12.6 QUADRIC SURFACES

In 12.1 we saw that the set of points $(x,y,z) \in \mathbb{R}^3$ for which

$$F(x,y,z) = 0$$

for some given function $F: \mathbb{R}^3 \to \mathbb{R}$

forms a <u>surface</u>, S, in $\mathbb{R}^3$.

If the f^n F is a <u>quadratic</u> function of x, y, z

we call S a <u>QUADRIC</u> <u>SURFACE</u>

$\boxed{\text{GOAL:}}$ SKETCH S GIVEN F.

EXAMPLES

(A) $F(x,y,z) = Ax + By + Cz + D = 0$ PLANE

(B) $F(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$ SPHERE

(C) [GENERALIZED CYLINDERS]

If F is independent of one of variables say z, Then S is a generalized cylinder

STRATEGY TO SKETCH S

$F(x,y) = 0$ is a CURVE C, in xy-plane

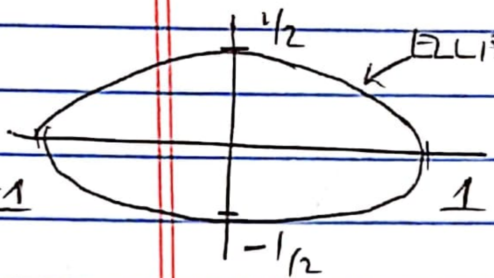
Translate C up + down z-axis to form surface S.

