12.3 THE DOT PRODUCT

DEF (CC) The DOT PRODUCT of vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ is the real #

$$\boxed{\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3}$$

EX $(1, 2, 3) \cdot (4, 5, 6) = 4 + 10 + 18 = 32$

SUPER EASY TO CALCULATE!

Q WHY DO WE CARE?

A The dot product can be used to
- (a) Find angle between 2 vectors
- (b) Test if $\vec{u} \perp \vec{v}$
- (c) Find the projection (shadow) of $\vec{u}$ onto $\vec{v}$.

3 FUNDAMENTAL ALGEBRAIC PROPERTIES

(1) $|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$ LENGTH FORMULA

(2) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ DISTRIBUTIVE LAW

(3) $\vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v}$ COMMUTATIVITY or SYMMETRY LAW.

PF

1. $|\vec{u}|^2 = u_1^2 + u_2^2 + u_3^2 = \vec{u} \cdot \vec{u}$

2. , 3. Follows from distributive + commutativity laws for real numbers.

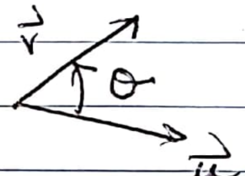
□

We can use these properties to do ALGEBRA WITH VECTORS, ie VECTOR ALGEBRA

For example, we can prove

THM (PHYSICS DEF OF DOT PRODUCT)

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$$



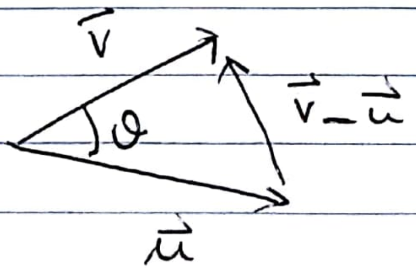
$\boxed{0 \leq \theta \leq \pi}$

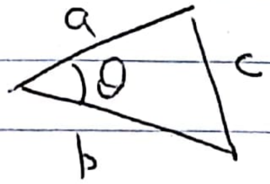
$\boxed{Q}$ How do you measure θ?

$\boxed{A}$ Place a protractor in the plane determined by $\vec{u}, \vec{v}$.

PF OF THM Uses Vector Algebra + Law of Cosines

$c^2 = a^2 + b^2 - 2ab\cos\theta$





So
$$|\vec{v}-\vec{u}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$

or
$$|\vec{u}||\vec{v}|\cos\theta = \frac{1}{2}\left[|\vec{u}|^2 + |\vec{v}|^2 - |\vec{v}-\vec{u}|^2\right]$$
$$= \frac{1}{2}\left[|\vec{u}|^2 + |\vec{v}|^2 - (\vec{v}-\vec{u})\cdot(\vec{v}-\vec{u})\right]$$
$$\overset{(2)}{=} \frac{1}{2}\left[|\vec{u}|^2 + |\vec{v}|^2 - (\vec{v}\cdot\vec{v} - \vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u} + \vec{u}\cdot\vec{u})\right]$$
$$\overset{(1),(3)}{=} \frac{1}{2}\left[|\vec{u}|^2 + |\vec{v}|^2 - (|\vec{v}|^2 - 2\vec{u}\cdot\vec{v} + |\vec{u}|^2)\right]$$
$$= \vec{u}\cdot\vec{v}. \qquad \square$$

Using Thm we have

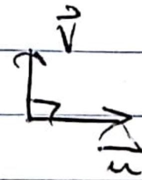
3 FUNDAMENTAL GEOMETRIC PROPERTIES

(4) If $|\vec{u}| = |\vec{v}| = 1$ Then $\vec{u}\cdot\vec{v} = \cos\theta$

$\boxed{\text{DOT PRODUCT IS COSINE IN DISGUISE}}$

(5) TEST FOR ORTHOGONALITY

Suppose $\vec{u} \neq \vec{0}$, $\vec{v} \neq \vec{0}$.
Then
$$\boxed{\vec{u} \perp \vec{v} \iff \vec{u}\cdot\vec{v} = 0.}$$



