

# **Model Uncertainty, Limited Market Participation, and Asset Prices**

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We demonstrate that limited participation can arise endogenously in the presence of model uncertainty and heterogeneous uncertainty-averse investors. When uncertainty dispersion among investors is small, full participation prevails in equilibrium. Equity premium is related to the average uncertainty among investors and a conglomerate trades at a price equal to the sum of its single-segment components. When uncertainty dispersion is large, investors with high uncertainty choose not to participate in the stock market, resulting in limited market participation. When limited participation occurs, participation rate and equity premium can decrease in uncertainty dispersion and a conglomerate trades at a discount.

It has been well documented that a significant proportion of the U.S. households does not participate in the stock markets. On the basis of 1984 Panel Study of Income Dynamics (PSID) data, Mankiw and Zeldes (1991) find that among a sample of 2998 families, only 27.6% of the households own stocks. For families with liquid assets of \$100,000 or more, only 47.7% own stocks. More recent surveys show that even during the 1990s with the tremendous growth of the U.S. stock markets, such limited market participation still exists.<sup>1</sup>

Several explanations have been proposed in the literature to explain the phenomenon. Allen and Gale (1994) and Williamson (1994) show how transaction cost and liquidity needs create limited market participation.

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<sup>1</sup> For instance, the 1998 Survey of Consumer Finances shows that less than 50% of U.S households own stocks and/or stock mutual funds (including holdings in their retirement accounts).

Vissing-Jorgensen (1999) and Yaron and Zhang (2000) examine the effect of fixed entry cost on investors' participation decisions and find that while entry cost is a significant factor in investors' market participation decision, it alone does not provide the full picture of why there is limited market participation.<sup>2</sup> Haliassos and Bertaut (1995) explore the potential of risk aversion, heterogeneous beliefs, habit persistence, time-nonseparability, borrowing constraints, differential borrowing and lending rates, and minimum-investment requirements as causes for limited market participation and find that they are unable to account for the observed phenomenon.

Recently, studies show that disappointment aversion [Ang, Bekaert, and Liu (2002)] and uncertainty about the *probability distribution* of asset payoff, that is, model uncertainty (Dow and Werlang (1992) and Epstein and Schneider (2002)) can be important determinants in investors' market-participation decision. These studies, however, are conducted in partial equilibrium settings with a representative agent. As limited market participation is a heterogeneous agent issue in nature, a model with heterogeneous agents is thus needed to address the limited participation issue.

Building on the work of Dow and Werlang (1992) and Epstein and Schneider (2002), this paper examines how limited participation may arise in equilibrium with heterogeneous uncertainty averse agents in the presence of model uncertainty and its implications for equilibrium asset prices. Specifically, we first examine the equilibrium implication of investors' endogenous market participation decision for equity premium. We show that when there is limited market participation, equity premium can be lower than in the economy with full participation. Furthermore, to explore the broader implication of the uncertainty-induced limited market participation for asset prices, we focus on the widely-debated conglomerate discount issue.<sup>3</sup> We show that when there is limited

<sup>2</sup> For example, entry cost can not explain why families with substantial liquid asset (over \$100,000) may not participate in stock markets as shown in Mankiw and Zeldes (1991). Such evidence is also a challenge to theories based on other forms of transaction cost and liquidity need.

<sup>3</sup> For example, Lang and Stulz (1994), Berger and Ofek (1995), Rajan, Servaes, and Zingales (2000) show that conglomerates exhibit on average negative excess value of 10–15%. As the flip side of the conglomerate discount, Hite and Owers (1983), Miles and Rosenfeld (1983) and Schipper and Smith (1983) document a 2–3% increase in shareholder value around the corporate spin-offs. Villalonga (2004), using the Business Information Tracking Series (BITS) data that cover the whole U.S. economy at the establishment level, shows that diversified firms actually trade at a significant premium. There is also debate on what causes the discount. Rajan, Servaes, and Zingales (2000) and Scharfstein and Stein (2000) argue that diversification discount can be attributed to inefficient internal capital markets and/or inefficient resource allocation. A recent study by Bevelander (2001) finds that most diversified firms have Tobin's *q*s comparable to those of focused firms when firm age is explicitly accounted for. Using plant level observations from the Longitudinal Research Database, Schoar (2002) shows that conglomerates are more productive than stand-alone firms. However, conglomerates still trade at a discount of about 10 percent. She partially attributes this discount to higher wages than stand-alone firms. Recent studies by Graham, Lemmon, and Wolf (2002) and Maksimovic and Phillips (2002) did not find empirical evidence that would suggest inefficient resource allocation across divisions of a conglomerate.

participation a merged firm can trade at a value lower than the sum of the values of its component firms, that is, there is a diversification (conglomerate) discount.

Our study follows the Knightian approach to model uncertainty which differs from the more common Bayesian approach. In the Bayesian approach, when facing model uncertainty, the investor uses a prior distribution over a set of models.<sup>4</sup> His decision is based on the expected utility evaluated with respect to the predictive distribution. As a result, the investor is neutral with respect to model uncertainty, although he can be averse to risk. In the Knightian approach,<sup>5</sup> while the investor may still use von-Neumann-Morgenstern expected utility when facing risk, he no longer uses a single prior when facing uncertainty. In the multi-priors expected utility framework [Gilboa and Schmeidler (1989)], an uncertainty-averse investor evaluates an investment strategy according to the expected utility under the worst case probability distribution in a set of prior distributions. As a result, the investor is averse to both risk and model uncertainty.

We perform our analysis in a tractable mean–variance framework. The added feature of our study is that, in addition to the risk associated with asset payoff, investors are uncertain about the mean risky asset payoff and are heterogeneous in their levels of uncertainty.<sup>6</sup> The crucial feature of our framework is that for each uncertainty averse investor, there is a region of prices over which the investor holds neither long nor short position of a stock. Investors who are more uncertain about the mean payoff of the stock have larger nonparticipation regions than investors with less uncertainty. As a result, when uncertainty dispersion is sufficiently large, in equilibrium, investors with relatively high uncertainty choose not to participate in the stock market, giving rise to limited participation.

In our model, equity premium can be decomposed into two components: the risk premium and the uncertainty premium. The risk premium corresponds to the equity premium obtained in standard expected utility models. The uncertainty premium arises from investors' aversion to the uncertainty about the distribution of the stock payoff. When uncertainty dispersion is large, a fraction of investors choose not to participate in the stock market and hence there are fewer investors who hold the stock. Because there are fewer investors to bear the risk, the risk premium has

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<sup>4</sup> Wang (2005) shows that an investor's portfolio decision is highly sensitive to the choice of prior.

<sup>5</sup> See Dow and Werlang (1992), Epstein and Wang (1994, 1995), Anderson, Hansen and Sargent (1999), Maenhout (2004), Chen and Epstein (2001), Routledge and Zin (2002), Trojani and Vanini (2002), Epstein and Miao (2003), Uppal and Wang (2003), Liu, Pan and Wang (2004), among others.

<sup>6</sup> This modeling strategy of uncertainty has been used in many studies including Epstein and Miao (2003), and Kogan and Wang (2002), among others. Kogan and Wang (2002) is the closest to ours in terms of modeling framework but we consider an equilibrium model with heterogeneous agents.

to rise to induce these few investors to hold the stock. On the other hand, because investors participating in the stock market have lower uncertainty, a lower uncertainty premium is sufficient to induce these investors to hold the stock. Since these two effects work in the opposite direction, the net effect of uncertainty dispersion on equity premium depends on the relative dominance of these two opposing effects. In our setting, the decrease in the uncertainty premium dominates, leading to a lower equity premium than that in the economy with full market participation.

The intuition behind the relationship between limited participation and diversification discount is also straightforward. When there is limited participation, a firm's pre-merger price is determined by investors with less uncertainty for that firm. After the merger, investors have to buy both firms as a bundle. An investor who is less uncertain about one firm is not necessarily less uncertain about the other firm and hence is only willing to offer a lower price for the combined firm than the sum of the prices of the two firms traded separately. Thus, when there is limited participation, a merger results in a discount in the price of the combined firm.

Our work is directly related to studies that examine the limited market participation phenomenon and its implication for the equity premium. In addition to the papers cited earlier, Mankiw and Zeldes (1991), Attanasio, Banks, and Tanner (2002), Brav, Constantinides, and Geczy (2002), and Vissing-Jorgenson (2002) empirically demonstrate that if the sub-sample of stockholders' observations is used, the standard asset-pricing models such as Lucas (1978) can match the observed equity premium using a much lower risk aversion coefficient than the one used in Mehra and Prescott (1985). Basak and Cuoco (1998) show theoretically that equity premium increases as more investors are excluded from the risky asset markets.<sup>7</sup> The limited market participation in these studies, however, is *exogenously* specified. Moreover, these studies are all based on standard expected utility maximization models which abstract from model uncertainty. Our result on equity premium suggests that the prediction that limited market participation leads to higher equity premium may not be robust in a general equilibrium setting that takes account of model uncertainty.

More broadly, our paper belongs to the literature on market participation and its implication for asset prices. While the bulk of this literature is on the implication of limited participation for equity premium-related issues, the implication for other asset-pricing issues are beginning to be explored. Some recent examples include Chen, Hong, and Stein (2002) and Hong and Stein (2003), both of which study the

<sup>7</sup> Shapiro (2002) extends Basak and Cuoco's model to multiple stocks.

impact of exclusion of investors from the market due to short-sale constraints on asset prices. Our contribution to that literature is to show that diversification discount may also be related to limited participation due to the heterogeneity in investors' uncertainty about firms' asset returns.<sup>8</sup>

The rest of the paper is organized as follows. Section 1 presents the model. Section 2 describes investors' portfolio choices and their participation decisions. Section 3 analyzes how the limited market participation relates to equity premium, while Section 4 extends the model to allow for two stocks and investigates the relation between limited participation and the diversification discount. Section 5 discusses the robustness of our results in a more general setting with richer factor structure for asset payoffs. Section 6 extends the study to a simple dynamic setting. Section 7 concludes. Proofs are collected in the appendix.

## 1. The Model

To highlight the intuition, we first consider a market with one stock and one bond. The case with multiple stocks is considered in Section 4.

### 1.1 The setting

We assume a one-period heterogeneous agent economy. Investment decisions are made at the beginning of the period, while consumption takes place at the end period. Each investor is endowed with  $x > 0$  shares of the stock. The risk-free rate is set at zero. The stock can be viewed as the market portfolio in a single-factor economy in which the market portfolio is the only factor. The payoff of the stock, denoted  $r$ , is normally distributed with mean  $\mu$  and variance  $\sigma^2$ . We assume that investors have a precise estimate of the variance  $\sigma^2$ , but do not know exactly the mean of  $r$ . This is motivated by both the analytical tractability and the empirical evidence on the predictability of the volatility of stock returns [Bollerslev, Chou, and Kroner, (1992)], and the difficulty in estimating precisely the expected stock returns.

### 1.2 The preferences

Each agent in the economy has a CARA utility function  $u(W)$ , where  $W$  is the end of period wealth. Due to the lack of perfect knowledge of the

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<sup>8</sup> There are alternative theories about the diversification discount and spin-off premium. For example, Miller (1977) argues informally that short-sale constraint and differences of opinion can explain the diversification discount. Habib, Johnsen, and Naik (1999) offer an explanation of spin-off premium based on higher incentive to collect information associated with spin-offs which reduces the risk premium. Gomes and Livdan (2003) develop a dynamic model of firm diversification focusing on the role of diminishing marginal productivity of technology and synergies of different activities in reducing fixed cost of production.

probability law of stock payoff, the agent's preference is represented by a multi-priors expected utility (Gilboa and Schmeidler, 1989)

$$\min_{Q \in \mathcal{P}} \{E^Q[u(W)]\}, \quad (1)$$

where  $E^Q$  denotes the expectation under the probability measure  $Q$  and  $\mathcal{P}$  is a set of probability measures. The basic intuition behind multi-priors expected utility preferences is that in the real world, investors do not know precisely the probability distribution of stock pay-off. While they may have some information about the true objective probability law, which could result from some econometric analysis, they are not completely sure what the true law is. As a result, instead of restricting themselves to using any particular distribution as the proxy for the true probability distribution, they are willing to consider a set  $\mathcal{P}$  of alternative probability distributions. These investors are averse to the uncertainty about the true probability distribution. They evaluate the expected utility of a random consumption under the worst case scenario in the set  $\mathcal{P}$ .