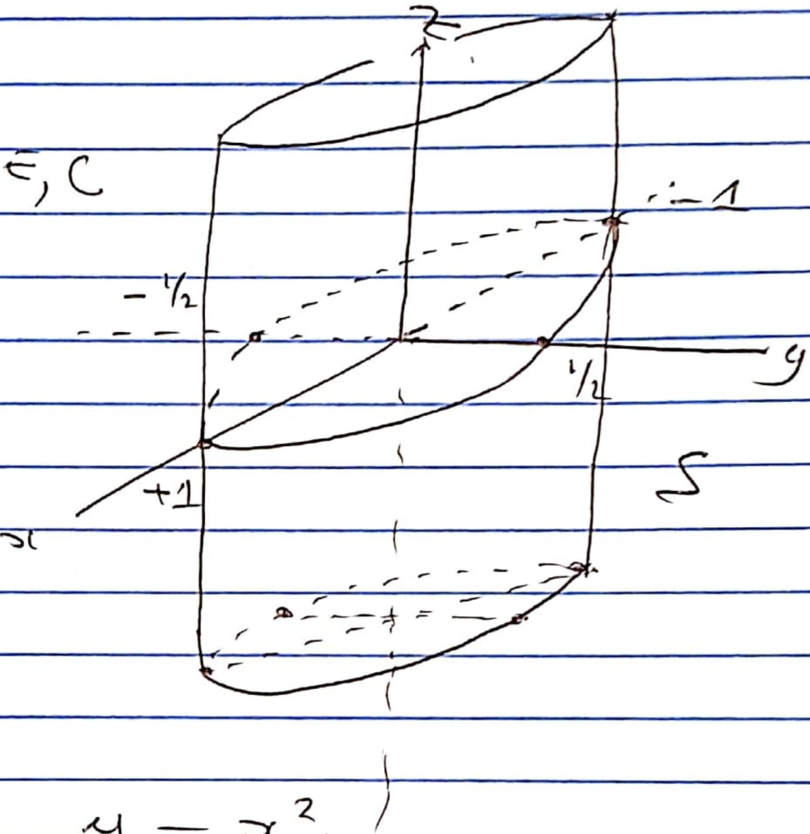
~~Q $z = x^2 + y^2$~~

~~A $z = x^2$~~

Q $x^2 + 4y^2 = 1$

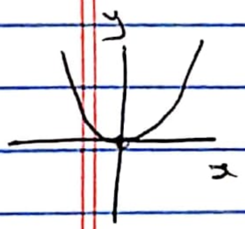
A

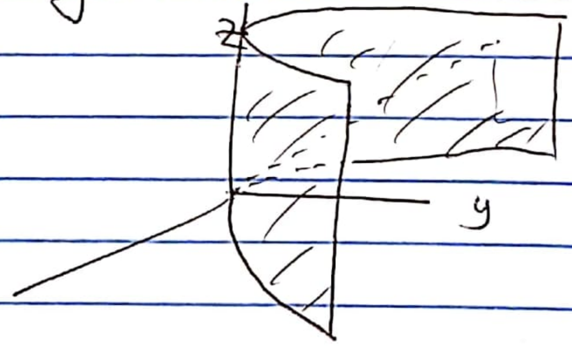




Q (DO AFTER Q AKS) $y = x^2$.

A





(D) SURFACES OF REVOLUTION (VIA CYL COORDS)

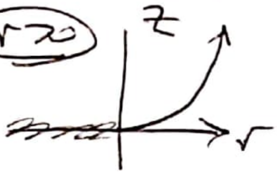
If x, y are always combined as x^2+y^2 in formula for F

Then S is a Surface of Revolution.

SKETCHING STRATEGY: - CONVERT TO CYL COORDS
- PLOT C IN (r, z) - SPACE
- ROTATE ABOUT z-AXIS TO GET S

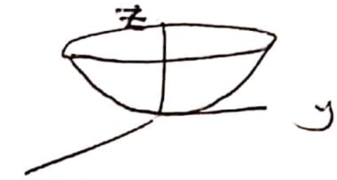
Q $z = x^2 + y^2$ $(r \geq 0)$

A $z = r^2$



PARABOLA

ROTATE $\longrightarrow$

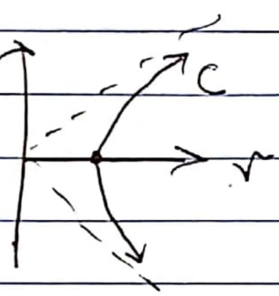


PARABOLOID

Q $x^2 + y^2 - z^2 = 1$ RECT

A $r^2 - z^2 = 1$ CYL (INDEPT OF θ !!)

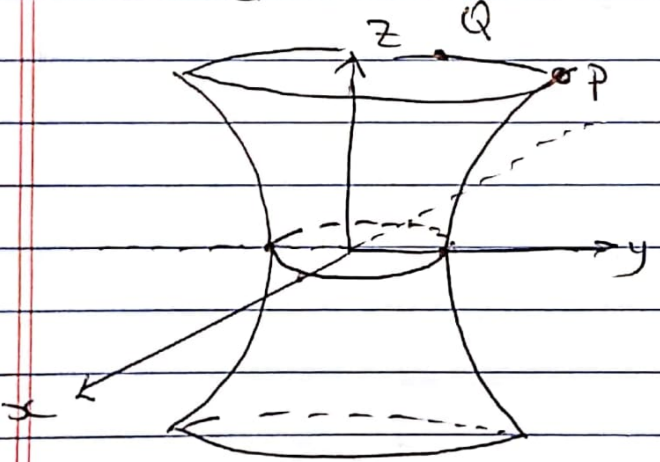
RECALL $r \geq 0$



1-ARMED HYPERBOLA

Rotate r axis about z axis to get xy-plane

Rotate curve C ——————— surface S



1-SHEETED HYPERBOLOID.

Since P, Q have same (r, z) (but different θ's)

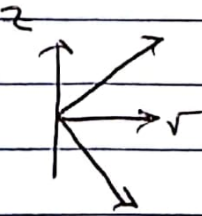
If P lies on curve $r^2 - z^2 = 1$ in (r, z)-space

Then P, Q both lie on surface $r^2 - z^2 = 1$ in $\mathbb{R}^3$.

