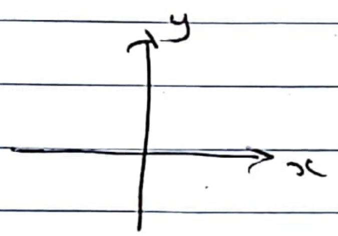
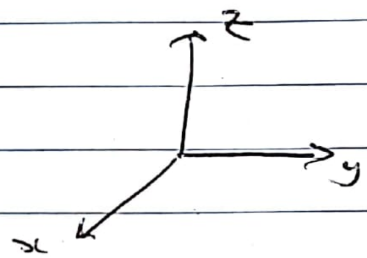
12.1, 12.2 EUCLIDEAN SPACE AND VECTORS

UNIVERSE

THE PLANE, $\mathbb{R}^2$ or SPACE, $\mathbb{R}^3$





POINTS

PHYSICS DEF^N: A POINT, p, IS A POSITION IN $\mathbb{R}^n$ $(n = 2$ or $3)$

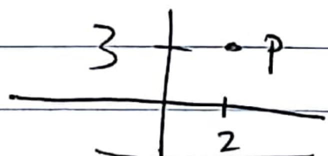
COMPUTER SCIENCE (CS) DEF^N:

A POINT, p, IS AN ARRAY (ORDERED LIST) OF n REAL NUMBERS $(n = 2$ or $3)$

$\mathbb{R}^2$ $\quad p = \begin{pmatrix} x \\ y \end{pmatrix}$

EX $\quad p = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\boxed{\text{CS}}$

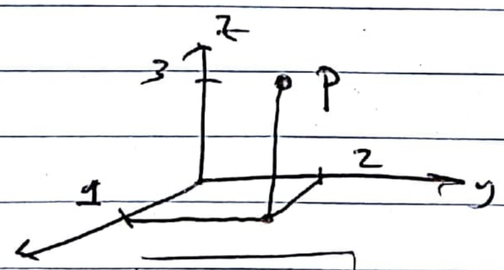


$\boxed{\text{PHYSICS}}$

$\mathbb{R}^3$ $\quad p = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$p = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

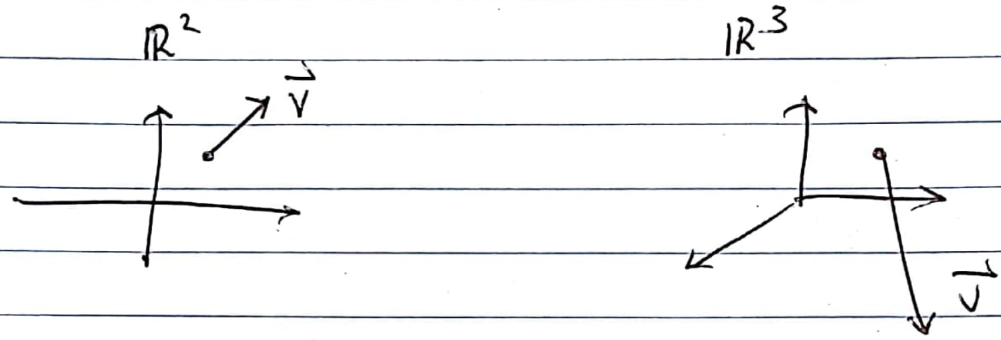
$\boxed{\text{CS}}$



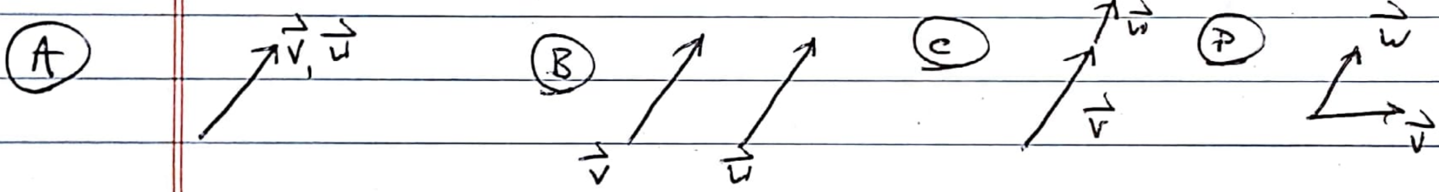
$\boxed{\text{PHYSICS}}$

VECTORS

PHYSICS DEF: A VECTOR, $\vec{v}$, is a quantity with MAGNITUDE and DIRECTION



QUESTION Which pairs of vectors are the same?