The specification of the set  $\mathcal{P}$  follows Kogan and Wang (2002). Let  $P$  be a reference probability distribution obtained, say, from econometric analysis.  $\mathcal{P}$  is defined as the collection of all probability distributions  $Q$ , satisfying  $E^Q[\ln(dQ/dP)] < \eta$ . The intuition behind this specification of  $\mathcal{P}$  is that, since  $E^Q[\ln(dQ/dP)]$  is the log likelihood ratio under  $Q$ ,  $\mathcal{P}$  can be viewed as a confidence region around  $P$  and  $\eta$  can be viewed as the critical value for, say, 95% confidence. For a fixed level of confidence, the size of  $\mathcal{P}$  captures an investor's uncertainty about the reference probability distribution  $P$ . If the investor has more information about,  $P$ , he would be able to estimate the true probability law more precisely. Then for the same level of confidence,  $\eta$  will be small. Conversely, if the investor has little information about  $P$ , than  $\eta$  will be large. In this sense,  $\eta$  can be viewed as a measure of uncertainty.

Kogan and Wang (2002) show that for normal distributions with a common known variance, the confidence region can always be described by a set of quadratic inequalities. For the specific case of this section, let  $(\mu + v)$  denote the mean of  $r$  under  $Q$  distribution. Then the confidence region is described by:

$$v^2 \leq \phi^2, \quad (2)$$

where  $\phi$  is a parameter that captures the investor's uncertainty about the mean of  $r$ . Therefore the set  $\mathcal{P}$  can be equivalently defined as the collection of all normal distributions with mean  $\mu + v$  and variance  $\sigma^2$  where  $v$  satisfies

Equation 2. All investors are assumed to have the same point estimate  $\mu$ .<sup>9</sup> Note that the higher the  $\phi$ , the wider the range for the expectation of  $r$ . Thus we call  $\phi$  the investor's level of uncertainty about the mean stock payoff.

In our economy, investors have the same risk aversion, but are heterogeneous in their levels of uncertainty.<sup>10</sup> Such heterogeneity can arise for a variety of reasons. For example, different investment firms may use their own proprietary models to analyze the data, which may lead to different levels of precision of the estimated mean. Differences in perceived familiarity about the stock market may also contribute to the heterogeneity in uncertainty [Huberman (2001)]. We abstract from the details of how such heterogeneity arises and simply assume that the difference is captured by different  $\phi$ 's. Furthermore, we assume that  $\phi$  is uniformly distributed among investors on the interval<sup>11</sup>

$$S_1 = [\bar{\phi} - \delta, \bar{\phi} + \delta],$$

with density  $1/(2\delta)$ , where  $\bar{\phi} \geq \delta$  and  $\delta$  measure the dispersion of the uncertainty among investors. When  $\bar{\phi} = \delta = 0$ , there is no model uncertainty, and our model collapses to the standard expected utility model.

The preference in Equation (1) exhibits aversion to uncertainty whereas preference represented by Savage's expected utility is uncertainty neutral. The simplest example that highlights the difference is the Ellsberg experiment [Ellsberg (1961)]. This difference between the multi-priors expected utility and the standard expected utility is crucial for understanding why our findings differ from the results of existing studies.

Before moving on to discuss individual investor's portfolio and market-participation decision, we make a few remarks on the modeling strategy that we have adopted regarding investors' utility function and their endowments. We assume that all investors have the same CARA utility and the same endowment of  $x$  shares of the stock. This assumption is made mainly to abstract from the endowment effect. We note that, just as in the standard expected utility framework, CARA utility does not have wealth effect even in the multi-priors expected utility framework, as will be seen shortly. As a result, the initial endowment of an investor does not affect his optimal portfolio holding. With other utility functions, however,

<sup>9</sup> Our results regarding limited participation, equity premium, and diversification discount hold without assuming the same point estimate for all investors. However, to focus on the effect of heterogeneity in model uncertainty, we abstract from the heterogeneity of the point estimate  $\mu$ , because heterogeneity in estimated expected returns is often interpreted as heterogeneity of opinion. Thus, by abstracting away such heterogeneity, we are trying to isolate the effect of model uncertainty from the effect of heterogeneous opinion described by Miller (1977). The assumption also allows us to obtain closed-form solutions which greatly facilitate the analysis.

<sup>10</sup> In an early version, we allowed for heterogeneity in investors' risk aversion. Because the qualitative nature of our results is not affected, we assume the same level of risk aversion for all investors for ease of exposition.

<sup>11</sup> In Section 6, we consider a more general distribution for uncertainty among investors.

this is often not the case. One obvious example is CRRA utility. Other more interesting examples include the semi-order preference introduced in Bewley (2002) and loss-aversion utility [Kahneman and Tversky (1979)]. For these two preferences, the starting point is important for the ranking of consumption bundles. Thus, in the portfolio-choice problem with such preferences, endowment affects an individual's portfolio choice not only through its impact on the initial wealth in the budget constraint, but also by entering directly into the utility of the individual. By abstracting away the endowment effect, our modeling strategy allows us to examine more cleanly the importance of model uncertainty. It also gives us the analytic tractability which greatly facilitates the analysis.<sup>12</sup>

## 2. Portfolio Choice and Equilibrium Market Participation

In this section, we first solve investors' optimal portfolio-choice problem and then derive the closed-form solution for the equilibrium. We show that when the dispersion in investor's uncertainty is large, there is limited market participation in equilibrium.

### 2.1 Portfolio choices

Let  $P$  denote the price of the stock, the investor's wealth constraint is

$$W_{1i} = W_{0i} + D_i(r - P),$$

where  $W_{0i} = xP$  and  $W_{1i}$  are investor  $i$ 's beginning of period and end of period wealth, respectively,  $D_i$  is investor  $i$ 's net demand for the stock, and  $r$  is the per share return on the stock. Under the assumption of normality and CARA utility, investor  $i$ 's utility maximization problem becomes:

$$\max_{D_i} \min_v D_i(\mu - v - P) - \frac{\gamma\sigma^2}{2} D_i^2,$$

where  $\gamma$  is the investor's risk-aversion coefficient. The first term of the objective function represents the expected excess payoff from investing in the stock with adjustment to the mean due to uncertainty, and the last term represents adjustment due to risk. Solving the inner minimization, the above problem reduces to

$$\max_{D_i} D_i[\mu - sgn(D_i)\phi_i - P] - \frac{\gamma\sigma^2}{2} D_i^2, \quad (3)$$

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<sup>12</sup> In a related paper, Cao, Hirshleifer, and Zhang (2004) investigate the endowment effect and investors' portfolio choices under model uncertainty using a preference relation that reflects aspects of Bewley (2002) and Gilboa and Schmeidler (1989), but which emphasizes fear of the unfamiliar as reflected in a reluctance to deviate from a specified status-quo action.

where  $sgn(\cdot)$  is an indicator function which takes the sign of its argument. The expression,  $\mu - sgn(D_i)\phi_i$ , represents the uncertainty-adjusted expected payoff under the worst case scenario for holding  $D_i$  shares of the stock. The intuition behind Equation (3) is the following. If investor  $i$  takes a long position ( $D_i > 0$  and  $sgn(D_i) = 1$ ), the worst case scenario for him is that the mean excess payoff is given by  $\mu - \phi_i - P$ . Similarly, if he takes a short position ( $D_i < 0$  and  $sgn(D_i) = -1$ ), the worst case scenario is that the mean excess payoff is  $\mu + \phi_i - P$ .

Summarizing, the investor's optimal holding in the stock is given by

$$D_i = \begin{cases} \frac{1}{\gamma\sigma^2}(\mu - \phi_i - P) & \text{if } \mu - P > \phi_i, \\ 0 & \text{if } -\phi_i \leq \mu - P \leq \phi_i, \\ \frac{1}{\gamma\sigma^2}(\mu + \phi_i - P) & \text{if } \mu - P < -\phi_i. \end{cases} \quad (4)$$

Thus, due to his uncertainty about the mean of  $r$ , investor  $i$  holds a long (short) position in the stock only when the equity premium is strictly positive (negative) in the worst case scenario,  $v = \phi_i$  ( $v = -\phi_i$ ). When the price  $P$  falls in the region,  $[\mu - \phi_i, \mu + \phi_i]$ , the investor optimally chooses not to participate in the market. The non-participation region for investor  $i$  is the interval between the investor's lowest and the highest expected payoffs.

## 2.2 Full participation

In equilibrium the market clears. As will be seen later, in equilibrium, the stock always sells at a discount, that is,  $\mu - P > 0$ . Thus, the expression of the demand function Equation (4) implies that no investor will hold short positions in the stock. In the full participation equilibrium, all investors participate in the stock market. The market clearing condition for the stock can thus be written as

$$x = \int_{\phi_i \in S_1} \frac{(\mu - \phi_i - P)}{\gamma\sigma^2} \frac{1}{2\delta} d\phi_i = \frac{1}{\gamma\sigma^2} (\mu - \bar{\phi} - P).$$

Solving for  $P$ , we arrive at the following equilibrium pricing equation,<sup>13</sup>

$$\mu - P = \gamma\sigma^2 x + \bar{\phi}. \quad (5)$$

The first term on the right hand side of Equation (5) is standard. It represents the risk premium, which is proportional to investors' risk-aversion coefficient and the variance of  $r$ . The second term is new. It represents the uncertainty premium and is proportional to the average level of uncertainty ( $\bar{\phi}$ ) of all investors about the mean stock payoff.

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<sup>13</sup> Chen and Epstein (2001) obtained a similar decomposition of equity premium in a representative agent model.

The full participation equilibrium prevails when all investors participate in the stock market. This requires all investors, including investors with the highest uncertainty, to hold long positions. Equation (4) thus implies

$$\mu - (\bar{\phi} + \delta) - P > 0.$$

Using Equation (5), we have that in equilibrium all investors participate in the market if

$$\gamma\sigma^2 x > \delta. \quad (6)$$

This full participation condition indicates that whether all investors participate in the stock market depends on the dispersion of investors' model uncertainty. When  $\delta$  is small, investors are relatively homogeneous and all investors will participate.

Under full market participation, investors' model uncertainty dispersion has no effect on price. What matters is the average uncertainty. Investors who are more uncertain about the mean stock payoff will hold less stock, while investors who are less uncertain will hold more. In aggregate, the market price behaves as if all investors have the average uncertainty.

### 2.3 Limited participation

When uncertainty dispersion among investors is sufficiently large such that condition (6) is not satisfied, investors with high uncertainty optimally choose not to participate in the stock market. Let  $\phi^*$  denote the lowest level of uncertainty at which investors do not hold the stock. Then

$$\mu - \phi^* - P = 0. \quad (7)$$

Using the market clearing condition,

$$x = \int_{\phi_i < \phi^*, \phi_i \in S_1} \frac{D_i}{2\delta} d\phi_i = \frac{1}{\gamma\sigma^2} \frac{(\phi^* - \bar{\phi} + \delta)}{2\delta} \left[ \mu - \frac{\phi^* - \bar{\phi} + \delta}{2} - P \right], \quad (8)$$

and solving for  $\phi^*$  yields

$$\phi^* = \bar{\phi} - \delta + 2\sigma\sqrt{\gamma\delta x}. \quad (9)$$

Combining this with Equation (7), we arrive at the following equilibrium price for the stock,

$$P = \mu - (\bar{\phi} - \delta) - 2\sigma\sqrt{\gamma\delta x}. \quad (10)$$

We now turn to the implication of limited market participation for equity premium.

