Introduction to the λ-calculus

CS6371: Advanced Programming Languages

First,
Some Mathematical History

Mathematics in Ancient Times

- Based on observational reasoning
  - Example: Does the sum of interior angles of a triangle always equal 180° ?
  - Draw a few triangles, measure the angles.
  - Seems to always work. "Proved."
- Unreliable, prone to error
  - Only works when future instances happen to coincide with prior sample cases
- Modern notion of "proof" did not exist

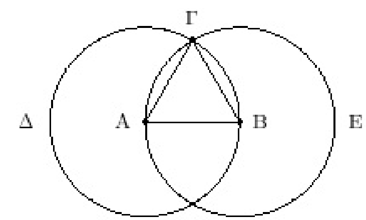
Deductive Logic

- Euclid's *The Elements*
  - written c. 300 B.C.
  - deductive reasoning: 23 definitions, 10 axioms
  - geometry, algebra, number theory
  - foundation of mathematics for about 2000 years

Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.

Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ.

Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι.



Κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ, καὶ πάλιν κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΑ κύκλος γεγράφθω ὁ ΑΓΕ, καὶ ἀπὸ τοῦ Γ σημείου, καθ᾽ ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπὶ τὰ Α, Β σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστιν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν.

Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον, καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς ΑΒ.

[Ἐπὶ τῆς δοθείσης ἄρα εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συνέσταται]· ὅπερ ἔδει ποιῆσαι.

- Problem: Some theorems unprovable from axioms
  - Example: Two circles with centers closer than the sum of their radii have an intersection point

Set Theory

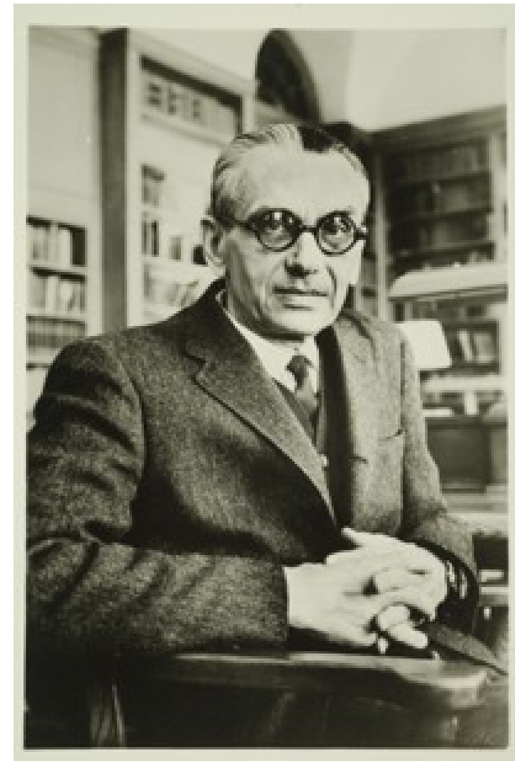
- First proposed by Georg Cantor in 1874
  - new foundation for mathematics
  - early version contained paradoxes
- Deductive Set Theory
  - axiomized by Zermelo and Fraenkel between 1908 and 1930
  - Zermelo-Fraenkel set theory with axiom of choice (ZFC)
- Problem: Some theorems still unprovable!
  - Example (Continuum Hypothesis): There is no set larger than $\mathbb{N}$ but smaller than $\mathbb{R}$

Hilbert's Program

- Proposed by David Hilbert in 1921
- Goals
  - Provide an unassailable foundation for all mathematics
  - Find a set of axioms and rules of logical inference sufficient to deductively prove all mathematical theorems
- Required properties
  - Soundness: No untrue statement provable
  - Completeness: All true statements provable
  - Decidability: Algorithm that can decide whether any mathematical statement is true or false

Gödel's Incompleteness Theorem

- Proved by Kurt Gödel in 1931
- Theorem: No finite collection of axioms is both sound and complete!
- Ramifications
  - Given any sound axiomization of mathematics, there are true statements that are unprovable.
  - There is no decision algorithm for mathematical truth.
- Essentially destroyed Hilbert's program



- Raised another question: What is decidable?

Theory of Computation

Alan Turing

- "Decide" = "Compute"
- 1936: Two models of "computation" proposed
  - Turing Machines
  - λ-calculus (Alonzo Church)
- Both models equivalent in power

Alonzo Church

- Church-Turing Thesis
  - All (reasonable) models of computation are equally powerful
- Birth of computer science
  - Turing Machines = imperative programming
  - λ-calculus = functional programming

Today: λ-calculus