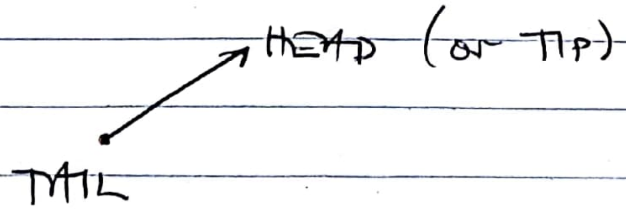
ANSWER A, B : In both cases $\vec{v}, \vec{w}$ have same magnitude + direction, so they are same vector.

In B it does not matter that $\vec{v}, \vec{w}$ have different starting (base) points.

C Different magnitudes $\Rightarrow \vec{v} \neq \vec{w}$

D Different directions $\Rightarrow \vec{v} \neq \vec{w}$.

PHYSICS TERMINOLOGY



CS DEFN of Vector is same as CS defn of point.

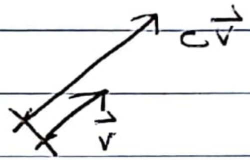
But vectors have special properties:

2 FUNDAMENTAL PROPERTIES OF VECTORS

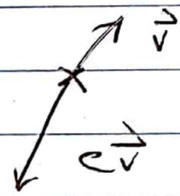
(a) SCALAR MULTIPLICATION

Given $\vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ There is a vector $c\vec{v} \in \mathbb{R}^n$ with

PHYSICS $\boxed{c>0}$



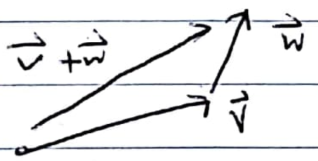
$\boxed{c<0}$



CS $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, $c\vec{v} = \begin{pmatrix} ca \\ cb \end{pmatrix}$

(b) VECTOR ADDITION

PHYSICS



CS IF $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\vec{w} = \begin{pmatrix} c \\ d \end{pmatrix}$

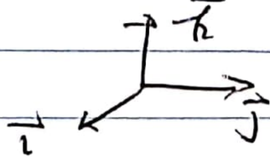
Then $\vec{v} + \vec{w} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$

SPECIAL VECTORS

(1) $\vec{i}, \vec{j}, \vec{k}$

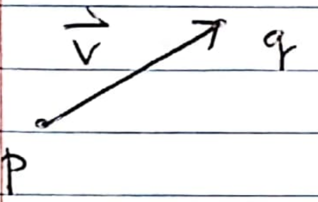
PHYSICS MAGNITUDE = 1, DIRECTED ALONG x, y, z AXES

CS $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



③ <u>DISPLACEMENT VECTOR</u> from point, p, to point, q

<u>PHYSICS</u>



<u>CS</u>

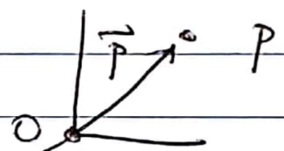
$\vec{v} = q - p$

<u>EX</u>

$p = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad q = \begin{pmatrix} 3 \\ 6 \end{pmatrix}, \quad \vec{v} = q - p = \begin{pmatrix} 3-1 \\ 6-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

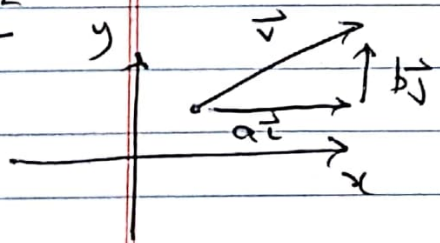
③ <u>POSITION VECTOR</u>, $\vec{p}$, of point p is Displacement vector from origin to p:

$\vec{p} = p - 0 = p$



CONVERTING FROM PHYSICS TO CS DEFNS

<u>$\mathbb{R}^2$</u>



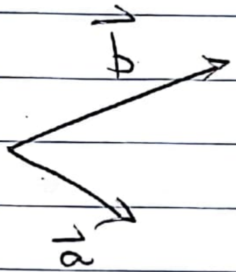
Any $\vec{v} \in \mathbb{R}^2$ can be expressed as

