Exercise 1. Consider the following recursively defined function \( f : \mathbb{Z} \rightarrow \mathbb{Z} \).

\[
f(x) = (x = 0 \rightarrow 0 \mid x > 0 \rightarrow 2 - f(1 - x) \mid x < 0 \rightarrow f(-x))
\]

Find a closed-form definition of \( f \) and prove your answer.

To find a closed-form definition (i.e., one that is non-recursive and does not use \( \text{fix} \)), it is often useful to define functional \( F \) and then construct the graph of the least fixed point of \( F \). Recall that functional \( F \) is defined by

\[
F(g) = \lambda x . (x = 0 \rightarrow 0 \mid x > 0 \rightarrow 2 - g(1 - x) \mid x < 0 \rightarrow g(-x))
\]

The graph of the least fixed point of \( F \) is the set of input-output pairs that comprises \( \text{fix}(F) \). We can construct it incrementally by applying \( F \) to itself starting with \( \perp \):

\[
\begin{align*}
F^0(\perp) &= \{\} \\
F^1(\perp) &= \{(0,0)\} \\
F^2(\perp) &= \{(0,0), (1,2)\} \\
F^3(\perp) &= \{(-1,2), (0,0), (1,2)\} \\
F^4(\perp) &= \{(-1,2), (0,0), (1,2), (2,0)\} \\
F^5(\perp) &= \{(-2,0), (-1,2), (0,0), (1,2), (2,0)\} \\
F^6(\perp) &= \{(-2,0), (-1,2), (0,0), (1,2), (2,0), (3,2)\} \\
F^7(\perp) &= \{(-3,2), (-2,0), (-1,2), (0,0), (1,2), (2,0), (3,2)\}
\end{align*}
\]

As you can see, eventually a pattern starts to emerge. Function \( f \) appears to return 2 on odd inputs and 0 on even inputs. Thus, we conjecture that \( f = h \) where \( h \) is the following closed-form definition:

\[
h(x) = \begin{cases} 
2 & \text{if } x \text{ is odd} \\
0 & \text{if } x \text{ is even}
\end{cases}
\]

This does not constitute a proof; it is merely a conjecture. We can prove the \( f \subseteq h \) half of the conjecture using fixed point induction.

Proof. Define property \( P \) by \( P(g) \equiv \forall x \in g^\rightarrow . g(x) = h(x) \). We wish to prove \( P(f) \). Define functional \( F \) as above, and observe that \( \text{fix}(F) = f \) by the definition of recursion. Thus, to prove \( P(f) \) it suffices to prove \( P(\text{fix}(F)) \) by fixed-point induction.
**Base Case:** \( P(\bot) \) holds vacuously.

**Inductive Hypothesis:** Assume that \( P(g) \) holds for some arbitrary function \( g \). That is, assume that \( \forall x \in g^{-} \cdot g(x) = h(x) \).

**Inductive Case:** We will prove that \( P(F(g)) \) holds. Let \( x \in F(g)^{-} \) be given. Looking at the definition of \( F \), there are three cases to consider:

**Case 1:** Suppose \( x = 0 \). Then by definition of \( F \), \( F(g)(x) = 0 = h(x) \).

**Case 2:** Suppose \( x > 0 \). Then by definition of \( F \), \( F(g)(x) = 2 - g(1 - x) \). By inductive hypothesis, \( g(1 - x) = 2 \) if \( 1 - x \) is odd and \( 0 \) if \( 1 - x \) is even. If \( x \) is odd then \( 1 - x \) is even, so \( g(1 - x) = 0 \); thus \( 2 - g(1 - x) = 2 = h(x) \). If \( x \) is even then \( 1 - x \) is odd, so \( g(1 - x) = 2 \); thus \( 2 - g(1 - x) = 0 = h(x) \). Either way, \( F(g)(x) = 2 - g(1 - x) = h(x) \).

**Case 3:** Suppose \( x < 0 \). Then by definition of \( F \), \( F(g)(x) = g(-x) \). By inductive hypothesis, \( g(-x) = 2 \) if \( -x \) is odd and \( 0 \) if \( -x \) is even. Since \(-x\) has the same parity as \( x \), it follows that \( F(g)(x) = 2 \) if \( x \) is odd and \( 0 \) if \( x \) is even. Hence, \( F(g)(x) = h(x) \).

Functions of multiple arguments can be treated as functions of a single pair argument.

**Exercise 2.** Consider the following recursively defined function \( f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \).

\[
f(x, y) = (x = 0 \to y) \ | \ y = 0 \to x \ | \ x, y > 0 \to f(x - 1, y - 1) + 1
\]

Prove that \( f \subseteq \text{max} \).

**Proof.** Define property \( P \) by \( P(g) \equiv \forall (x, y) \in g^{-} \cdot g(x, y) = \text{max}(x, y) \). We wish to prove \( P(f) \).

Define functional \( F \) in the usual way:

\[
F(g) = \lambda(x, y).\ (x = 0 \to y) \ | \ y = 0 \to x \ | \ x, y > 0 \to g(x - 1, y - 1) + 1
\]

To prove \( P(f) \) it suffices to prove \( P(\text{fix}(F)) \) by fixed-point induction.

**Base Case:** \( P(\bot) \) holds vacuously.

**Inductive Hypothesis:** Assume that \( P(g) \) holds for some arbitrary function \( g \). We will prove that \( P(F(g)) \) holds. Let \( (x, y) \in F(g)^{-} \) be given.

**Case 1:** Suppose \( x = 0 \). Then by definition of \( F \), \( F(g)(x, y) = y = \text{max}(x, y) \).

**Case 2:** Suppose \( y = 0 \). Then by definition of \( F \), \( F(g)(x, y) = x = \text{max}(x, y) \).

**Case 3:** Suppose \( x, y > 0 \). Then by definition of \( F \), \( F(g)(x, y) = g(x - 1, y - 1) + 1 \). By inductive hypothesis, \( F(g)(x) = \text{max}(x - 1, y - 1) + 1 \). If \( x \geq y \) then \( \text{max}(x - 1, y - 1) = x - 1 \), so \( F(g)(x, y) = x - 1 + 1 = x \). If \( x < y \) then \( \text{max}(x - 1, y - 1) = y - 1 \), so \( F(g)(x, y) = y - 1 + 1 = y \). In either case \( F(g)(x, y) = \text{max}(x, y) \).