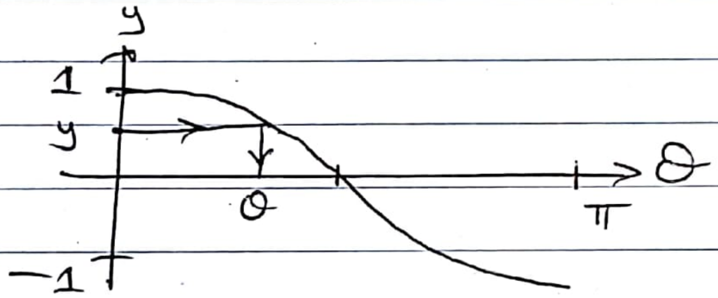
REASON $\quad \cos \pi/2 = 0.$

6 IF $\vec{u}, \vec{v}$ are both non-zero THEN

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}$$

or

$$\boxed{\theta = \arccos\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right)}$$



$y = \cos\theta$
$\theta = \arccos(y)$

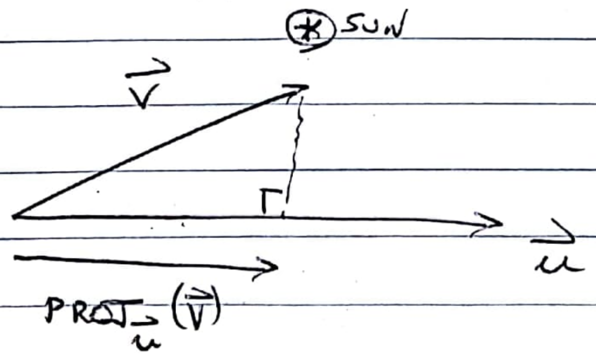
NOTE By def^n of arccos we always have

$$0 \le \theta \le \pi \qquad \text{Good}$$

PROJECTIONS (SHADOWS)

DEF

THE VECTOR PROJECTION of $\vec{v}$ ONTO $\vec{u}$ IS the vector $\text{PROJ}_{\vec{u}}(\vec{v})$ GIVEN BY

