1 Current Flow and Conductivity

1.1 Electron Gas Model

The first model of electron motion in a metal is known as the electron-gas model (also known a the Drude Model, circa 1900). The analogy was the motion of gas molecules in a container. The theory assumes that electrons are in continuous motion within the metal with their direction being changed at each collision with the heavy (almost stationary) ions. The average distance between collisions is called the *mean free path*. Since the motion is random, then on average there are as many electrons passing through some unit area in one direction as in the opposite direction in a given time. Hence the **average** current is zero. Note that the **net** flux (or flow) of particles through any surface of an open system in thermal equilibrium must be zero due to the principle of *Detailed Balance*.

If a constant electric field, E, is applied, electrostatic force due to the field accelerates the electrons. The electron velocities would increase indefinitely with time if it were not for collisions with ions. With each inelastic collision, the electron loses energy and a steady state condition is reached with a finite *drift speed*, v. The drift velocity is in the direction opposite to the electric field. The speed at time t between collisions is at where $a = q\mathsf{E}/m$ is the acceleration. Hence the average speed is proportional to the field. Thus $\mathsf{v} = \mu\mathsf{E}$, where μ is called the mobility. The steady-state drift speed is superimposed upon the random thermal motion, leading to directed flow of electrons which constitutes a dc current.

Suppose N electrons are contained in a length L of a metal conductor and it takes an electron a period of time T to travel the distance L, then the total number of electrons passing through any cross section of a uniform conductor in a unit time is N/T. Then the total charge per second flowing through any section of conductor is the definition of current;

$$I = \frac{Nq}{T} = \frac{Nq\mathsf{v}}{L} \ , \tag{1}$$

because L/T is the average, or drift, speed of the electrons. The definition of current density is current per unit area.

$$J = \frac{I}{A} = \frac{Nq\mathsf{v}}{LA} \ . \tag{2}$$

Note in equation (2) that LA is the volume containing N electrons, thus N/LA is the electron concentration n. Therefore equation (2) can be rewritten as

$$J = nq\mathsf{v} = \rho\mathsf{v} \ , \tag{3}$$

where $\rho = nq$ is the charge density.

1.2 Carrier Motion with Collisions

Electrons in a solid move constantly due to their thermal energy that is imparted to them from the ambient temperature. An individual electron in motion will randomly scatter from lattice atoms, impurities, defects and other electrons. There is no net motion of a group of n electrons/cm^3 over any period of time. If an electric field E is applied to the material, each electron will experience a net force. The magnitude of the component of the force in the x direction will be $-q\mathsf{E}_x$. This force may not appreciably alter the random path of an individual electron; however the effect when averaged over all electrons is a net motion of the group in the $-x$ direction. If p_x is the x component of the total momentum of the group, then the force due to the field on the n electrons/cm^3 by Newton's 2$^{\text{nd}}$ Law of motion ($\mathbf{F} = \dot{\mathbf{p}}$) leads to the following equation

$$-nq\mathsf{E}_x = \left.\frac{dp_x}{dt}\right|_{\text{field}} . \tag{4}$$

Equation (4) indicates a continuous acceleration of the electron in the $-x$ direction. In actuality this is not the case, because the net acceleration is just balanced in steady state by the collision processes on going in the material. Thus while the constant electric field E_x does produce a net momentum p_{-x}, the net rate of change of momentum when collisions are included must be zero in the case of steady state current flow. To find the total rate of momentum change from collisions, collision probabilities must be investigated.

If the collisions are truly random, there will be a constant probably of collision at any time for each electron. Consider a group of N_o electrons at time $t = 0$ and define $N(t)$ as the number of electrons which it have not undergone a collision by time t. The rate of decrease in $N(t)$ at any time t is proportional to the number left unscattered at t, this rate equation can be written down as

$$\frac{dN(t)}{dt} = \frac{1}{\bar{t}}\,N(t) , \tag{5}$$

where $\bar{t}^{-1}$ is a constant of proportionality. The solution to this first order differential equation is

$$N(t) = N_o\,\boldsymbol{e}^{-t/\bar{t}} . \tag{6}$$

From equation (6) it can be seen that $\bar{t}$ represents the mean time between scattering events, called the *mean free time*. The probability that any electron has a collision in the time interval dt is $dt/\bar{t}$. Thus the differential change in p_x due to collisions in time dt is

$$dp_x = -\,p_x\,dt/\bar{t} . \tag{7}$$

The rate of change of p_x due to the decelerating effect of collisions is

$$\left.\frac{dp_x}{dt}\right|_{\text{collision}} = -\frac{p_x}{\bar{t}} \ . \tag{8}$$

The sum of acceleration effects due to the field and deceleration effects due to collisions must be zero for steady state, therefore equating equations (4) and (8) yields

$$-\frac{p_x}{\bar{t}} - nq\mathsf{E}_x = 0 \ . \tag{9}$$

From equation (9), we can find the average momentum per electron as

$$\langle p_x \rangle = \frac{p_x}{n} = -q\bar{t}\mathsf{E}_x \ , \tag{10}$$

where $\langle\cdot\rangle$ bracket indicates an average over the entire group of electrons. As needed for steady state, the electrons have "on the average" a constant net velocity in the negative x direction. Therefore we can write

$$\langle \mathsf{v}_x \rangle = \frac{\langle p_x \rangle}{m_n^*} = -\frac{q\bar{t}}{m_n^*}\mathsf{E}_x \ , \tag{11}$$

where m_n^* is the effective mass the the electron in the conduction band of the material. Individual electrons move in many directions by thermal motion during a given time interval, but equation (11) gives the **net drift** of an average electron in response to the electric field. This drift speed is usually much smaller than the random speed due to thermal motion.

