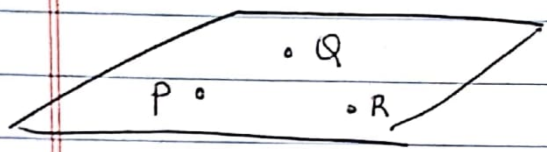
12.5B PLANES.

3 WAYS TO SPECIFY A PLANE

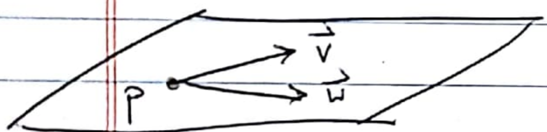
CONVERSIONS

Ⓐ 3 POINTS



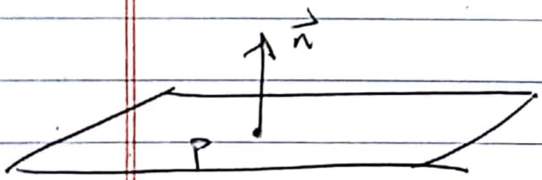
$$Q = P + \vec{v}$$
$$R = P + \vec{w}$$

Ⓑ POINT + 2 VECTORS



$$\vec{v} = Q - P$$
$$\vec{w} = R - P$$

Ⓒ POINT + NORMAL VECTOR



$$\vec{n} = \vec{w} \times \vec{v}$$

3 FORMULAE FOR A PLANE

Ⅰ PARAMETRIZATION $\vec{r}(s,t) = \vec{p} + s\vec{v} + t\vec{w}$ $\quad s,t \in \mathbb{R}$

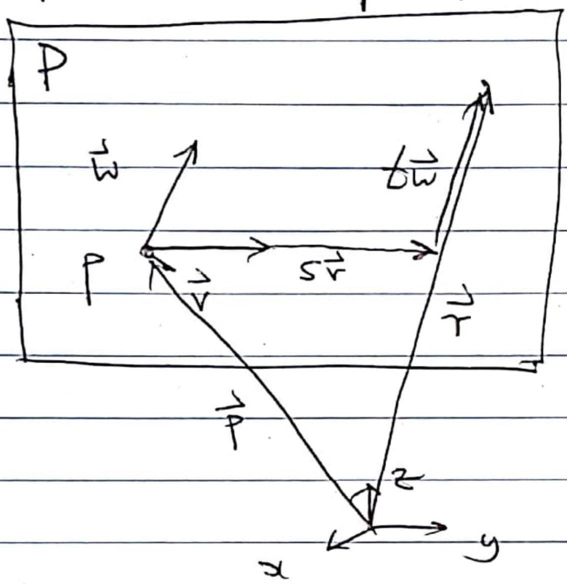
Ⅱ LEVEL SURFACE EQN $(\vec{r} - \vec{p}) \cdot \vec{n} = 0$

or $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Ⅲ GRAPH OF A FUNCTION $z = Ax + By + C$

I PARAMETRIZATIONS OF PLANES

Let P be plane thru pt p containing vectors $\vec{v}, \vec{w}$



If $\vec{r}$ is any point in P then there are reals s, t so that

$$\boxed{\vec{r}(s,t) = \vec{p} + s\vec{v} + t\vec{w}.}$$

Use Vector Add^n!

EX 1 Parametrize the plane thru the 3 points
$P = (1,2,3)$, $Q = (2,4,5)$, $R = (-1,3,-7)$

$\vec{p} = (1,2,3)$
$\vec{v} = Q - P = (1,2,2)$ $\quad \vec{w} = R - P = (-2,1,-10)$

So

$$\vec{r}(s,t) = (1,2,3) + s(1,2,2) + t(-2,1,-10)$$
$$= (1+s-2t,\ 2+2s+t,\ 3+2s-10t)$$