$\vec{v} = a\vec{i} + b\vec{j}$ for some $a, b \in \mathbb{R}$

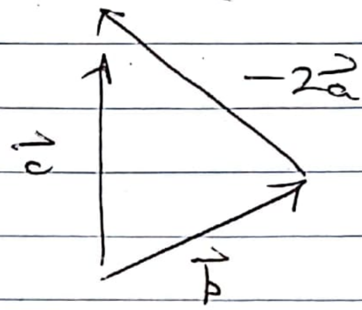
<u>CS</u> $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$

QUESTIONS USE THE 2 FUNDAMENTAL PROPERTIES OF VECTORS TO SOLVE:

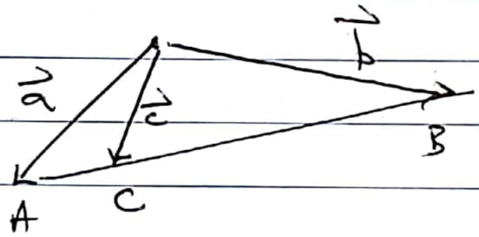
① DRAW $\vec{c} = \vec{b} - 2\vec{a} = \vec{b} + (-2\vec{a})$



SOLN

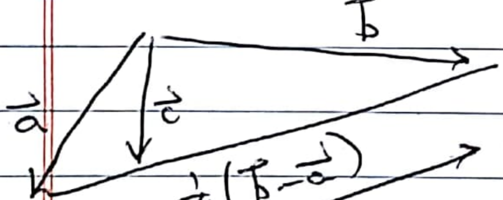


② FIND FORMULA for $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$ given



$|AC| = \frac{1}{4}|AB|$

SOLN



$\vec{c} = \vec{a} + \frac{1}{4}(\vec{b} - \vec{a})$

$$\boxed{\vec{c} = \frac{3}{4}\vec{a} + \frac{1}{4}\vec{b}}$$

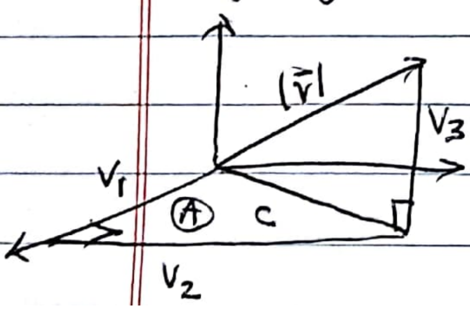
— o —

LENGTH OF $\vec{v}$ is $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

PF Use Pythagoras Twice

Ⓐ $c^2 = v_1^2 + v_2^2$

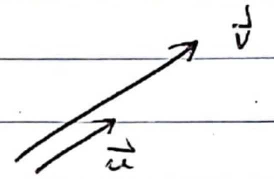
Ⓑ $|\vec{v}|^2 = c^2 + v_3^2 = v_1^2 + v_2^2 + v_3^2$.



CONVERTING FROM CS TO PHYSICS DEFNS

CS Suppose $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

PHYSICS
- MAGNITUDE of $\vec{v}$ = LENGTH of $\vec{v}$ $= |\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- DIRECTION of $\vec{v}$ is $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|}$



NOTE
- $\vec{u}$ and $\vec{v}$ are parallel and point in same direction
- $|\vec{u}| = 1$

$$\boxed{\text{PROOF THAT } |\vec{u}| = 1}$$

- PROPERTY OF LENGTH $|c\vec{v}| = |c|\,|\vec{v}|$

as $|c\vec{v}| = \sqrt{(cv_1)^2 + (cv_2)^2 + (cv_3)^2} = \sqrt{c^2}\sqrt{v_1^2 + v_2^2 + v_3^2}$
$= |c|\,|\vec{v}|$

- So

$$|\vec{u}| = \left|\frac{1}{|\vec{v}|}\vec{v}\right| = \frac{1}{|\vec{v}|}|\vec{v}| \quad \text{using } c = \frac{1}{|\vec{v}|}$$
$$= 1$$

