

# *Partial Regression Coefficients.*

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## INTRODUCTION

The partial regression coefficient is also called regression coefficient, regression weight, partial regression weight, slope coefficient or partial slope coefficient. It is used in the context of multiple linear regression (MLR) analysis and gives the amount by which the dependent variable (DV) increases when one independent variable (IV) is increased by one unit and all the other independent variables are held constant. This coefficient is called partial because its value depends, in general, upon the other independent variables. Specifically, the value of the partial coefficient for one independent variable will vary, in general, depending upon the other independent variables included in the regression equation

## MULTIPLE REGRESSION FRAMEWORK

In MLR, the goal is to predict, knowing the measurements collected on  $N$  subjects, the value of the dependent variable  $Y$  from a set of  $K$  independent variables  $\{X_1, \dots, X_k, \dots, X_K\}$ . We denote by  $\mathbf{X}$  the  $N \times (K + 1)$  augmented matrix collecting the data for the independent variables (this matrix is called augmented because the first column is composed only of ones), and by  $\mathbf{y}$  the  $N \times 1$  vector of observations for the dependent variable (see Figure 1).

Multiple regression finds a set of partial regression coefficients  $b_k$  such that the dependent variable could be approximated as well as possible by a linear combination of the independent variables (with the  $b_j$ 's being the weights of the combination). Therefore, a predicted value, denoted  $\hat{Y}$ , of the dependent variable is obtained as:

$$\hat{Y} = b_0 + b_1X_1 + b_2X_2 + \dots + b_kX_k + \dots + b_KX_K. \quad (1)$$

The value of the partial coefficients are found using ordinary least squares (OLS). It is often convenient to express the MLR equation using matrix notation. In

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<sup>1</sup>In: Lewis-Beck M., Bryman, A., Futing T. (Eds.) (2003). *Encyclopedia of Social Sciences Research Methods*. Thousand Oaks (CA): Sage.

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$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} & \dots & x_{1k} & \dots & x_{1K} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \dots & x_{nk} & \dots & x_{nK} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 1 & x_{N1} & \dots & x_{Nk} & \dots & x_{NK} \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \\ \vdots \\ y_N \end{bmatrix}$$

Figure 1: The structure of the  $\mathbf{X}$  and  $\mathbf{y}$  matrices.

this framework, the predicted values of the dependent variable are collected in a vector denoted  $\hat{\mathbf{y}}$  and are obtained using MLR as:

$$\hat{\mathbf{y}} = \mathbf{X}\mathbf{b} \quad \text{with } \mathbf{b} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{y}. \quad (2)$$

The quality of the prediction is evaluated by computing the multiple coefficient of correlation  $R_{Y,1,\dots,k,\dots,K}^2$  which is the coefficient of correlation between the dependent variable ( $Y$ ) and the predicted dependent variable ( $\hat{Y}$ ). The specific contribution of each IV to the regression equation is assessed by the partial coefficient of correlation associated to each variable. This coefficient, closely associated to the partial regression coefficient, corresponds to the increment in explained variance obtained by adding this variable to the regression equation after all the other IV's have been already included.

#### PARTIAL REGRESSION COEFFICIENT AND REGRESSION COEFFICIENT

When the independent variables are pairwise orthogonal, the effect of each of them in the regression is assessed by computing the slope of the regression between this independent variable and the dependent variable. In this case, (i.e., orthogonality of the IV's), the partial regression coefficients are equal to the regression coefficients. In all other cases, the regression coefficient will differ from the partial regression coefficients.

For example, consider the data given in Table 1 where the dependent variable  $Y$  is to be predicted from the independent variables  $X_1$  and  $X_2$ . In this example,  $Y$  is a child's memory span (i.e., number of words a child can remember in a set of short term memory tasks) which we want to predict from the child's age ( $X_1$ ) and the child's speech rate ( $X_2$ ). The prediction equation (using Equation 2) is

$$Y = 1.67 + X_1 + 9.50X_2 ; \quad (3)$$

where  $b_1$  is equal to 1 and  $b_2$  is equal to 9.50. This means that children increase their memory span by one word every year and by 9.50 words for every additional word they can pronounce (i.e., the faster they speak, the more they can

Table 1: A set of data:  $Y$  is to be predicted from  $X_1$  and  $X_2$  (data from Abdi et al., 2002).  $Y$  is the number of digits a child can remember for a short time (the “memory span”),  $X_1$  is the age of the child, and  $X_2$  is the speech rate of the child (how many words the child can pronounce in a given time). Six children were tested.