VECTOR PARAMETRIC EQN

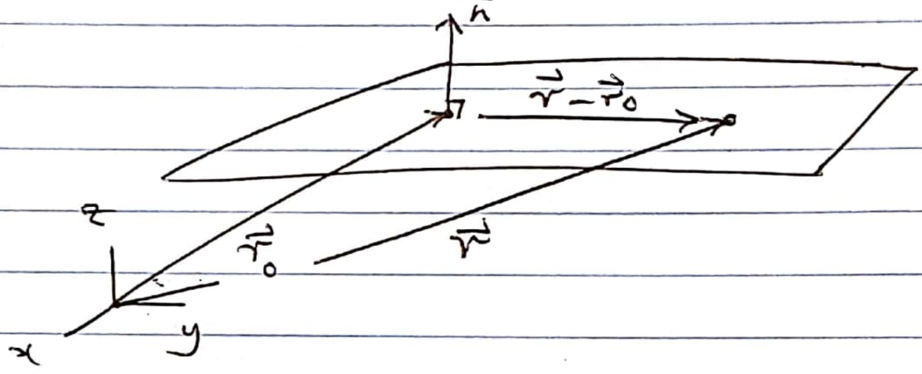
or

$$\begin{cases} x = 1+s-2t \\ y = 2+2s+t \\ z = 3+2s-10t \end{cases}$$

SCALAR PARAMETRIC EQNS

B LEVEL SET EQN

Let P be plane through $\vec{r}_0$ with normal vector $\vec{n}$.



Let $\vec{r} = (x, y, z)$ be an arbitrary point in P.
Then

$$\vec{r} - \vec{r}_0 \perp \vec{n}$$

So $$\boxed{(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0}$$ VECTOR FORM OF LEVEL SET EQUATION

CONVERT TO SCALAR FORM:

$\vec{r} = (x, y, z)$, $\vec{r}_0 = (x_0, y_0, z_0)$, $\vec{n} = (a, b, c)$

PLUG IN:

$$(x - x_0, y - y_0, z - z_0) \cdot (a, b, c) = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$ SCALAR FORM OF LEVEL SET EQN.

NOTE In general a level set eqn is one of form
$$F(x, y, z) = 0$$ (See #14)

EX2 $\vec{n} = (7,2,3)$ $\quad \vec{r}_0 = (-1,3,4)$, $\vec{r} = (x,y,z)$

$$(x+1,\ y-3,\ z-4)\cdot(7,2,3) = 0$$

$$7(x+1) + 2(y-3) + 3(z-4) = 0$$

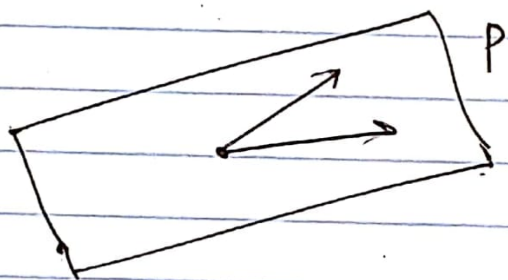
$$\boxed{7x + 2y + 3z - 11 = 0} \quad (*)$$

INTUITION

1 EQN IN 3 UNKNOWNS

So $3-1=2$ DEGREES of freedom

Corresponds ~~to~~ 2 independent directions ~~to~~ move in the plane.



(C) GRAPH OF FUNCTION

EX3.

Lets solve $(*)$ for z:

$$z = \frac{1}{3}(-7x - 2y + 11) = g(x,y)$$

In general

$$z = g(x,y) = Ax + By + C \quad \text{is a plane.}$$

Q Can we always solve for z?

A No. In general we have

$$ax + by + cz = d.$$

If $c = 0$ can't solve for z.

But $\vec{n} = (a, b, c) \neq (0, 0, 0)$ (Normal Vector $\neq \vec{0}$)