The current density resulting from this net drift is the number of electrons crossing a unit area per unit time multiplied by the charge on the electron.

$$J_x = -qn\langle \mathsf{v}_x \rangle \ . \tag{12}$$

After substituting equation (11) for the velocity into equation (12) we get

$$J_x = \frac{nq^2\bar{t}}{m_n^*}\mathsf{E}_x \ . \tag{13}$$

Thus the current density is proportional to the electric field, as expected from Ohm's Law.

$$J_x = \sigma\mathsf{E}_x \ , \quad \text{where } \sigma = \frac{nq^2\bar{t}}{m_n^*} \ . \tag{14}$$

The electrical conductivity σ can be define as

$$\sigma = qn\mu_n \ , \quad \text{where } \mu_n = \frac{q\bar{t}}{m_n^*} \ , \tag{15}$$

and μ_n is called the electron mobility. Remembering that: $J_x = \sigma \mathsf{E}_x = -qn\langle \mathsf{v}_x \rangle$ and rearranging equation (15) and substituting yields

$$\mu_n = \frac{\sigma}{qn} = -\frac{\langle \mathsf{v}_x \rangle}{\mathsf{E}_x} \ . \tag{16}$$

The mobility can be expressed as the average particle drift velocity per unit electric field; $J_x = qn\mu_n \mathsf{E}_x$. For holes in a semiconductor, the mobility is $\mu_p = +\langle \mathsf{v}_x \rangle / \mathsf{E}_x$. For semiconductors the total electron and hole current density can be written as

$$J_x = q(n\mu_n + p\mu_p)\mathsf{E}_x = \sigma \mathsf{E}_x \ . \tag{17}$$

1.3 Collision Time

Now we will look a little deeper into the physics to justify some of the previous sections. The following is summerized from Reif. Consider an electron with velocity $\mathbf{v}$. Let $P(t)$ be the probability that the electron survives a time t **without** suffering a collision. Since the electron certainly will survive a vanishingly short time before any collision, we see that this will require as time $t \to 0$, $P(0) = 1$. But $P(t)$ must decrease as time t increases, since the electron is in constant danger of colliding with a scattering center (e.g. lattice, ion, crystal defect); hence its probability of surviving a time t without suffering such a fate decreases as time goes on. Finally, $P(t) \to 0$ as $t \to \infty$. To describe the collisions, let $\wp\, dt$ equal the probability that an electron collides between t and $t + dt$.Thus the quantity $\wp$ is the probability per unit time that an electron suffers a collision which is called the "collision time." We will assume that it does not matter when the electron last experienced a collision, so that the probability $\wp$ is *independent* of the electron's past history.

Knowing the collision probability $\wp$, it is possible to calculate the survival probability $P(t)$, by noting that [the probability that an electron survives a time $t+dt$ **without** suffering a collision] must be equal to [the probability that this electron survives a time t **without** suffering a collision] multiplied by [the probability that it does not suffer a collision in the time interval between $t + dt$]. This can be written down mathematically as

$$P(t + dt) = P(t)(1 - \wp\, dt) \ . \tag{18}$$

$$P(t) + \frac{dP}{dt}\, dt = P(t) - P(t)\, \wp\, dt \ . \tag{19}$$

$$\frac{1}{P}\frac{dP}{dt} = -\,\wp \ . \tag{20}$$

It is assumed between collisions the speed v of the electron does not change. Even if it experiences an electromagnetic field, the speed changes by a relatively small amount in the short time $\wp^{-1}$. Hence the probability $\wp$, even if it is a function of $\boldsymbol{v}$, can be considered a constant independent of time. After separation of variables, integration and some algebra applied to equation (18) yields

$$P = C\,\boldsymbol{e}^{-\,\wp t}\ . \tag{21}$$

The constant of integration C can be determined by the condition that $P(0) = 1$. Thhus $C = 1$ so

$$P(t) = \boldsymbol{e}^{-\,\wp t}\ . \tag{22}$$

Now the probability that an electron, after surviving without collisions for a time t, suffers a collision in the time interval between $t + dt$ is given by

$$\mathcal{P}(t)dt = P(t) - P(t+dt) = -\frac{dP}{dt}\,dt = \boldsymbol{e}^{-\,\wp t}\,\wp\,dt\ . \tag{23}$$

This function is normalized such that integration over all time gives unity, meaning that the electron collides at *some* time.

Let $\tau \equiv \bar{t}$ be the mean time between collisions. This is also called the "collision time" of the electron. We can write the equation for the statistical mean

$$\tau \equiv \bar{t} = \int_0^\infty t\,\mathcal{P}(t)dt = \int_0^\infty t\,\boldsymbol{e}^{-\,\wp t}\,\wp\,dt = \frac{1}{\wp}\ . \tag{24}$$

Therefore the the probability that an electron remains unscattered after time t is

$$P(t) = \boldsymbol{e}^{-t/\tau}\ , \tag{25}$$

and the probability that an electron is scattered in the moment after time t is

$$\mathcal{P}(t) = \frac{\boldsymbol{e}^{-t/\tau}}{\tau}\ . \tag{26}$$