$Y$ (Memory span)	14	23	30	50	39	67
$X_1$ (age)	4	4	7	7	10	10
$X_2$ (Speech rate)	1	2	2	4	3	6

remember). A multiple linear regression analysis of this data set gives a multiple coefficient of correlation of  $R_{Y,12}^2 = .9866$ . The coefficient of correlation between  $X_1$  and  $X_2$  is equal to  $r_{1,2} = .7500$ , between  $X_1$  and  $Y$  is equal to  $r_{Y,1} = .8028$ , and between  $X_2$  and  $Y$  is equal to  $r_{Y,2} = .9890$ . The squared partial regression coefficient between  $X_1$  and  $Y$  is computed as

$$r_{Y,1|2}^2 = R_{Y,12}^2 - r_{Y,2}^2 = .9866 - .9890^2 = .0085, \quad (4)$$

likewise

$$r_{Y,2|1}^2 = R_{Y,12}^2 - r_{Y,1}^2 = .9866 - .8028^2 = .3421. \quad (5)$$

#### $F$ AND $t$ TESTS FOR THE PARTIAL REGRESSION COEFFICIENT

The null hypothesis stating that a partial regression coefficient is equal to zero can be tested by using a standard  $F$ -test which tests the equivalent null hypothesis stating that the associated partial coefficient of correlation is zero. This  $F$ -test has  $\nu_1 = 1$  and  $\nu_2 = N - K - 1$  degrees of freedom (with  $N$  being the number of observations and  $K$  being the number of predictors). Because  $\nu_1$  is equal to 1, the square root of  $F$  gives a Student- $t$  test. The computation of  $F$  is best described with an example: The  $F$  for the variable  $X_1$  in our example is obtained as

$$F_{y,1|2} = \frac{r_{Y,1|2}^2}{1 - R_{Y,12}^2} \times df_{\text{regression}} = \frac{r_{Y,1|2}^2}{1 - R_{Y,12}^2} \times (N - K - 1) = \frac{.0085}{1 - .9866} = 1.91. \quad (6)$$

This value of  $F$  is smaller than the critical value for  $\nu_1 = 1$  and  $\nu_2 = N - K - 1 = 3$  which is equal to 10.13 for an alpha level of  $\alpha = .05$ . Therefore  $b_1$  (and  $r_{Y,1|2}^2$ ) cannot be considered different from zero.

#### STANDARD ERROR AND CONFIDENCE INTERVAL

The standard error of the partial regression coefficient is useful to compute confidence intervals and perform additional statistical tests. The standard error of coefficient  $b_k$  is denoted  $S_{b_k}$ . It can be computed directly from  $F$  as

$$S_{b_k} = b_k \sqrt{\frac{1}{F_k}}, \quad (7)$$

where  $F_k$  is the value of the  $F$ -test for  $b_k$ . For example, we find for the first variable that:

$$S_{b_k} = 1 \times \sqrt{\frac{1}{1.91}} = .72 , \quad (8)$$

The confidence interval of  $b_k$  is computed as

$$b_k \pm S_{b_k} \sqrt{F_{\text{critical}}} \quad (9)$$

with  $F_{\text{critical}}$  being the critical value for the  $F$ -test for  $b_k$ . For example, the confidence interval of  $b_1$  is equal to  $1 \pm .72\sqrt{10.13} = 1 \pm 2.29$ , and therefore the 95% confidence interval for  $b_1$  goes from  $-1.29$  to  $+3.29$ . This interval encompasses the zero-value and this corresponds to the failure to reject the null hypothesis.

#### $\beta$ WEIGHTS AND PARTIAL REGRESSION COEFFICIENTS

There is some confusion here because depending upon the context,  $\beta$  is the parameter estimated by  $b$ , whereas in some other contexts  $\beta$  is the regression weight obtained when the regression is performed with all variables being expressed in  $Z$  scores. In the latter case,  $\beta$  is called the standardized partial regression coefficient or the  $\beta$ -weight. These weights have the advantage of being comparable from one independent variable to the other because the unit of measurement has been eliminated. The  $\beta$ -weights can easily be computed from the  $b$ 's as

$$\beta_k = \frac{S_k}{S_y} \times b_k , \quad (10)$$

(with  $S_k$  being the standard deviation of the  $k$ th independent variable, and  $S_y$  being the standard deviation of the dependent variable).

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