### 3. Market Participation and Equity Premium

A widely investigate issue in the literature on limited market participation is how equity premium relates to limited market participation. Several studies [Mankiw and Zeldes (1991), Basak and Cuoco (1998), Attanasio, Banks, and Tanner (2002), Brav, Constantinides, and Geczy (2002), and Vissing-Jorgensen (2002)] find that equity premium increases as fewer investors participate in stock markets. These studies, however, assume that some investors are *exogenously* excluded from participating in the stock markets. In this section, we re-examine the relation between the equity premium and limited market participation by allowing investors to optimally choose whether or not to participate in the market so that limited market participation arises *endogenously*.

To facilitate the analysis, we re-write the pricing Equation (10) as

$$\mu - P = \frac{\gamma\sigma^2 x}{\alpha} + \phi_p, \quad (11)$$

where

$$\alpha = \frac{\phi^* - (\bar{\phi} - \delta)}{2\delta} = \sigma\sqrt{\gamma x/\delta} \quad (12)$$

is the proportion of investors holding long positions in the stock, and

$$\phi_p = \frac{1}{2}(\phi^* + \bar{\phi} - \delta) = \bar{\phi} - \delta + \sigma\sqrt{\gamma\delta x} \quad (13)$$

is the average level of uncertainty among market participants. Thus, Equation (11) says that equity premium can be decomposed into two components: the risk premium and the uncertainty premium. The lower the participation rate, the higher the risk premium, which is consistent with what is known in the literature [e.g., Basak and Cuoco (1998)].

The following is the main result of this section.

**Proposition 1.** Suppose that  $\gamma\sigma^2 x < \delta$ . Then there is limited market participation in equilibrium. Furthermore,

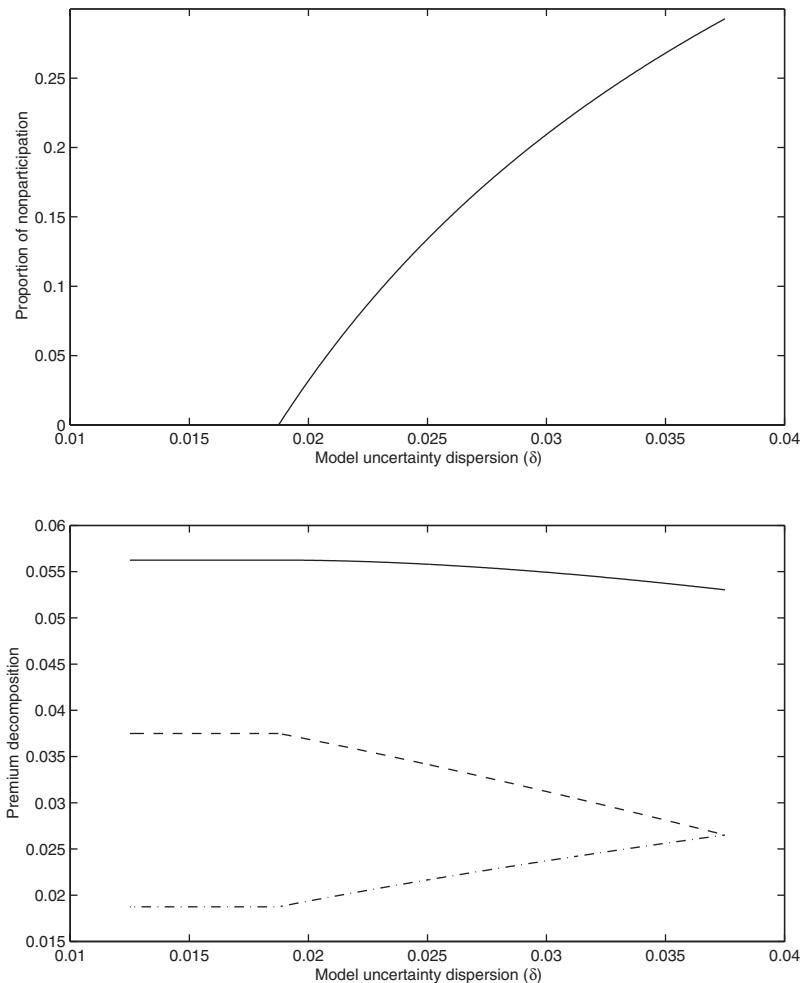
- (a) investors' participation rate  $\alpha$  decreases in uncertainty dispersion, i.e.,  $\partial\alpha/\partial\delta < 0$ ;
- (b) the average uncertainty of market participants decreases in uncertainty dispersion, i.e.,  $\partial\phi_p/\partial\delta < 0$ ; and
- (c) the equity premium decreases in uncertainty dispersion, i.e.,  $\partial(\mu - P)/\partial\delta < 0$ .

We have shown earlier that with full market participation, equity premium does not depend on uncertainty dispersion among investors. With limited market participation, however, equity premium decreases in

uncertainty dispersion. The intuition is as follows. According to the demand functions [Equation (4)], when the stock sells at discount ( $\mu - P > 0$ ), no investor will hold short position in the stock. Thus, in the equilibrium with limited participation, there are fewer investors holding the stock than in the equilibrium with full participation. Since fewer investors in the market have to bear all the market risk, they demand a higher risk premium. This, on the surface, suggests that price of the stock is lower and premium higher in the equilibrium with limited participation. However, in the equilibrium with limited participation, only investors with relatively low uncertainty remain in the stock market. These investors on average are willing to accept a lower uncertainty premium as shown in Proposition 1(b). The net effect on equity premium depend on which of the two forces dominates. In our model, as uncertainty dispersion increases, according to Proposition 1(a), market-participation rate decreases and risk premium increases. At the same time, uncertainty premium decreases according to Proposition 1(b). Moreover, when there is limited market participation, the uncertainty premium effect dominates the risk premium effect, leading to a lower equity premium as shown in Proposition 1(c).

Figure 1 illustrates the percentage of non-participation in the stock market (top panel) and the equity premium (solid line) decomposed into the uncertainty premium (dashed line) and the risk premium (dash-dotted line) (bottom panel) as a function of the uncertainty dispersion among investors ( $\delta$ ). At very low level of the uncertainty dispersion, full participation prevails in equilibrium. The equity premium, uncertainty premium, and risk premium all remain constant. As the model uncertainty dispersion increases, investors with high uncertainty stay sidelined, the risk premium rises, the uncertainty premium falls, and the total equity premium declines.

Figure 2 provides some empirical observations on the relation between the stock market participation and stock returns for France, Germany, Italy, the Netherlands, Sweden, the United Kingdom, and the United States. The stock market participation measures are based on 1998 microeconomic surveys in respective countries except for Sweden, which is based on a 1999 survey. We use the total annual returns in U.S. dollars for the equity market in respective countries for the period between 1986 and 1997. Two measures of the stock market participation are used: direct participation which considers traded and non-traded stocks held directly (top panel) and total participation which also includes mutual funds and managed investment accounts (bottom panel). Both the top and bottom panels suggest that the stock market participation and stock returns are positively correlated. This is reflected by an upward sloping least-square line fitted to the data for both the direct participation and stock return relation and total participation and

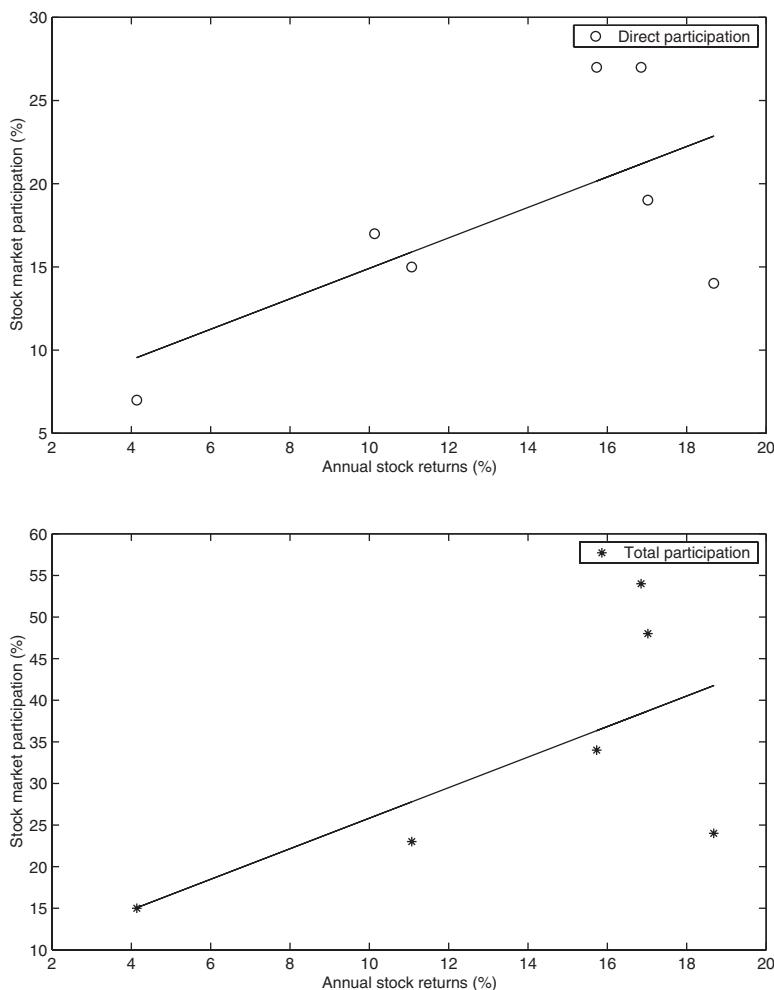
**Figure 1**

The top panel shows the proportion of investors not participating the asset market as a function of uncertainty dispersion ( $\delta$ )

The lower panel illustrates the decomposition of the equity premium (solid line) into uncertainty premium (dashed line) and risk premium (dash-dotted line) as a function of uncertainty dispersion ( $\delta$ ). The parameters used are as follows:  $x = 0.3$ ,  $\sigma = 0.25$ ,  $\mu = 1.06$ ,  $\gamma = 1$ , and  $\bar{\phi} = 0.0375$ .

stock return relation.<sup>14</sup> However, because in our model, both the stock market participation and stock returns are determined by the heterogeneity of model uncertainty, there is no causality between the two

<sup>14</sup> The test statistics show that the slope is positive at conventional significance level. However, we do not report the test statistics and the corresponding  $p$  values due to the limited number of observations.



**Figure 2**

**Stock market participation and total annual stock returns for France, Germany, Italy, the Netherlands, Sweden, the United Kingdom, and the United States**

The stock return is the percent annual change in the MSCI market index in the U.S. dollars for respective countries with dividends reinvested between 1986 and 1997. Two market participation measures are used: Direct participation represented by "o" (top panel) and total participation denoted by "\*" (bottom panel). The market participation measures are based on 1998 microeconomic surveys conducted in respective countries except for Sweden which is based on the 1999 survey. For Germany, only direct participation measure is available. The data are from Tables 4 and 11 of Guiso, Haliassos, and Jappelli (2003).

variables. The evidence contrasts with the results obtained by treating the stock market participation as exogenous and then focusing on how the implied stock returns respond to different levels of the stock market participation.

#### 4. Multiple Stocks and the Diversification Discount

In this section, we examine how, in the *absence* of any synergy between two firms, the market value of the merged firm (the conglomerate) depends upon investors' uncertainty dispersion and how it relates to limited participation at firm level. Our study differs but complements studies based on production and synergy such as Gomes and Livdan (2004).