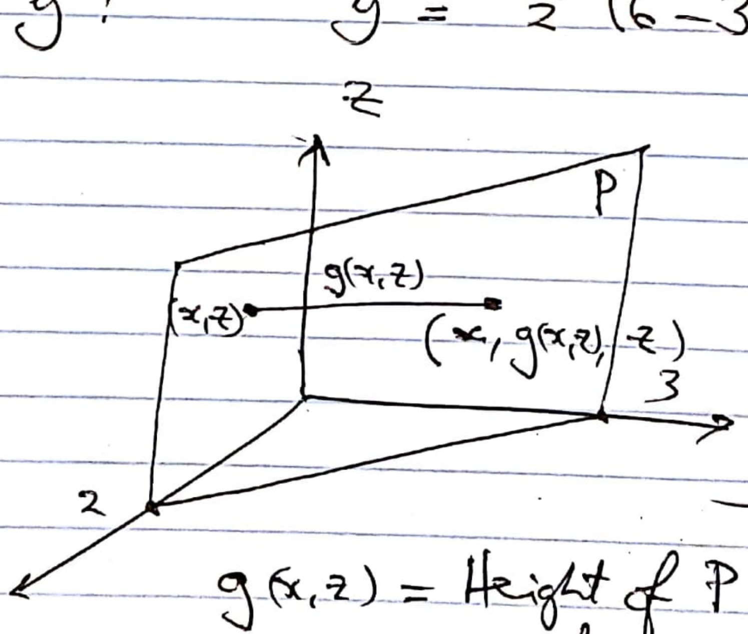
So at least one of a, b, c is non zero

So can solve for at least one of x, y, z.

$\boxed{\text{Ex } 4}$

$$3x + 2y = 6 \quad \text{in } \mathbb{R}^3$$

Solve for y: $\quad y = \frac{1}{2}(6 - 3x) = g(x, z)$



$g(x, z)$ = Height of P "above" xz plane.

CONVERTING BETWEEN THE 3 FORMULAE

LEVEL SET EQN $\xrightarrow{\text{EX } 3,4}$ GRAPH OF FUNCTION

GRAPH OF FUNCTION $\xrightarrow{\text{EX } 5}$ PARAMETERIZATION

PARAMETERIZATION $\xrightarrow{\text{EX } 6}$ LEVEL SET EQN

$\boxed{\text{EX 5}}$ GRAPH $\longrightarrow$ PARM

Suppose $z = g(x, y) = 4x + 5y + 6$

SCALAR FORM

Choose
$$x = x(s,t) = s$$
$$y = y(s,t) = t$$
$$z = z(s,t) = g(s,t) = 4s + 5t + 6$$

VECTOR FORM

$$\vec{r}(s,t) = x(s,t)\,\vec{i} + y(s,t)\,\vec{j} + z(s,t)\,\vec{k}$$
$$= s\vec{i} + t\vec{j} + (4s + 5t + 6)\vec{k}$$
$$= s(1, 0, 4) + t(0, 1, 5) + (0, 0, 6)$$
$$\vec{r}(s,t) = s\vec{v} + t\vec{w} + \vec{p}.$$

So $\vec{v} = (1, 0, 4)$, $\vec{w} = (0, 1, 5)$ are 2 vectors in P
$\vec{p} = (0, 0, 6)$ is a point in P

$\boxed{\text{EX 6}}$ PARM $\rightarrow$ LEVEL SET.

$\vec{r}(s,t) = \vec{p} + s\vec{v} + t\vec{w}$ $\qquad$ $\vec{p}, \vec{v}, \vec{w}$ as in EX 5.

Need $\quad (\vec{r} - \vec{r}_0) \cdot \vec{n} = 0.$

Choose $\quad \vec{r}_0 = \vec{p} = (0,0,6)$ $\qquad$ pt on P

$\vec{n} = \vec{v} \times \vec{w}$ $\qquad$ normal to P

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 4 \\ 0 & 1 & 5 \end{vmatrix}$$

$$\vec{n} = -4\vec{i} - 5\vec{j} + \vec{k}$$

So get

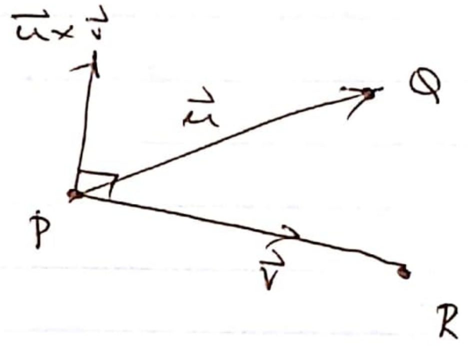
$$(-4, -5, 1) \cdot (x-0, y-0, z-6) = 0$$

$$-4x - 5y + z - 6 = 0 \qquad \text{LEVEL SET}$$

$$z = 4x + 5y + 6 \qquad \text{GRAPH} \checkmark$$

EX7 Find a unit vector $\vec{n}$ $\perp$ to plane through points

$$P = (2, 0, -3), \quad Q = (3, 1, 0), \quad R = (5, 2, 2)$$



$$\vec{u} = \overrightarrow{PQ} = Q - P = (1, 1, 3)$$

$$\vec{v} = \overrightarrow{PR} = R - P = (3, 2, 5)$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 3 \\ 3 & 2 & 5 \end{vmatrix} = (-1, 4, -1)$$

$$\vec{n} = \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|} = \frac{1}{\sqrt{18}}(-1, 4, -1)$$

