

Lecture #9: Complete Partial Orders

CS 6371: Advanced Programming Languages

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The denotational semantics of loops is part of a more general mathematical theory of complete partial orders and continuous functions. Some of the basics of that theory are presented below, culminating in the Fixed-Point Theorem. We use the Fixed-Point Theorem to prove that our denotational definition of `while` loops is a well-formed mathematical definition and constitutes the least fixed point of the functional Γ . We begin with important definitions.

Definition: A *partial order* (p.o.) is a set P on which there is a binary relation \sqsubseteq which is

- (i) reflexive: $\forall p \in P. p \sqsubseteq p$,
- (ii) transitive: $\forall p, q, r \in P. (p \sqsubseteq q) \wedge (q \sqsubseteq r) \implies (p \sqsubseteq r)$, and
- (iii) antisymmetric: $\forall p, q \in P. (p \sqsubseteq q) \wedge (q \sqsubseteq p) \implies (p = q)$.

Definition: A p.o. (P, \sqsubseteq) has a *bottom* element $\perp_P \in P$ iff $\perp_P \sqsubseteq p$ for all $p \in P$.

Observe that $(\Sigma \rightarrow \Sigma, \subseteq)$ is a partial order because the subset relation \subseteq is reflexive, transitive, and antisymmetric. The empty set $\{\}$ (i.e., the partial function that is undefined for all inputs) is a bottom element of this partial order because the empty set is a subset of every set.

Definition: For a partial order (P, \sqsubseteq) and a subset $X \subseteq P$, we say p is an *upper bound* of X iff $\forall q \in X. q \sqsubseteq p$.

Note that not every pair of partial functions $f, g : \Sigma \rightarrow \Sigma$ have an upper bound. For example, if $f(\sigma) \neq g(\sigma)$, then there is no function h such that $f \subseteq h$ and $g \subseteq h$. However, for any two partial functions such that $f \subseteq g$, then obviously g is an upper bound for f and g .

Definition: We say p is a *least upper bound* of X , written $p = \bigsqcup X$, if p is an upper bound of X and $p \sqsubseteq q$ for all upper bounds q of X . We also denote the least upper bound of two elements $p, q \in P$ as $p \sqcup q$.

In the above example, g is also a least upper bound for f and g because $g = f \cup g$.

Definition: An ω -chain of a partial order (P, \sqsubseteq) is an infinite sequence $p_0, p_1, \dots \in P$ such that $p_0 \sqsubseteq p_1 \sqsubseteq \dots$.

Recall that we proved in class that $\perp \subseteq \Gamma(\perp) \subseteq \Gamma^2(\perp) \subseteq \dots$ is a family of nested subsets. Therefore, $\perp, \Gamma(\perp), \Gamma^2(\perp), \dots$ is an ω -chain for $(\Sigma \rightarrow \Sigma, \subseteq)$.

Definition: A partial order (P, \sqsubseteq) is a *complete partial order* (cpo) iff every ω -chain $p_0, p_1, \dots \in P$ has a least upper bound $\bigsqcup_{i \geq 0} p_i \in P$.

Observe that $(\Sigma \rightarrow \Sigma, \subseteq)$ is a cpo because for every ω -chain, the infinite union of all partial functions in the chain is also a partial function in $\Sigma \rightarrow \Sigma$. That infinite union is a least upper bound of the chain. For example, $\bigcup_{i \geq 0} \Gamma^i(\perp)$ is a least upper bound for the chain $\perp, \Gamma(\perp), \Gamma^2(\perp), \dots \in \Sigma \rightarrow \Sigma$.

Definition: A function $f : P \rightarrow P$ is *monotonic* iff for all $p, q \in P$, $p \sqsubseteq q \implies f(p) \sqsubseteq f(q)$.

It is easy to prove that the functional Γ is monotonic.

Definition: A function $f : P \rightarrow P$ is *continuous* iff it is monotonic and for all ω -chains $p_0, p_1, \dots \in P$, we have

$$\bigsqcup_{i \geq 0} f(p_i) = f\left(\bigsqcup_{i \geq 0} p_i\right)$$

We can prove that Γ is continuous in the following way:

Proof. Let $p_0, p_1, p_2, \dots \in \Sigma \rightarrow \Sigma$ be an arbitrary ω -chain in cpo $(\Sigma \rightarrow \Sigma, \subseteq)$. The proof that Γ is continuous consists of two parts: First we prove that if $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$ then $(\sigma, \sigma') \in \Gamma(\bigcup_{i \geq 0} p_i)$. This proves that $\bigcup_{i \geq 0} \Gamma(p_i) \subseteq \Gamma(\bigcup_{i \geq 0} p_i)$. Next we prove that if $(\sigma, \sigma') \in \Gamma(\bigcup_{i \geq 0} p_i)$ then $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$. This proves that $\bigcup_{i \geq 0} \Gamma(p_i) \supseteq \Gamma(\bigcup_{i \geq 0} p_i)$. We conclude therefore that $\bigcup_{i \geq 0} \Gamma(p_i) = \Gamma(\bigcup_{i \geq 0} p_i)$.

Proof of \subseteq direction: Let $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$ be given. Thus, there exists $n \geq 0$ such that $(\sigma, \sigma') \in \Gamma(p_n)$. Since $p_n \subseteq \bigcup_{i \geq 0} p_i$, it follows from the monotonicity of Γ that $\Gamma(p_n) \subseteq \Gamma(\bigcup_{i \geq 0} p_i)$. Therefore $(\sigma, \sigma') \in \Gamma(\bigcup_{i \geq 0} p_i)$.

Proof of \supseteq direction: Now instead let $(\sigma, \sigma') \in \Gamma(\bigcup_{i \geq 0} p_i)$ be given. From the definition of Γ , we know there are two possible cases:

Case 1: If $\mathcal{B}[b](\sigma) = F$ then $\sigma' = \sigma$. Since $\{(\sigma, \sigma) \mid \mathcal{B}[b](\sigma) = F\}$ is a subset of $\Gamma(x)$ for every set x , it follows that $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$.

Case 2: If $\mathcal{B}[b](\sigma) = T$ then $\sigma' = (\bigcup_{i \geq 0} p_i)(\mathcal{C}[c](\sigma))$. Thus, $(\mathcal{C}[c](\sigma), \sigma') \in \bigcup_{i \geq 0} p_i$, so there exists $n \geq 0$ such that $(\mathcal{C}[c](\sigma), \sigma') \in p_n$. That is to say, $p_n(\mathcal{C}[c](\sigma)) = \sigma'$. Since $\mathcal{B}[b](\sigma) = T$, it follows from the definition of Γ that $(\sigma, \sigma') \in \Gamma(p_n)$. We conclude that $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$. \square

Definition: Let $f : P \rightarrow P$ be a continuous function on a cpo P . A *fixed point* of f is an element $p \in P$ such that $f(p) = p$.

Theorem (Fixed-Point Theorem): Let $f : P \rightarrow P$ be a continuous function on a cpo P with bottom \perp . Then $\bigsqcup_{i \geq 0} f^i(\perp)$ is a least fixed point of f .

From the fixed-point theorem we conclude that $\bigcup_{i \geq 0} \Gamma^i(\perp)$ is a least fixed point of Γ .