We extend the model in Section 1 to allow for two stocks. The payoffs on the two stocks follow a two factor model

$$r_1 = f_A + \rho f_B, \quad (14)$$

$$r_2 = \rho f_A + f_B, \quad (15)$$

where  $r_1$  and  $r_2$  are payoff on stocks 1 and 2, respectively,  $f_A$  and  $f_B$  are two factors, and  $\rho$  represents the factor loading. We assume  $|\rho| < 1$ . The total supplies of the two stocks are given by  $x_1$  and  $x_2$ , with  $x_1 = x_2 = x > 0$ . All variables are jointly normally distributed. Investors know that the factors,  $f_A$  and  $f_B$ , follow a joint normal distribution with a non-degenerate variance–covariance matrix  $\Omega_F$ . Without loss of generality, we assume that  $\Omega_F$  is diagonal with common element  $\sigma^2$ . However, investors do not know exactly the mean of  $f_A$  or  $f_B$ . Let  $\mu_A = \mu_B = \mu$  be the point estimates of the means of  $f_A$  and  $f_B$ , respectively. As before, let  $\mu_A + v_A$  and  $\mu_B + v_B$  be the the means of the factors under an alternative distribution  $Q$  and define the set  $\mathcal{P}$  by the following constraints on  $v_A$  and  $v_B$ :

$$v_A^2 \leq \phi_A^2, \quad (16)$$

and

$$v_B^2 \leq \phi_B^2, \quad (17)$$

where  $\phi_A$  and  $\phi_B$  are the parameters that capture the investor's uncertainty about the means of factors  $A$  and  $B$ , respectively. Investors believe the means of factor  $f_A$  and  $f_B$  fall into the set

$\{(\mu_A + v_A, \mu_B + v_B) : v_A \text{ and } v_B \text{ satisfy Equations (16) and (17), respectively}\}$ .

We assume that  $\phi = (\phi_A, \phi_B)$  is uniformly distributed on the square

$$S_2 = [\bar{\phi} - \delta, \bar{\phi} + \delta] \times [\bar{\phi} - \delta, \bar{\phi} + \delta],$$

where  $\bar{\phi} \geq \delta$ .

When there is a merger, firms combine their operations to form a conglomerate denoted by  $M$ . The cash flows are unaffected by the merger so that the payoff to holding firm  $M$ 's stock is  $r_M = r_1 + r_2 =$

$(1 + \rho)(f_A + f_B)$ .<sup>15</sup> After the merger, investors can no longer trade stocks 1 and 2 and can only hold asset  $M$ .

For ease of exposition, we introduce factor portfolios tracking the two factors  $A$  and  $B$ . Let  $P_A$  and  $P_B$  be the prices of factor portfolios,  $A$  and  $B$ , respectively.  $P_j$  be the price of stock  $j$ , for  $j = 1, 2$ , and  $P_M$  be the market price of the merged firm. The supply of each factor portfolio is then  $\hat{x} \equiv (1 + \rho)x$ . Since the two factors are orthogonal to each other and investors have CARA utility function, the prices of the factors ( $P_A$  and  $P_B$ ) can be determined similarly as in the case with a single stock in Section 2. The prices of the stocks when traded separately can be obtained using the relation  $P_1 = P_A + \rho P_B$  and  $P_2 = \rho P_A + P_B$ .

The following proposition summarizes the main result on the relationship between limited market participation and the diversification discount.

**Proposition 2.** Suppose  $(1 + \rho)x\gamma\sigma^2 < \delta$ . Then there is limited market participation in equilibrium. Furthermore, when stocks 1 and 2 are traded separately, their prices are given by

$$P_1 = P_2 = (1 + \rho)[\mu - (\bar{\phi} - \delta) - 2\sigma\sqrt{\gamma(1 + \rho)x\delta}].$$

When the firms are merged, the price of the merged firm is given by

$$P_M = 2(1 + \rho)\{\mu - g^{-1}[\gamma(1 + \rho)x\sigma^2]/2\},$$

where

$$g(y) = \left\{ (y - 2\bar{\phi} + 2\delta)^3 - 2[(y - 2\bar{\phi})^+]^3 \right\} / (48\delta^2),$$

and  $P_M < P_1 + P_2$ , i.e., there is a diversification discount. Moreover, the diversification discount increases in uncertainty dispersion ( $\delta$ ).

## 5. Generalization

In this section, we extend the model in Section 4 to allow for a more general factor structure and demonstrate that our basic results are robust in the more general setting. Specifically, the payoffs of stocks 1 and 2 are now given by

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \beta^\top \begin{pmatrix} f_A \\ f_B \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}, \quad (18)$$

where

$$\beta = \begin{pmatrix} \beta_{1A} & \beta_{2A} \\ \beta_{1B} & \beta_{2B} \end{pmatrix}. \quad (19)$$

<sup>15</sup> We abstract from agency problems such as inefficient internal capital markets and rent-seeking at division-level considered by Rajan, Servaes, and Zingales (2002) and Scharfstein and Stein (2000).

The matrix  $\beta$  represents the factor loadings and is assumed to be invertible. The factors  $f_A$  and  $f_B$  are the systematic factors as before. The random shocks  $\epsilon_1$  and  $\epsilon_2$  are the idiosyncratic risk for stocks 1 and 2, respectively.

All variables are jointly normally distributed. The idiosyncratic factors are uncorrelated with each other and are uncorrelated with the systematic factors. As in Section 4, investors do not have perfect knowledge of the distribution of the stock payoffs. They know that the factors  $(f_A, f_B, \epsilon_1, \epsilon_2)^\top$  follow a joint normal distribution. The risk of stock payoffs is summarized by the non-degenerate variance–covariance matrix  $\Omega_F$ . We assume that investors have a precise estimated of  $\Omega_F$ . Without loss generality, we assume that  $f_A$  and  $f_B$  have the same variance  $\sigma^2$ , and  $\epsilon_1$ , and  $\epsilon_2$  have variances  $\sigma_1^2$  and  $\sigma_2^2$ , respectively. However, investors do not know exactly the means of  $f_A$  or  $f_B$ .<sup>16</sup> Investors' uncertainty is represented by the constraints

$$v_{iA}^2 \leq \phi_{iA}^2, \quad v_{iB}^2 \leq \phi_{iB}^2.$$

Similar to Section 4, investors' heterogeneity is characterized by  $\phi_i = (\phi_{iA}, \phi_{iB})^\top$ . We assume that  $\phi_{iA}$  and  $\phi_{iB}$  have a joint distribution defined on  $S_3 \equiv [\phi_{lA}, \phi_{uA}] \times [\phi_{lB}, \phi_{uB}]$  with a continuous probability density function  $h(\phi_{iA}, \phi_{iB})$ . The average of  $\phi_i$  is denoted by  $\bar{\phi} = (\bar{\phi}_A, \bar{\phi}_B)^\top$ . Let  $x_1$  and  $x_2$  be the supplies of stocks 1 and 2, respectively. The corresponding factor portfolio supplies are given by

$$x \equiv \begin{bmatrix} x_A \\ x_B \end{bmatrix} = \beta \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

## 5.1 Portfolio choice and market participation

With CARA utility, investor  $i$ 's maximization problem is:

$$\begin{aligned} \max_{D_{i1}, D_{i2}} & \min_{v_{iA}, v_{iB}} -D_{i1}P_1 - D_{i2}P_2 + (\beta_{1A}D_{i1} + \beta_{2A}D_{i2})(\mu_A - v_{iA}) \\ & + (\beta_{1B}D_{i1} + \beta_{2B}D_{i2})(\mu_B - v_{iB}) \\ & - \frac{\gamma}{2} [(\beta_{1A}D_{i1} + \beta_{2A}D_{i2})^2\sigma^2 + (\beta_{1B}D_{i1} + \beta_{2B}D_{i2})^2\sigma^2 + D_{i1}^2\sigma_1^2 + D_{i2}^2\sigma_2^2]. \end{aligned} \tag{20}$$

The demand for factor portfolios are given by:

$$D_i \equiv \begin{pmatrix} D_{iA} \\ D_{iB} \end{pmatrix} \equiv \beta \begin{pmatrix} D_{i1} \\ D_{i2} \end{pmatrix}. \tag{21}$$

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<sup>16</sup> By assumption, the idiosyncratic factors have zero mean.

Since the matrix  $\beta$  is invertible, there is a one-to-one correspondence between the holdings of factor portfolios and those of the stocks 1 and 2. In particular, if an investor does not participate in the factor portfolio markets, he does not participate in the stock markets.

The first-order condition for investor  $i$ 's optimal holdings of the factor portfolios is given by

$$\begin{bmatrix} \mu_A - \phi_{iA} + \lambda_{iA} \\ \mu_B - \phi_{iB} + \lambda_{iB} \end{bmatrix} - P = \gamma \left( \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} + (\beta^{-1})^\top \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \beta^{-1} \right) \begin{bmatrix} D_{iA} \\ D_{iB} \end{bmatrix},$$

where  $\lambda_{iA}$  and  $\lambda_{iB}$  are non-negative numbers such that  $0 \leq \lambda_{iA} \leq 2\phi_{iA}$  and  $0 \leq \lambda_{iB} \leq 2\phi_{iB}$ , and  $P \equiv (P_A, P_B)^\top = (\beta^{-1})^\top (P_1, P_2)^\top$ . Let  $\lambda_i = (\lambda_{iA}, \lambda_{iB})^\top$ . Then the demand for factor portfolios is given by

$$D_i = (\gamma \hat{\Omega})^{-1} [\mu - P] - (\gamma \hat{\Omega})^{-1} [\phi_i - \lambda_i], \quad (22)$$

where

$$\hat{\Omega} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} + (\beta^{-1})^\top \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \beta^{-1}$$

is the variance–covariance matrix of the factor portfolios. Without loss of generality, we assume that  $\hat{\Omega}$  is positive.<sup>17</sup> The first term on the right hand side of Equation (22) is the standard mean–variance demand, and the second term is the adjustment to the mean–variance demand due to model uncertainty. Given the demand for the factor portfolios, the demand for individual stocks can be determined by

$$(D_{i1}, D_{i2})^\top = \beta^{-1} D_i. \quad (23)$$

We summarize the result on participation in the following proposition.

**Proposition 3.** *If  $\min\{\phi_{uA}, \phi_{uB}\} > M \equiv \max\{|\mu_A - P_A|, |\mu_B - P_B|\}$ , then investors with uncertainty level in  $S = [M, \phi_{uA}] \times [M, \phi_{uB}]$  will take no position in the stock markets.*

## 5.2 Equilibrium and market participation

In the general setting of this section, we do not have a closed-form solution for the equilibrium. However, the existence of equilibrium is guaranteed by the following proposition.

**Proposition 4.** *There exists an equilibrium. In addition, if the supply of both factor portfolios are strictly positive, the equilibrium is unique. Moreover, both factor portfolios are traded at a discount, i.e.,  $\mu_A - P_A > 0$  and  $\mu_B - P_B > 0$ .*

<sup>17</sup> Otherwise, we can redefine a new set of factors  $f'_A = -f_A, f'_B = f_B$  such that  $\hat{\Omega}$  is positive.

### 5.3 Equity premium under limited participation

In equilibrium, the asset markets must clear, or equivalently, the factor portfolio markets must clear, that is.,

$$x_A = \int \int_{D_{iA} \neq 0, \phi_i \in S_3} D_{iA} h(\phi_{iA}, \phi_{iB}) d\phi_{iA} d\phi_{iB}, \quad (24)$$

$$x_B = \int \int_{D_{iB} \neq 0, \phi_i \in S_3} D_{iB} h(\phi_{iA}, \phi_{iB}) d\phi_{iA} d\phi_{iB}. \quad (25)$$

We say that is limited participation in factor portfolio  $j$ ,  $j = A, B$ , when the measure of non-participants in factor portfolio  $j$  is positive. We have the following results regarding the relation between limited participation and asset prices.