V☐ ✓

$$\vec{n} \cdot \vec{u} = 0 = \vec{n} \cdot \vec{v}.$$

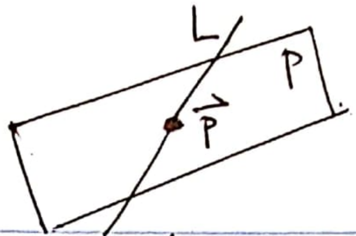
ALSO LEVEL SURFACE EQN of our plane is

$$(-1, 4, -1) \cdot (x - 2, y - 0, z + 3) = 0$$

$$-(x - 2) + 4y - (z + 3) = 0$$

$$\boxed{x - 4y + z = -1} \quad (**)$$

V☐ ✓ P, Q, R all satisfy (**)



Ex 8 A line L and plane P usually intersect in a point. $\vec{p}$

To find the point it is best to use parametrization of line and level set eqn of plane.

Ex ① $\vec{r}(t) = (1+4t,\ 2+5t,\ 3+6t)$ LINE

② $x + 2y - z = 10$ PLANE

Plug ① into ② and solve for t:

$$1+4t + 2(2+5t) - (3+6t) = 10$$

$$2 + 8t = 10$$

$$t = 1$$

So $\vec{p} = \vec{r}(1) = (5, 7, 9)$

CHECK Plug $\vec{p}$ INTO ②:

$$5 + 14 - 9 = 10 \checkmark$$

EX 9

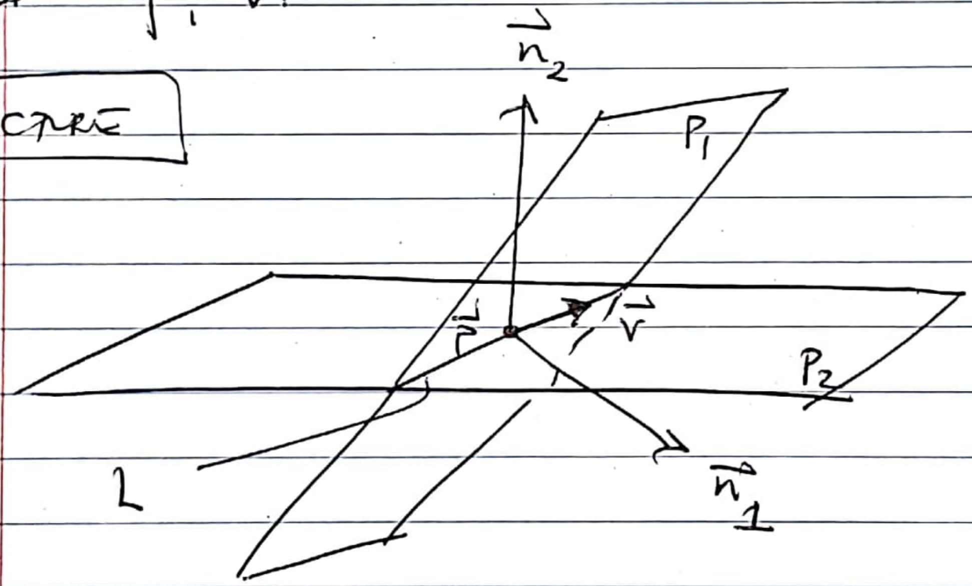
Usually 2 planes, P_1 and P_2, intersect in a line L.

Q What strategy should we use to find parametrization of L?

L : $\vec{r}(t) = \vec{p} + t\vec{v}$

Find $\vec{p}$, $\vec{v}$.

A $\boxed{\text{PICTURE}}$



(a) $\vec{v} = \vec{n}_1 \times \vec{n}_2$

(b) $\vec{p}$ is a point on both P_1 and P_2.

If P_1 is $a_1x + b_1y + c_1z = d_1$
and P_2 is $a_2x + b_2y + c_2z = d_2$

Then try setting $z = 0$ and solving 2 linear eqns for (x, y).