow (\lambda g.(g(\lambda x.(g(x x)))(\lambda x.(g(x x)))) g) \end{aligned}$$

$$\begin{aligned} &\Rightarrow (\lambda g.(g(\lambda x1.(g(x1 x1)))(\lambda x.(g(x x)))) g) \\ &\Rightarrow (\lambda g.(g([\lambda x.(g(x x)) / x1](g(x1 x1)))) g) \\ &\Rightarrow (\lambda g.(g(g(\lambda x.(g(x x)))(\lambda x.(g(x x)))) g) g) \\ &\Rightarrow (\lambda g.(g(g(\lambda x2.(g(x2 x2)))(\lambda x.(g(x x)))) g) g) \\ &\Rightarrow (\lambda g.(g(g([\lambda x.(g(x x)) / x2]\lambda x2.(g(x2 x2)))) g) g) \\ &\Rightarrow (\lambda g.(g(g(g(\lambda x.(g(x x)))(\lambda x.(g(x x)))) g) g) g) \end{aligned}$$

and so on, not terminating.

e. If  $S=(((\lambda x. \lambda y. \lambda z. xy(yz)))$

$$K=(\lambda x. \lambda y.x)$$

$$I=(\lambda x.x)$$

Since (A)  $SKK=I$ , (B)  $SKKK=K$

$$(a) I(SKK)(SKKK)=I(I)K = I K = K$$

$$(b) (SKK)IK=I I K = K$$

$$\begin{aligned} SKK &= ((\lambda x. \lambda y. \lambda z. xy(yz)) (\lambda x. \lambda y. x)) (\lambda x. \lambda y. x) \\ &\Rightarrow ((\lambda x. \lambda y. \lambda z. xy(yz)) (\lambda x1. \lambda y1.x1)) (\lambda x. \lambda y.x) \\ &\Rightarrow (([\lambda x1. \lambda y1.x1] / x](\lambda x. \lambda y. \lambda z. xy(yz))) (\lambda x. \lambda y.x)) \\ &\Rightarrow (\lambda y. \lambda z. (\lambda x1. \lambda y1.x1) y(yz)) (\lambda x. \lambda y.x) \\ &\Rightarrow ([(\lambda x2. \lambda y2.x2) / y]\lambda z. (\lambda x. \lambda y1.x) y(yz)) \\ &\Rightarrow \lambda z. (\lambda x. \lambda y1.x) (\lambda x2. \lambda y2.x2) ((\lambda x2. \lambda y2.x2) z) \\ &\Rightarrow \lambda z. (\lambda x. \lambda y1.x) (\lambda x2. \lambda y2.x2) ([z/x2]\lambda y2.x2) \\ &\Rightarrow (\lambda z. (\lambda x. \lambda y1.x) (\lambda x2. \lambda y2.x2) (\lambda y2.z)) \\ &\Rightarrow (\lambda z. [(\lambda x2. \lambda y2.x2) / x]\lambda y1.x) \\ &\Rightarrow (\lambda z. \lambda y1. (\lambda x2. \lambda y2.x2)) \\ &\Rightarrow (\lambda x. \lambda y1.x) (\lambda x2. \lambda y2.x2) \\ &\Rightarrow (\lambda x2. \lambda y2.x2) \\ &\Rightarrow (\lambda x. x) \end{aligned}$$

$$SKKK = (((\lambda x. \lambda y. \lambda z. xy(yz)) (\lambda x. \lambda y.x)) (\lambda x. \lambda y.x)) (\lambda x. \lambda y.x))$$

$$SKKK = (SKK)K = IK = K$$