Q $z^2 = x^2 + y^2$ RECT

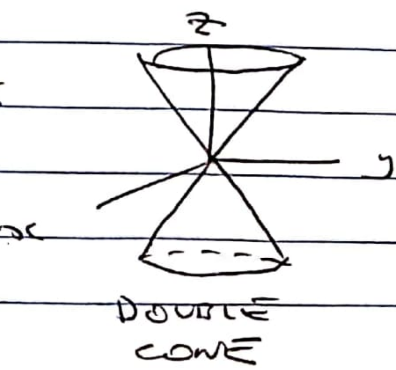
A $z^2 = r^2$ CYL

$z = \pm r$ θ-INDEPT



PAIR OF LINES

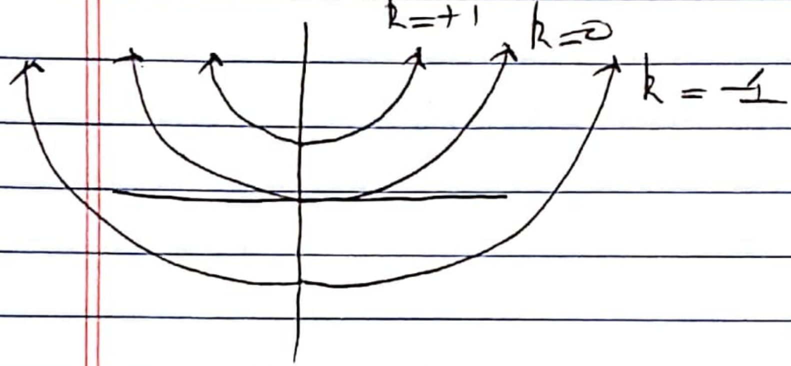
ROTATE $\rightsquigarrow$



DOUBLE CONE

INTERLUDE: FAMILIES OF QUADRIC SURFACES IN $\mathbb{R}^2$

EXS ① PARABOLAS $y = x^2 + k$ for different PARAMETERS, k



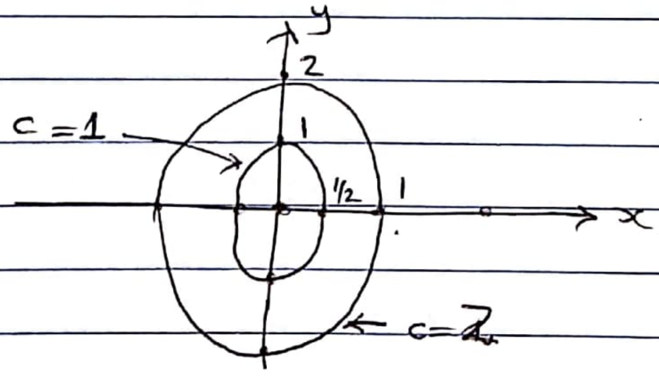
Translate $y = x^2$ up by k.

② ELLIPSES

$$4x^2 + y^2 = c^2$$ for parameter c.

INTERCEPTS: $y = 0 \Rightarrow x = \pm \frac{c}{2}$

$x = 0 \Rightarrow y = \pm c$



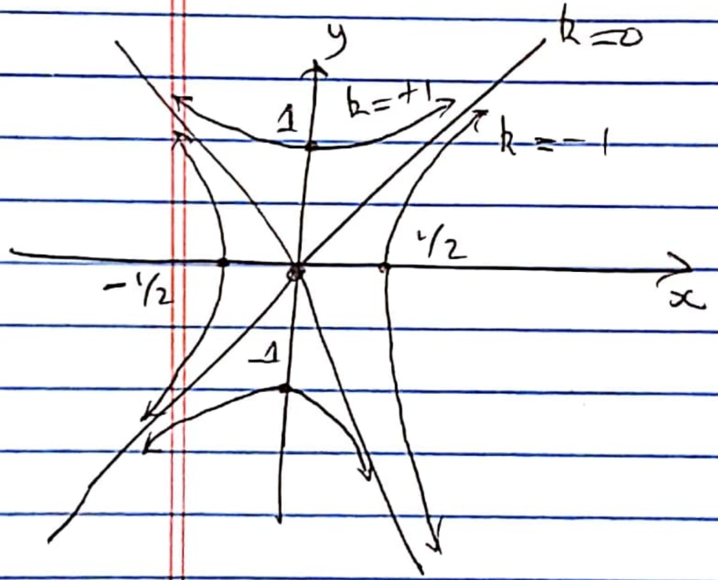
$c = 0$ is ORIGIN.