**Proposition 5.** Suppose that  $x_A > 0$  and  $x_B > 0$ . Let  $\omega_{ij}$  denote the  $(i, j)$ th component of  $(\gamma \hat{\Omega})^{-1}$ .

- (a) If  $x \geq (\omega_{11}(\phi_{uA} - \bar{\phi}_A) + \omega_{12}(\phi_{uB} - \bar{\phi}_B), \omega_{12}(\phi_{uA} - \bar{\phi}_A) + \omega_{22}(\phi_{uB} - \bar{\phi}_B))^\top$ , then all investors hold both factor  $A$  and factor  $B$  portfolios. In addition, the full participation equilibrium prices for the factor portfolios,  $P^f \equiv (P_A^f, P_B^f)^\top$ , are given by

$$P^f = \mu - \bar{\phi} - \gamma \hat{\Omega} x. \quad (26)$$

- (b) If there is limited participation in factor portfolio  $A$  ( $B$ ) but full participation in factor portfolio  $B$  ( $A$ ), then  $P_A > P_A^f$  and  $P_B = P_B^f$  ( $P_A = P_A^f$  and  $P_B > P_B^f$ );  
(c) If there is limited participation in both factor portfolios,  $P_A > P_A^f$  and  $P_B > P_B^f$ .

Market participation clearly depends on the dispersion of investors' uncertainty. When investors are relatively homogeneous,  $\phi_{uA} - \phi_{IA}$  and  $\phi_{uB} - \phi_{IB}$  are small, a full participation equilibrium prevails. When investors are relatively homogeneous with respect to one factor but not the other, there could be limited participation with respect to one of the factor portfolios. Finally, when investors' uncertainty is very heterogeneous with respect to both factors, some investors do not participate in either factor portfolio and we have limited participation in both markets.

## 5.4 Diversification discount

In this subsection, we show that our results on the diversification discount still hold in the general setting. Without loss of generality, we normalize the share supply of the conglomerate to one. Then the payoff of the conglomerate is  $r_M = x_1 r_1 + x_2 r_2$ . We summarize the results in the following proposition.

**Proposition 6.** *When there is full participation, there is no diversification discount, i.e.,  $P_M = x_1 P_1 + x_2 P_2$ . When there is limited participation, there is diversification discount, i.e.,  $P_M \leq x_1 P_1 + x_2 P_2$ . Moreover, the diversification discount is strictly positive when a positive measure of investors participate only in one of the factor portfolios.*

## 6. A Dynamic Model

It is well understood that in a standard single-period model with CARA utility, investors' asset demand is determined by the main-variance tradeoff, while in a dynamic model, the demand consists of the mean-variance trade-off component and a hedging component. So far, we have conducted our analysis in a single-trading session model and demonstrated that in equilibrium some investors may choose not to participate in the stock market. A natural question is: in a dynamic model where investors trade also for hedging purpose, will there still be limited market participation in equilibrium [Epstein and Schneider (2002)]? A full dynamic model is clearly beyond the scope of this paper. However, we will demonstrate in this section with a simple two-period model that while hedging need generates some interesting dynamics in investor's asset demand, limited market participation can still arise in equilibrium for the same reason as in the static model analyzed in previous sections.

There are two periods. Investors trade at times  $t = 0$  and 1 and consumption occurs at  $t = 2$ . There are one stock and one bond. The risk-free rate is set at zero. Each investor is endowed with  $x > 0$  shares of the stock as before. The stock pays dividends,  $r_1$  and  $r_2$  at times  $t = 1$  and 2, respectively. For simplicity, we assume that  $r_1$  and  $r_2$  are independent and normally distributed with variances  $\sigma_1^2$  and  $\sigma_2^2$ , respectively. There is a continuum of investors with multi-priors CARA expected utility function of the terminal wealth and the same risk-aversion coefficient  $\gamma$ . Investors are uncertain about the expected values of the dividends. Investor  $i$ 's uncertainty is described in the same spirit as in previous sections by two intervals,  $[\mu_1 - \phi_{1i}, \mu_1 + \phi_{1i}]$  and  $[\mu_2 - \phi_{2i}, \mu_2 + \phi_{2i}]$ , respectively. Thus

$$\begin{aligned}\mathcal{P}_i = \{Q : (r_1, r_2) \text{ is jointly normal under } Q \text{ and } E^Q[r_j] \\ \in [\mu_j - \phi_{ji}, \mu_j + \phi_{ji}], j = 1, 2\}.\end{aligned}$$

The difference in investors' uncertainty measured by  $(\phi_{1i}, \phi_{2i})$  is assumed to be uniformly distributed in a rectangle  $[\bar{\phi}_1 - \delta_1, \bar{\phi}_1 + \delta_1] \times [\bar{\phi}_2 - \delta_2, \bar{\phi}_2 + \delta_2]$ , with density  $1/(4\delta_1\delta_2)$ . Each investor also receives a non-tradable wealth at time  $t = 1$  in the amount of  $er_1 + \xi$ , with  $\xi$  being normally distributed and independent of  $r_1$  and  $r_2$ . This non-tradable wealth can be viewed as the human capital of the investor and is assumed to be correlated with the stock payoff to introduce a hedging motive for the investor. We assume that  $e > 0$ . The analysis is similar for  $e < 0$ . Finally, a public noisy signal  $y$  arrives in the market at  $t = 1$  with  $y = r_2 + \epsilon$ , where the noise term  $\epsilon$  is normally distributed with mean zero and variance  $\sigma_\epsilon^2$ .<sup>18</sup>

The optimization problem of investor  $i$  is to maximize his utility,

$$\min_{Q \in \mathcal{P}_i} E^Q[-\exp(-\gamma W_2)] \quad (27)$$

subject to the budget constraints

$$\begin{aligned} W_1 &= W_0 + er_1 + \xi + D_{0i}(P_1 + r_1 - P_0) \\ W_2 &= W_1 + D_{1i}(r_2 - P_1). \end{aligned}$$

To solve the utility maximization problem, compute for each  $Q \in \mathcal{P}_i$  the conditional distribution after observing the public signal  $y$  at  $t = 1$ ,

$\mathcal{P}_i(y) = \{Q(y) : Q \in \mathcal{P}_i, Q(y) \text{ is the conditional distribution of } (r_1, r_2) \text{ given } y\}$ .

It can be verified that  $\mathcal{P}_i$  and  $\mathcal{P}_i(y)$  together satisfy the generalized Bayes rule and hence the preference Equation (27) is dynamically consistent [Epstein and Schneider (2003) and Wang (2003)]. Consequently, we can solve the utility maximization problem by backward induction. At time 1, there is only one period left. Using the result from Section 2, investor  $i$ 's demand is

$$D_{1i} = \begin{cases} \frac{\sigma_2^2 + \sigma_\epsilon^2}{\gamma\sigma_2^2\sigma_\epsilon^2} \left( \frac{\sigma_\epsilon^2(\mu_2 - \phi_{2i}) + \sigma_2^2 y}{\sigma_2^2 + \sigma_\epsilon^2} - P_1 \right) & \text{if } \frac{\sigma_\epsilon^2\mu_2 + \sigma_2^2 y}{\sigma_2^2 + \sigma_\epsilon^2} - P_1 > \phi_{2i} \frac{\sigma_\epsilon^2}{\sigma_2^2 + \sigma_\epsilon^2}, \\ 0 & \text{if } \left| \frac{\sigma_\epsilon^2\mu_2 + \sigma_2^2 y}{\sigma_2^2 + \sigma_\epsilon^2} - P_1 \right| < \phi_{2i} \frac{\sigma_\epsilon^2}{\sigma_2^2 + \sigma_\epsilon^2}, \\ \frac{\sigma_2^2 + \sigma_\epsilon^2}{\gamma\sigma_2^2\sigma_\epsilon^2} \left( \frac{\sigma_\epsilon^2(\mu_2 - \phi_{2i}) + \sigma_2^2 y}{\sigma_2^2 + \sigma_\epsilon^2} - P_1 \right) & \text{if } \frac{\sigma_\epsilon^2\mu_2 + \sigma_2^2 y}{\sigma_2^2 + \sigma_\epsilon^2} - P_1 < -\phi_{2i} \frac{\sigma_\epsilon^2}{\sigma_2^2 + \sigma_\epsilon^2}. \end{cases} \quad (28)$$

Note that, conditional on  $y$ , the variance of  $r_2$  is  $\sigma_2^2\sigma_\epsilon^2/(\sigma_2^2 + \sigma_\epsilon^2)$  and expected return in worst case scenario is  $[\sigma_\epsilon^2(\mu_2 - \phi_{2i}) + \sigma_2^2 y]/(\sigma_2^2 + \sigma_\epsilon^2)$ . It can be verified that when  $\delta_2 > \gamma\sigma_2^2 x$ , some investors will choose not to participate in the stock market and the equilibrium price is given by

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<sup>18</sup> For simplicity, we assume that there is no uncertainty about the mean of the noise term.

$$P_1 = \frac{\sigma_\epsilon^2 [\mu_2 - \phi_{2p}] + \sigma_2^2 y}{\sigma_2^2 + \sigma_\epsilon^2} - \frac{\gamma x}{\alpha_{2p}} \frac{\sigma_2^2 \sigma_\epsilon^2}{\sigma_2^2 + \sigma_\epsilon^2}. \quad (29)$$

where

$$\alpha_{2p} = \sigma_2 \sqrt{x\gamma/\delta_2} \quad \text{and} \quad \phi_{2p} = \bar{\phi}_2 - \delta_2 + 2\sigma_2 \sqrt{\delta_2 x\gamma}$$

are the proportion of investors participating in the stock market and average uncertainty about the mean of  $r_2$  of the stock market participants, respectively. The pricing formula is the same as in Section 2 except that the mean and variance are their conditional counterparts.

We now turn to the portfolio choice and market-participation decision of investor  $i$  at time  $t = 0$ . Let  $J_i(W_1, y)$  denote the indirect utility function at  $t = 1$ . Using expression (28), it can be verified that

$$J_i(W_1, y) = -\exp \left\{ -\gamma W_1 - \frac{\sigma_\epsilon^2}{2\sigma_2^2(\sigma_2^2 + \sigma_\epsilon^2)} \left( \phi_{2p} - \phi_{2i} + \frac{\gamma x}{\alpha_{2p}} \sigma_2^2 \right)^2 \right\}.$$

At time  $t = 0$ , the utility maximization problem Equation (27) can be written as

$$\max_{D_{0i}} \min_{Q \in \mathcal{P}_i} E^Q[J_i(W_1, y)] \quad (30)$$

subject to  $W_1 = W_0 + er_1 + \xi + D_{0i}(P_1 + r_1 - P_0)$ , which is equivalent to

$$\begin{aligned} \max_{D_{0i}} D_{0i} & \left( \frac{\sigma_\epsilon^2 [\mu_2 - \phi_{2p}] + \sigma_2^2 [\mu_2 - sgn(D_{0i})\phi_{2i}]}{\sigma_2^2 + \sigma_\epsilon^2} - \frac{\gamma x}{\alpha_{2p}} \frac{\sigma_2^2 \sigma_\epsilon^2}{\sigma_2^2 + \sigma_\epsilon^2} - P_0 \right) \\ & + (D_{0i} + e)[\mu_1 - sgn(D_{0i} + e)\phi_{1i}] - \frac{\gamma}{2} \left( [D_{0i} + e]^2 \sigma_1^2 + D_{0i}^2 \frac{\sigma_2^4}{\sigma_2^2 + \sigma_\epsilon^2} \right). \end{aligned} \quad (31)$$

Before the formal analysis of investor  $i$ 's market-participation decision, it is useful to examine intuitively what gives rise to the hedging demand. The investor has a non-traded wealth risk at time  $t = 1$ . Naturally, he wants to hedge against this risk. Since the stock pays a dividend of  $r_1$  dollars which is perfectly correlated with a component of his non-traded wealth, a simple hedging strategy that can completely hedge this component is to sell  $e$  shares of the stock and unwind the position at time 1. Of course, whether the investor wants to completely hedge depends on his overall portfolio-choice problem. Nonetheless, the presence of non-traded wealth creates a hedging need that would make his  $t = 0$  demand for the stock different from his mean-variance demand.