NOTE

$$\vec{v} = |\vec{v}| \cdot \frac{\vec{v}}{|\vec{v}|}$$

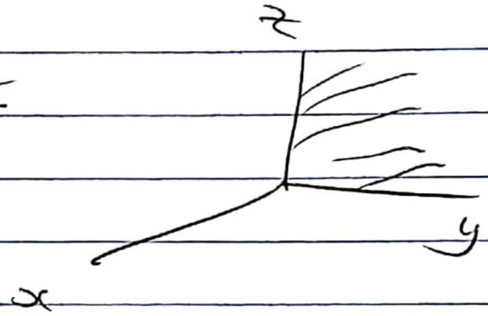
VECTOR = (LENGTH) x (DIRECTION)

INTRODUCTION TO SURFACES IN $\mathbb{R}^3$

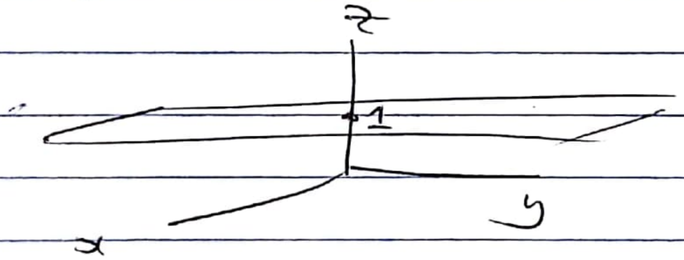
THE SET OF POINTS $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ SATISFYING ONE EQUATION IN THE 3 UNKNOWNS x, y, z FORMS A 2-DIMENSIONAL (2D) SURFACE IN $\mathbb{R}^3$.

EXS

① $x = 0$ IS yz-PLANE



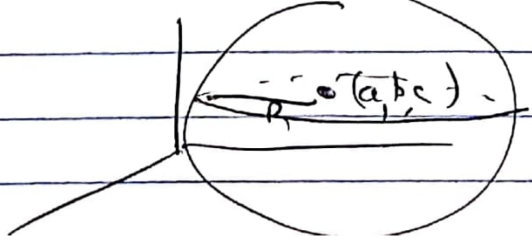
② $z = 1$ IS HORIZONTAL PLANE $\parallel$ TO xy-PLANE



③ $(x-a)^2 + (y-b)^2 + (z-c)^2 = R^3$

IS SET OF POINTS (x,y,z) whose distance from (a,b,c) is R.

SPHERE center (a,b,c) radius R



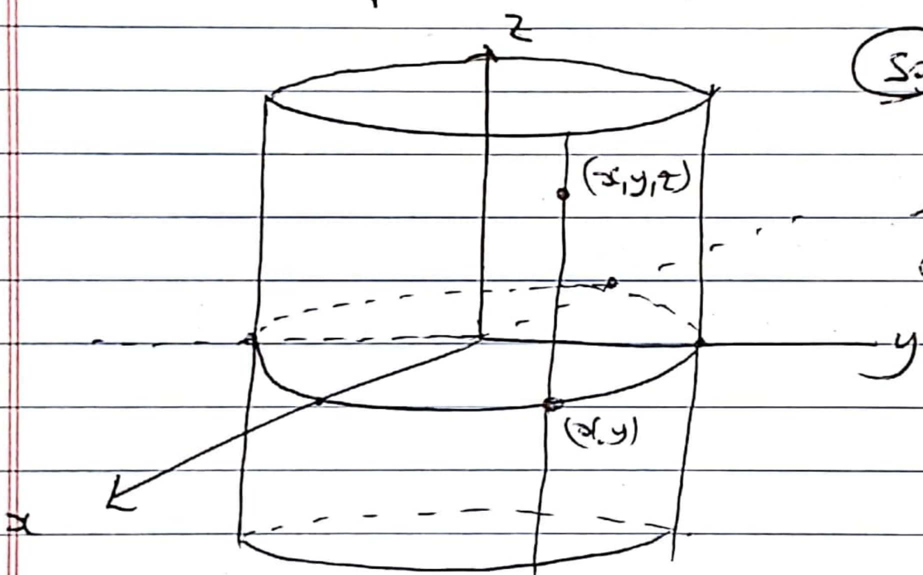
Q

(4) What is surface, S, consisting of all (8)

pts $(x, y, z) \in \mathbb{R}^3$ with $x^2 + y^2 = 1$?

A $(x, y, z) \in S$ precisely when shadow

(x, y) in plane $z = 0$ lies on circle $x^2 + y^2 = 1$



So

S is curved surface of ∞ cylinder