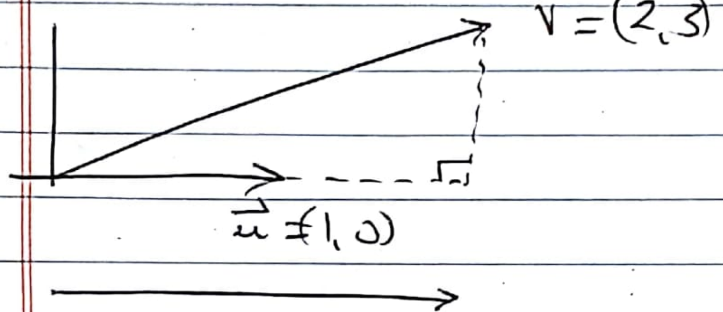


NOTE

$\text{PROJ}_{\vec{u}}(\vec{v}) \parallel \vec{u}$

ANS

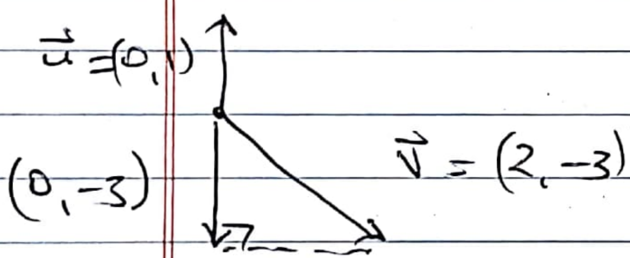
(A) $PROJ_{(1,0)}(2,3) = ?$



$PROJ_{\vec{u}}(\vec{v}) = (2, 0)$

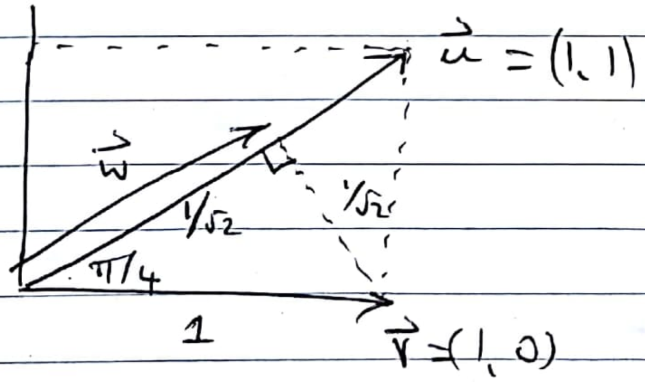
(B) $PROJ_{(0,1)}(2, -3) = ?$

ANS $= (0, -3)$



(C) $PROJ_{(1,1)}(1, 0) = ? = \vec{w}$

$|\vec{w}| = \frac{1}{\sqrt{2}}$



DIRN OF $\vec{w} = \frac{\vec{u}}{|\vec{u}|}$

$= \frac{1}{\sqrt{2}}(1,1)$

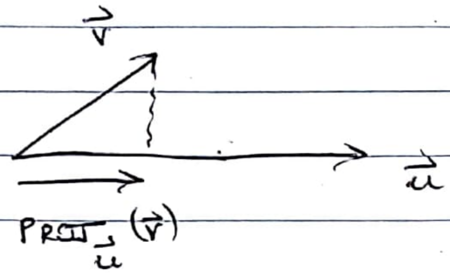
So

$\vec{w} = |\vec{w}| \frac{\vec{w}}{|\vec{w}|}$

$= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}(1,1) = \left(\frac{1}{2}, \frac{1}{2}\right)$

(*) SUN.

FORMULAE FOR PROJECTIONS



VECTOR PROJECTION:

$$\text{PROJ}_{\vec{u}}(\vec{v}) = \underbrace{\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}}_{\text{SCALAR}} \times \underbrace{\frac{\vec{u}}{|\vec{u}|}}_{\text{DIRECTION}} = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\,\vec{u}$$

SCALAR x DIRECTION

SCALAR PROJECTION / COMPONENT

$$\text{COMP}_{\vec{u}}(\vec{v}) = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|}$$

EX

(B) $$\text{COMP}_{(0,1)}(2,-3) = \frac{(0,1)\cdot(2,-3)}{|(0,1)|} = \frac{-3}{1} = -3$$

(C) $$\text{PROJ}_{(1,1)}(1,0) = \frac{(1,1)\cdot(1,0)}{|(1,1)|}\,\frac{(1,1)}{|(1,1)|}$$

$$= \frac{(1,1)\cdot(1,0)}{|(1,1)|^2}\,(1,1) \qquad \text{(BETTER)}$$

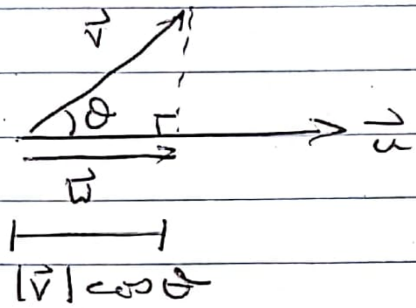
$$= \frac{1}{2}(1,1) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

as before!!

WHY IS PROJ^N FORMULA TRUE?

$\boxed{\text{CASE} \quad 0 < \theta < \pi/2}$

LET $\quad \vec{w} = \text{PROJ}_{\vec{u}}(\vec{v})$



$$|\vec{w}| = |\vec{v}|\cos\theta$$

$$\text{DIRN}(\vec{w}) = \frac{\vec{u}}{|\vec{u}|}$$

So $\quad \vec{w} = |\vec{w}| . \text{DIRN}(\vec{w}) = |\vec{v}|\cos\theta \, \frac{\vec{u}}{|\vec{u}|}$

$$= \frac{|\vec{u}|\,|\vec{v}|\cos\theta}{|\vec{u}|^2}\,\vec{u}$$

$$= \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\,\vec{u}$$