Investor  $i$ 's market participation decision at time  $t = 0$  depends on the right and left derivatives of his utility function at  $D_{0i} = 0$ . The right derivative of the objective function in Equation (31) at  $D_{0i} = 0$  is

$$\frac{\sigma_\epsilon^2(\mu_2 - \phi_{2p}) + \sigma_2^2(\mu_2 - \phi_{2i})}{\sigma_2^2 + \sigma_\epsilon^2} - \frac{\gamma x}{\alpha_{2p}} \frac{\sigma_\epsilon^2 \sigma_2^2}{\sigma_2^2 + \sigma_\epsilon^2} + (\mu_1 - \phi_{1i}) - P_0 - \gamma e \sigma_1^2, \quad (32)$$

and the left derivative is

$$\frac{\sigma_\epsilon^2(\mu_2 - \phi_{2p}) + \sigma_2^2(\mu_2 - \phi_{2i})}{\sigma_2^2 + \sigma_\epsilon^2} - \frac{\gamma x}{\alpha_{2p}} \frac{\sigma_\epsilon^2 \sigma_2^2}{\sigma_2^2 + \sigma_\epsilon^2} + (\mu_1 - \phi_{1i}) - P_0 - \gamma e \sigma_1^2. \quad (33)$$

If the expression in Equation (32) is negative and that in Equation (33) is positive, then investor  $i$  will not participate in the stock market even in the presence of the hedging demand.

To gain insight into investor  $i$ 's market participation decision and to see how the extra trading motive from hedging need interacts with the investor  $i$ 's aversion to model uncertainty, it is useful to examine first the case without signal  $y$ . This is the case when  $\sigma_\epsilon = \infty$ . In this case, the price  $P_1$  is constant and hence there is no future price uncertainty. Also, Equations (32) and (33) are of the same sign. Thus, while there is uncertainty about the mean of the first dividend,  $r_1$ , hedging demand dominates the aversion to uncertainty and the investor always participates in the stock market, except for the uninteresting case where the two expressions in Equations (32) and (33) happen to be equal to zero. Now, in the presence of the noisy signal  $y$ , the price of the stock at time  $t = 1$  becomes random. From pricing Equation (29), it is readily seen that the first two terms in Equation (32) give the expectation of  $P_1$  in the worst case scenario while those in Equation (33) give that in the best case scenario. That is, when  $P_1$  is random, in addition to uncertainty about the dividend  $r_1$ , the uncertainty about the future price of the stock is something that the investor also has to be concerned about in his stock-holding decision at time zero. As a result, the investor faces a tradeoff: he can satisfy his hedging need, completely or partially, by taking a position in the stock, but by doing so, he exposes himself to the future price uncertainty. Because of his aversion to such uncertainty, he may choose not to participate in the stock market even in the presence of hedging need.

The analysis above is for individual investors. The following proposition provides a set of conditions under which there is limited market participation in equilibrium. The conditions are the counterpart of the condition for limited market participation in Section 2.

**Proposition 7.** Suppose that  $\sigma_1^2 \gamma x < \delta_1$  and  $\sigma_2^2 \gamma x < \delta_2$ . Then in equilibrium there is limited market participation in both trading periods of the stock market.

## 7. Conclusion

In this paper, we study the relation among three well-documented empirical phenomena, limited participation, the equity premium, and

the conglomerate diversification discount, in a general equilibrium framework with model uncertainty. We find that the three phenomena can be fundamentally connected through the heterogeneity in investors' uncertainty about the distribution of stock payoffs. Specifically, we find that when dispersion in investors' uncertainty about stock payoffs is large, there is limited market participation and equity premium can be lower than that in an economy with full market participation. This is in contrast to the understanding in the literature that limited participation leads to higher equity premium. We also find that diversification discount is related to limited market participation and larger dispersion in investors' uncertainty about the stock payoff leads to lower market participation and larger discount. This latter finding is novel and can be empirically tested in future research.

## Appendix

*Proof of Proposition 2.* The proof for the case without merger is very similar to the single stock case. Let  $\hat{x} = (1 + \rho)x$ . When  $\gamma\hat{x}\sigma^2 \geq \delta$ , there is full market participation. The prices for the factor portfolios are given by

$$P_A = P_B = \mu - \bar{\phi} - \gamma\hat{x}\sigma^2.$$

As a result we have

$$P_1 = P_A + \rho P_B = P_B + \rho P_A = P_2 = (1 + \rho)[\mu - \bar{\phi} - \gamma\hat{x}\sigma^2].$$

In the economy with merger, the maximization problem is

$$\max_{D_{iM}} D_{iM}[2(1 + \rho)\mu - (1 + \rho)sgn(D_{iM})(\phi_{iA} + \phi_{iB}) - P_M] - \gamma D_{iM}^2(1 + \rho)^2\sigma^2.$$

The demand function is thus

$$D_{iM} = \frac{2(1 + \rho)\mu - (1 + \rho)(\phi_{iA} + \phi_{iB}) - P_M}{2(1 + \rho)^2\gamma\sigma^2}.$$

When  $\gamma\hat{x}\sigma^2 \geq \delta$ , there is full market participation. Aggregating the demands across all investors and equating the aggregate demand to aggregate supply, we arrive at

$$x = \frac{2(1 + \rho)\mu - 2(1 + \rho)\bar{\phi} - P_M}{2(1 + \rho)^2\gamma\sigma^2},$$

which implies

$$P_M = 2(1 + \rho)[\mu - \bar{\phi} - \gamma\hat{x}\sigma^2] = (1 + \rho)(P_A + P_B) = P_1 + P_2,$$

and there is no diversification discount.

When  $\gamma\hat{x}\sigma^2 < \delta$ , in the economy without merger, there is limited participation. Following Equation (10), we get

$$P_A = P_B = \mu - (\bar{\phi} - \delta) - 2\sigma\sqrt{\gamma\hat{x}\delta}.$$

As a result, we have

$$P_1 = P_A + \rho P_B = P_B + \rho P_A = P_2 = (1 + \rho)[\mu - (\bar{\phi} - \delta) - 2\sigma\sqrt{\gamma\hat{x}\delta}]. \quad (\text{A1})$$

When  $\gamma\hat{x}\sigma < \delta$ , in the economy with merger, there is also limited participation. Thus, there exists a  $\hat{\phi}$  such that when  $\phi_{iA} + \phi_{iB} \geq \hat{\phi}$ , investor  $i$  will not invest in the conglomerate. This  $\hat{\phi}$  is determined by,

$$2(1 + \rho)\mu - (1 + \rho)\hat{\phi} - P_M = 0. \quad (\text{A2})$$

Notice that no investor will sell short the conglomerate and thus  $D_{iM} > 0$  for investors participating in the market. Consequently,  $\text{sgn}(D_{iM}) = 1$  for participating investors. Summing up investors' demand, we have

$$x = \int \int_{\phi_{iA} + \phi_{iB} < \hat{\phi}, \phi_i \in S_2} \frac{1}{8(1 + \rho)^2 \gamma \sigma^2 \delta^2} [2(1 + \rho)\mu - (1 + \rho)(\phi_{iA} + \phi_{iB}) - P_M] d\phi_{iA} d\phi_{iB}.$$

Utilizing equation (A2), we arrive at the following market clearing condition

$$(1 + \rho)\gamma\sigma^2 x = \int \int_{\phi_{iA} + \phi_{iB} < \hat{\phi}, \phi_i \in S_2} \frac{[\hat{\phi} - (\phi_{iA} + \phi_{iB})]}{8\delta^2} d\phi_{iA} d\phi_{iB} \equiv g(\hat{\phi}), \quad (\text{A3})$$

where

$$g(\hat{\phi}) = \frac{(\hat{\phi} - 2\bar{\phi} + 2\delta)^3 - 2[(\hat{\phi} - 2\bar{\phi})^+]^3}{48\delta^2}. \quad (\text{A4})$$

It is straightforward to verify that  $g$  is an increasing function of  $\hat{\phi}$ . Thus, there is a unique  $\hat{\phi}$  that satisfies Equation (A3) and the equilibrium price of the conglomerate is thus given by

$$P_M = 2(1 + \rho)[\mu - g^{-1}(\gamma\hat{x}\sigma^2)/2]. \quad (\text{A5})$$

It can be verified by tedious calculation that  $P_M < P_1 + P_2$ . Since the inequality is shown for a more general setting in Proposition 6, we omit the details here.

For the second half of the proposition, we prove the case for  $\rho = 0$ ,  $\gamma = 1$ ,  $x = 1$ ,  $\sigma = 1$ . The general case can be dealt with similarly. When  $\delta = 6$ , we have from Equation (A3) that  $g(\hat{\phi}) = 1$ . Also by Equation (A4),  $\delta/6 = g(2\bar{\phi})$ . Since  $g$  is an increasing function, we must have  $\hat{\phi} \leq 2\bar{\phi}$  for  $\delta \geq 6$ . Consequently,  $g(\hat{\phi}) = 1$  implies that  $\hat{\phi} = 2\sqrt[3]{6\delta^2} + 2(\bar{\phi} - \delta)$  for  $\delta \geq 6$ . By Equations (A1) and (A5), the diversification discount is  $2(\sqrt[3]{6\delta^2} - 2\sqrt{\delta})$ , which is increasing in  $\delta$ .

In the case where  $1 < \delta < 6$ , we have  $\hat{\phi} > 2\bar{\phi}$ . The diversification discount is  $z = \hat{\phi} - 2(\bar{\phi} - \delta) - 4\sqrt{\delta}$ . Substituting this back into  $g(\hat{\phi}) = 1$ , we have

$$F(z, \delta) \equiv (z + 4\sqrt{\delta})^3 - 2(z - 2\delta + 4\sqrt{\delta})^3 - 48\delta^2 = 0.$$

Define

$$\psi \equiv \sqrt{\delta}, \quad \omega \equiv \frac{z + 4\psi}{2\psi^2}.$$

Since  $0 < \hat{\phi} - 2\bar{\phi} < 2\delta, 2\delta < z + 4\psi < 4\delta$  and  $1 < \omega < 2$ . Substituting into  $F$ , we have

$$G(\omega, \psi) \equiv \omega^3 - 2(\omega - 1)^3 - \frac{6}{\psi^2} = 0.$$

Since  $G_\omega(\omega, \psi) = -3(\omega - 2)^2 + 6$ ,  $G_\omega > 0$  for  $1 < \omega < 2$ . Thus,  $G(\omega, \psi) = 0$  implicitly defines a function  $\psi \rightarrow \omega$  for  $1 < \psi < \sqrt{6}$  and  $1 < \omega < 2$ . From the definition of  $\omega$ , we have

$$\frac{dz}{d\psi} = 4(\psi\omega - 1) + 2\psi^2 \frac{d\omega}{d\psi} = 4(\psi\omega - 1) - 2\psi^2 \frac{G_\psi}{G_\omega} = 4 \left[ \psi\omega - 1 - \frac{2}{\psi(-\omega^2 + 4\omega - 2)} \right].$$

With some algebraic derivation, the inequality  $dz/d\psi > 0$  is equivalent to

$$\omega - \frac{\omega^3 - 2(\omega - 1)^3}{3(-\omega^2 + 4\omega - 2)} > \sqrt{\frac{\omega^3 - 2(\omega - 1)^3}{6}}. \quad (\text{A6})$$

From Equation (A6) we get

$$\left[ \omega - \frac{\omega^3 - 2(\omega - 1)^3}{3(-\omega^2 + 4\omega - 2)} \right]^2 > \frac{\omega^3 - 2(\omega - 1)^3}{6}. \quad (\text{A7})$$

Using one to subtract both sides of Equation (A7), we get

$$\left[ 1 - \omega + \frac{\omega^3 - 2(\omega - 1)^3}{3(-\omega^2 + 4\omega - 2)} \right] \left[ 1 + \omega - \frac{\omega^3 - 2(\omega - 1)^3}{3(-\omega^2 + 4\omega - 2)} \right] > \frac{6 - \omega^3 + 2(\omega - 1)^3}{6}. \quad (\text{A8})$$

Multiplying both sides of Equation (A8) by  $18/(2 - \omega)$ , we get

$$\frac{a(\omega)b(\omega)}{c(\omega)} < d(\omega), \quad (\text{A9})$$

where

$$\begin{aligned} a(\omega) &= -2\omega^3 + 3\omega^2 + 12\omega - 8, & b(\omega) &= \frac{9}{4} - \left( 2\omega - \frac{5}{2} \right)^2 \\ c(\omega) &= \left[ 2 - (2 - \omega)^2 \right]^2, & d(\omega) &= 3(6 - (2 - \omega))^2. \end{aligned}$$

It is easy to verify that  $a, c, d$  are increasing functions of  $\omega$  for  $1 < \omega < 2$ . The function  $b(\omega)$  is increasing for  $1 < \omega < 5/4$  but decreasing for  $5/4 \leq \omega < 2$  and maximized at  $\omega = 5/4$  with value  $9/4$ . Therefore, for  $1 < \omega \leq 9/8$ , we have

$$\frac{a(\omega)b(\omega)}{c(\omega)} < \frac{a(9/8)b(9/8)}{c(1)} < d(1) < d(\omega),$$

and, for  $9/8 < \omega \leq 3/2$ , we have

$$\frac{a(\omega)b(\omega)}{c(\omega)} < \frac{a(3/2)b(5/4)}{c(9/8)} < d(9/8) < d(\omega).$$

Finally, for  $3/2 < \omega \leq 2$ , we have

$$\frac{a(\omega)b(\omega)}{c(\omega)} < \frac{a(2)b(3/2)}{c(3/2)} < d(3/2) < d(\omega).$$

Consequently,  $z$  is an increasing function of  $\psi$  and thus an increasing function of  $\delta$ .

*Proof of Proposition 3.* The following information about  $\lambda_{iA}$  and  $\lambda_{iB}$  will be useful.  $\lambda_{iA} = 0$ , if  $D_{iA} > 0$ ,  $\lambda_{iA} = 2\phi_{iA}$  if  $D_{iA} < 0$  and  $0 \leq \lambda_{iA} \leq 2\phi_{iA}$  if  $D_{iA} = 0$ . Left multiply both sides of Equation 22 by  $(\mu - P + \lambda_i - \phi_i)^\top$ . Then the RHS is positive and the LHS is negative. Thus  $\mu - P + \lambda_i - \phi_i$  must be zero and  $D_i = 0$ . ■

*Proof of Proposition 4.* Notice that  $0 \leq \lambda_{iB} \leq 2\phi_{iB}$ . For existence, define the aggregate demand by the right side of Equations (24) and (25) and denote it by  $D(P)$ . Using the Maximum Theorem, it is readily shown that after the inner minimization, the objective function in Equation (20) is continuous and strictly concave in  $(D_A, D_B)$ . As a result, the demand function  $D(P)$  is continuous in  $P$ . Consider the following mapping

$$M(P) = P + \gamma\hat{\Omega}(D(P) - x) = [\mu - \bar{\phi} + \lambda(P)] - \gamma\hat{\Omega}x,$$

where

$$\lambda(P) = \int \int_{\phi_i \in S_3} \lambda_i(P) h(\phi_{iA}, \phi_{iB}) d\phi_{iA} d\phi_{iB}.$$

This mapping is continuous in  $P$ . Let

$$S = \left\{ [\mu - \bar{\phi} + \lambda] - \gamma\hat{\Omega}x, 0 \leq \lambda \leq 2(\bar{\phi} + \delta)(1,1)^\top \right\}.$$

Clearly,  $M(S) \in S$ . Applying the Brouwer's fixed point theorem, there exists a fixed point in which  $M(P) = P$ .

To prove uniqueness, first notice that investors' demand for portfolio  $A$  is decreasing in  $P_A$  and increasing in  $P_B$ . Suppose that there exists two sets of equilibrium prices,  $P \neq P'$ . Without loss of generality, we assume that  $P'_A > P_A$ . Let  $P'_B = P_B - \omega_{11}(P'_A - P_A)/\omega_{12}$ . Notice that  $\omega_{12}$  is off-diagonal element of  $(\gamma\hat{\Omega})^{-1}$  and is negative by assumption. Thus, we must have  $P'_B > P_B$ . We show that at price vector  $P^* = (P_A^*, P_B^*) = (P'_A, P'_B)$  the demand for factor portfolio  $A$  will be less than the supply. Suppose, to the contrary, that  $D_A^* > x_A = D_A$ . Then for some  $i$ ,  $D_{iA}^* > D_{iA}$ . It follows that  $\lambda_{iA} \geq \lambda_{iA}^*$  (This follows because  $D_{iA}^* > 0$  implies  $\lambda_{iA}^* = 0$ ;  $D_{iA}^* = 0$  implies  $D_{iA} < 0$  which implies that  $\lambda_{iA} = 2\phi_{iA}$ ; and finally  $D_{iA}^* < 0$  implies  $D_{iA} < 0$  which implies  $\lambda_{iA} = 2\phi_{iA} \geq \lambda_{iA}^*$ ). Noting that

$$0 > D_{iA} - D_{iA}^* = \omega_{11}(P_A^* - P_A + \lambda_A - \lambda_{iA}^*) + \omega_{12}(P_B^* - P_B + \lambda_B - \lambda_{iB}^*), \quad (\text{A10})$$

either we have a contradiction in the case when  $\omega_{12} = 0$ , or  $\lambda_{iB} > \lambda_{iB}^*$ , in the case  $\omega_{12} < 0$  (note that  $\omega_{12} \leq 0$ ) In the latter case,  $D_{iB} \leq 0$  and  $D_{iB}^* \geq 0$  hold. It follows that  $0 \geq D_{iB} - D_{iB}^*$ , and hence

$$-\omega_{21}(P_A^* - P_A + \lambda_A - \lambda_{iA}^*) > \omega_{22}(P_B^* - P_B + \lambda_B - \lambda_{iB}^*) > 0. \quad (\text{A11})$$

Noting that  $P_A^* > P_A$  and the two terms in the brackets of Equations (A10) and (A11) are all positive, we have  $\omega_{21}\omega_{21} > \omega_{11}\omega_{22}$ , which is a contradiction. Consequently, at price vector  $P^* = (P_A^*, P_B^*) = (P'_A - P_B)$  the demand for factor portfolio  $A$  will be less than the supply.

Since the supply is non-zero, with positive measure, some investors' demand curve for portfolio  $A$  must be strictly increasing with  $P_B$ . As a result, we have  $P'_B - P_B > P^*_B - P_B = -\omega_{11}(P'_A - P_A)/\omega_{12} > 0$ . Similarly, we can show that  $P'_A - P_A > -\omega_{22}(P'_B - P_B)/\omega_{12} > 0$ . These two inequalities together violates the positive definiteness of  $\hat{\Omega}^{-1}$ . Thus the equilibrium must be unique.

Next, we show that  $\mu_A > P_A$  and  $\mu_B > P_B$ . Since the supply of the factor portfolios are positive, there must be some investors who hold strictly long positions in factor portfolio  $A$ . Let  $i$  be such an investor. There are several cases. Suppose first that investor  $i$  also holds a strictly long position in factor portfolio  $B$ , we have

$$\mu - P = \gamma \hat{\Omega} D_i + \phi_i > 0.$$

Thus  $\mu_A - P_A > 0$  and  $\mu_B - P_B > 0$ .

Secondly, suppose that investor  $i$  holds positive amount in factor portfolio  $A$  and zero in factor portfolio  $B$ . Thus the equity premium for  $A$  must be positive as

$$\mu_A - P_A = \lambda \hat{\Omega}_{11} D_{iA} + \phi_{iA} > 0.$$

In the third scenario, suppose that investors who hold strictly long positions in factor portfolio  $A$  will only hold short positions in factor portfolio  $B$ . It follows that investors who hold long positions in factor portfolio  $B$  must hold non-positive positions in factor portfolio  $A$ . Suppose that there exists an investor  $j$  who hold strictly long positions in factor portfolio  $B$  but zero position in factor portfolio  $A$ , then we must have  $\mu_B - P_B = \gamma \hat{\Omega}_{22} D_{jB} + \phi_{jB} > 0$ . Moreover, we have  $\mu_A - P_A > \phi_{iA} > 0$ . Otherwise, we must have  $\mu_A - P_A - \phi_{iA} \leq 0$ , and

$$(\mu_A - P_A - \phi_{iA}, \mu_B - P_B + \phi_{iB}) D_i = (\mu_A - P_A - \phi_{iA}, \mu_B - P_B + \phi_{iB})(\gamma \hat{\Omega})^{-1} \\ \times (\mu_A - P_A - \phi_{iA}, \mu_B - P_B + \phi_{iB})^\top > 0,$$

and the RHS is strictly positive while the LHS is clearly negative.

Since investor  $i$  in the three cases above is arbitrary, we are left with the case where investors who hold strictly long positions in factor portfolio  $A$  hold short positions in factor portfolio  $B$ , and investors who hold long positions in factor portfolio  $B$  hold short positions in factor portfolio  $A$ . Let  $i$  be an investor who hold positive in  $A$ , negative in  $B$ , and let  $j$  be an investor who hold positive in  $B$  and negative in  $A$ . We have

$$(-\phi_{iA} - \phi_{jA}, \phi_{iB} + \phi_{jB})(D_i - D_j) = (-\phi_{iA} - \phi_{jA}, \phi_{iB} + \phi_{jB})(\gamma \hat{\Omega})^{-1} \\ (-\phi_{iA} - \phi_{jA}, \phi_{iB} + \phi_{jB})^\top.$$

The LHS is negative while RHS is strictly positive. Thus is a contradiction. Therefore, the last case is impossible. This finishes the proof that  $\mu_A - P > 0$ . Similarly,  $\mu_B - P_B > 0$ . ■

*Proof of Proposition 5.* It is easy to show that  $D_{iA}$  and  $D_{iB}$  are continuous functions of  $P_A$ ,  $P_B$ ,  $\phi_{iA}$ , and  $\phi_{iB}$ . Thus,  $\lambda_{iA}$  and  $\lambda_{iB}$  are also continuous functions of  $P_A$ ,  $P_B$ ,  $\phi_{iA}$ , and  $\phi_{iB}$ . We first prove that, in equilibrium, if almost all investors hold positions in both factor portfolios, then almost all investors will hold long positions in the factor portfolios. Suppose that there exist an investor  $i$  whose the demand for factor portfolio  $A$  is negative and thus  $\lambda_{iA}/\phi_{iA} = 2$ . Since the supply is positive, there must be an investor  $j$  whose demand for factor portfolio  $A$  is positive and thus  $\lambda_{jA}/\phi_{jA} = 0$ . Since  $\lambda_{iA}/\phi_{iA}$  is a continuous function of  $(\phi_{iA}, \phi_{iB})$ , there must exist an investor  $k$  such that  $0 < \lambda_{kA}/\phi_{kA} < 2$ . Moreover, since the inequality is strict, there exists an neighborhood of such investors

around  $k$ . This contradicts the assumption that almost all investors are participating in both factor portfolios.

Using the result just proved, it can be verified that if investors fully participate, the equilibrium prices for the factor portfolios are given by Equation (26) and that investors have strictly positive demand Equation (22) for both factor portfolios and markets clear. This proves part (a).

For (b) and (c), suppose that some investors do not participate in at least one of the factor portfolios. Multiplying Equation 22 by  $\gamma\hat{\Omega}$ , integrating and using Equation (26), we get

$$P - P^f = \lambda \equiv \int \int_{\phi_i \in S_3} \lambda_i h(\phi_{iA}, \phi_{iB}) d\phi_{iA} d\phi_{iB} \geq 0,$$

with at least one strict inequality.  $\blacksquare$

*Proof of Proposition 6.* Let  $x = (x_A, x_B)^\top$ ,  $P = (P_A, P_B)^\top$ . Let

$$\begin{bmatrix} r_A \\ r_B \end{bmatrix} = (\beta^\top)^{-1} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix},$$

which is the payoff vector of the factor portfolios. Clearly,  $x_1 r_1 + x_2 r_2 = x_A r_A + x_B r_B$  and  $x_1 P_1 + x_2 P_2 = x_A P_A + x_B P_B$ . So, we simply view the merged firm as having payoff  $x_A r_A + x_B r_B$  and give the proof in terms of factor portfolios.

Denote  $\mu_M \equiv x^\top \mu$ . We deal with the full participation case first. When the firms are merged,

$$1 = \int \int_{\phi_i \in S_3} D_{iM} d\phi_{iA} d\phi_{iB} = \int \int_{\phi_i \in S_3} (x^\top \hat{\Omega} x)^{-1} [\mu_M - P_M^f - x^\top \phi_i + \lambda_{iM}] d\phi_{iA} d\phi_{iB}.$$

As in single stock,  $\lambda_{iM} = 0$  in equilibrium. When the two stocks are traded separately, as shown in the proof of Theorem 5,  $\lambda_{iA} = \lambda_{iM} = 0$ , and hence

$$\gamma \hat{\Omega} x = \mu - P - \bar{\phi}, \quad \gamma x^\top \hat{\Omega} x = \mu_M - P_M - x^\top \bar{\phi}.$$

It is easy to check that our equilibrium demand satisfies the market-clearing condition.

With limited market participation, we have  $P = P^f + \lambda$  and  $P_M = P'_M + \lambda_M$ , where  $\lambda_M \equiv \int_{\phi_i \in S_3} \lambda_{iM} h(\phi_{iA}, \phi_{iB}) d\phi_{iA} d\phi_{iB}$ . We will show that at price  $\hat{P}_M = x^\top P$ , the market demand for the bundled stock is less than one. Since the demand curve is downward sloping, in equilibrium,  $P_M < \hat{P}_M = x^\top P$ .

We first prove that for all investors, the following condition holds:

$$\lambda_{iM} \leq x^\top \lambda_i. \tag{A12}$$

First of all, expression (A12) must hold when the demand for the merged firm is positive for investor  $i$  as  $\lambda_{iM} = 0$ . Now consider the case where investor  $i$  holds zero positions in both factor portfolios. In this case, the investor must hold zero positions in the merged firm and

$$0 = x^\top (\lambda \hat{\Omega}) D_i = \mu_M - \hat{P}_M - x^\top \phi_i + x^\top \lambda_i = \mu_M - \hat{P}_M - x^\top \phi_i + \lambda_{iM}.$$

Thus  $\lambda_{iM} = x^\top \lambda_i$ , and condition (A12) holds too.

We are left with case in which investors hold zero positions in the merged firm but non-zero positions when the stocks are traded separately. In the economy without the merger, since the excess payoffs of the two factor portfolios are both positive (Theorem 4), it is

impossible for any investor to hold negative positions in both factor portfolios or zero in one factor portfolio and negative in the other (recall that each component of  $\hat{\Omega}$  is positive). Moreover, it is not possible for investor  $i$ 's optimal demand for the two factor portfolios to be both positive. Otherwise, we have

$$D_{iM} = \left( x^\top \hat{\Omega} x \right)^{-1} \left[ \mu_M - P_M^f - x^\top \phi_i + \lambda_{iM} \right] > 0,$$

which contradicts with the assumption that  $D_{iM} = 0$ . Thus, the remaining case is that investor  $i$ 's holding is strictly positive for at least one factor portfolio and non-positive for the other. Without loss of generality, suppose that investor  $i$ 's for factor portfolio  $B$  is positive. Then, we must have  $\mu_B - P_B - \phi_{iB} > 0$ . Otherwise, multiplying both sides of Equation (22) by  $\mu - P - \phi_i + \lambda_i$ , would lead to a contradiction. Now suppose that investor  $i$ 's demand for factor portfolio  $A$  is negative. Then  $\lambda_{iB} = 0$ ,  $\lambda_{iA} = 2\phi_{iA}$ . Since  $D_{iM} = 0$ , we have  $\lambda_{iM} = -x^\top (\mu - P - \phi_i) < x_A [\phi_{iA} - (\mu_A - P_A)] < x_A \phi_{iA} < 2\phi_{iA} x_A = x^\top \lambda_i$ . Finally, if investor  $i$  holds zero position in factor portfolio  $A$ , then

$$\gamma \hat{\Omega}_{12} D_{iB} = \mu_A - P_A - \phi_{iA} + \lambda_{iA}, \quad \gamma \hat{\Omega}_{12} D_{iB} = \mu_B - P_B - \phi_{iB}.$$

Dividing yields  $\lambda_{iA} = -(\mu_A - P_A - \phi_{iA}) + \hat{\Omega}_{12}(\mu_B - P_B - \phi_{iB})/\hat{\Omega}_{22}$ . Hence

$$\begin{aligned} x_A \lambda_{iA} + x_B \lambda_{iB} &= -x_A (\mu_A - P_A - \phi_{iA}) + x_A \hat{\Omega}_{12} (\mu_B - P_B - \phi_{iB})/\hat{\Omega}_{22} \\ &> -x_A (\mu_A - P_A - \phi_{iA}) > -x^\top (\mu - P - \phi_i) = \lambda_{iM}. \end{aligned}$$

To summarize, condition (A12) holds for all cases. Consequently, the aggregate demand for the merged firm at price  $\hat{P}_M$  is:

$$\begin{aligned} D_M &= \int \int_{\phi_i \in S_3} D_{iM} h(\phi_{iA}, \phi_{iB}) d\phi_{iA} d\phi_{iB} = \left( x^\top \hat{\Omega} x \right)^{-1} [x^\top (\mu - P - \bar{\phi}) + \lambda_M] \\ &\leq \left( x^\top \hat{\Omega} x \right)^{-1} [x^\top (\mu - P - \bar{\phi}) + \lambda] = \left( x^\top \hat{\Omega} x \right)^{-1} \left( x^\top \hat{\Omega} x \right) = 1. \end{aligned} \quad (\text{A13})$$

Thus when there is limited participation, the market demand is less than 1 for the conglomerate at price  $\hat{P}_M = x^\top P$ . For the market to clear, the price of the conglomerate must be larger than  $x^\top P$ . Moreover, the inequality holds strictly when a positive measure of investors holds long positions in one factor portfolio but not the other. ■

*Proof of Proposition 7.* It can be verified from Equation (31) that the demand function of investor  $i$  is continuous and monotonic in  $P_0$ . Thus it is readily shown that equilibrium always exists. We only need to show that limited participation occurs under stated conditions. Suppose, to the contrary, that there is full participation in equilibrium. We first show that under the conditions of the theorem, if there is full market participation, then the equilibrium price must be given by

$$P_0 = \mu_1 - \bar{\phi}_1 + \mu_2 - \frac{\sigma_2^2 \bar{\phi}_2 + \sigma_\epsilon^2 \phi_{2p}}{\sigma_2^2 + \sigma_\epsilon^2} - \gamma(x + e)\sigma_1^2 - \frac{\gamma x \sigma_2^4}{\sigma_2^2 + \sigma_\epsilon^2} - \frac{\gamma x \sigma_2^2 \sigma_\epsilon^2}{\alpha_{2p}(\sigma_2^2 + \sigma_\epsilon^2)}. \quad (\text{A14})$$

It is readily verified that if every investor holds a long position in the stock, then Equation (A14) obtains from integration. So we show below that no investors in the full participation equilibrium hold short positions.

By Equation (31), investor  $i$ 's demand function is given by

$$D_{0i}(\phi_{1i}, \phi_{2i}, P_0) = \begin{cases} \frac{\mu_1 - \phi_{1i} + \frac{\sigma_e^2(\mu_2 - \phi_{2p}) + \sigma_2^2(\mu_2 - \phi_{2i})}{\sigma_2^2 + \sigma_e^2} - \gamma x \frac{\sigma_2^2 + \sigma_e^2}{\alpha_{2p}(\sigma_2^2 + \sigma_e^2)} - P_0}{\gamma \sigma_1^2 + \gamma \sigma_2^4 / (\sigma_2^2 + \sigma_e^2)} - e & \text{if } D_{0i} > 0 \\ 0 & \text{if } D_{0i} = 0 \\ \frac{\mu_1 - \phi_{1i} + \frac{\sigma_e^2(\mu_2 - \phi_{2p}) + \sigma_2^2(\mu_2 - \phi_{2i})}{\sigma_2^2 + \sigma_e^2} - \gamma x \frac{\sigma_2^2 + \sigma_e^2}{\alpha_{2p}(\sigma_2^2 + \sigma_e^2)} - P_0}{\gamma \sigma_1^2 + \gamma \sigma_2^4 / (\sigma_2^2 + \sigma_e^2)} - e & \text{if } D_{0i} < 0. \end{cases} \quad (\text{A15})$$

Suppose that there are some investors who in the full participation equilibrium takes a short position. That is, there is at least one  $(\hat{\phi}_{1i}, \hat{\phi}_{2i})$  for which  $D_{0i} < 0$ . Since  $x > 0$ , in equilibrium, there is also at least one investor who holds a long position, that is, there is at least one  $(\check{\phi}_{1i}, \check{\phi}_{2i})$  for which  $D_{0i}(\check{\phi}_{1i}, \check{\phi}_{2i}, P_0) > 0 > D_{0i}(\hat{\phi}_{1i}, \hat{\phi}_{2i}, P_0)$ . Then, the continuity of  $D_{0i}$  in  $(\phi_{1i}, \phi_{2i})$  implies that there is some  $(\phi_{1i}, \phi_{2i})$  such that  $D_{0i}(\phi_{1i}, \phi_{2i}, P_0) = 0$ . In fact, by Equation (A15), there must be a non-zero measure set of such  $(\phi_{1i}, \phi_{2i})$ . This is contrary to the full participation assumption.

Next we show that at the price given in Equation (A14), some investors would not hold long positions either. For the investor  $i$  with  $\phi_{1i} = \bar{\phi}_1 + \delta_1$  and  $\phi_{2i} = \bar{\phi}_2 + \delta_2$ , the right derivative of the objective function [Equation (31)] at  $D_{0i} = 0$  is,

$$-\delta_1 - \frac{\sigma_2^2 \delta_2}{\sigma_2^2 + \sigma_e^2} + \gamma x \sigma_1^2 + \frac{\gamma x \sigma_2^4}{\sigma_2^2 + \sigma_e^2}, \quad (\text{A16})$$

which by the assumption of the proposition is strictly negative. Thus, this investor and investors whose uncertainty levels are very close to this investor will not hold long positions in the stock. This is a contradiction to full participation. ■

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