③ HYPERBOLAE $y^2 - 4x^2 = k$

$\boxed{k=0}$ $y = \pm 2x$ Pair of Lines (Asymptotes)

$\boxed{k=+1}$ GOES THRU $x=0$, $y = \pm 1$

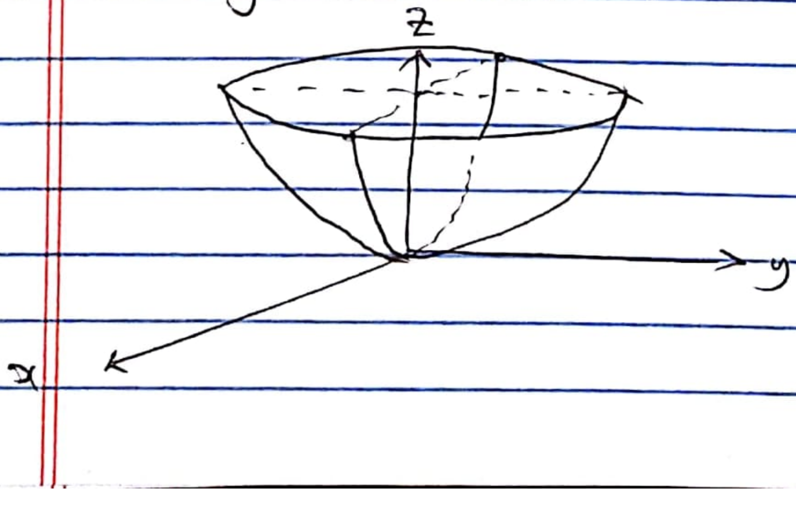
$\boxed{k=-1}$ GOES THRU $y=0$, $x = \pm \frac{1}{2}$



As $|k| \uparrow$
Intercepts move out along axes.
+ asymptotes remain same.

(E) GENERAL QUADRIC SURFACES (VIA TRACES/SLICES)

① $z = 4x^2 + y^2$. ELLIPTIC PARABOLOID



TO OBTAIN THIS SKETCH

SLICE Surface by planes parallel to 3 coord planes.

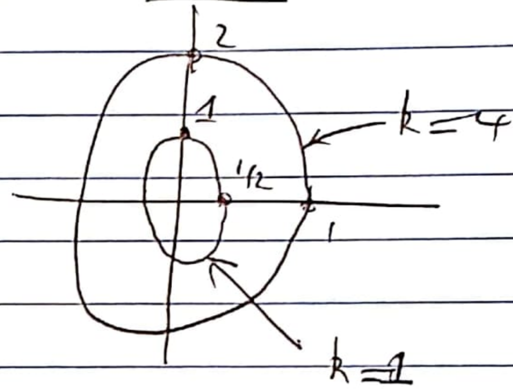
$\boxed{z=k}$ $4x^2+y^2=k$.

See Ellipses etc above. ($k=c^2$)

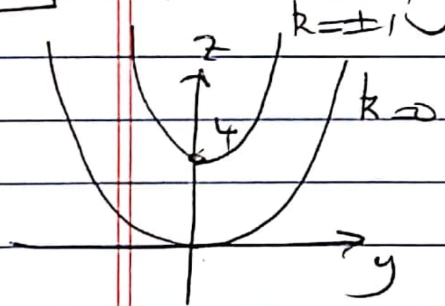
$k<0$ EMPTY SET

$k=0$ ORIGIN

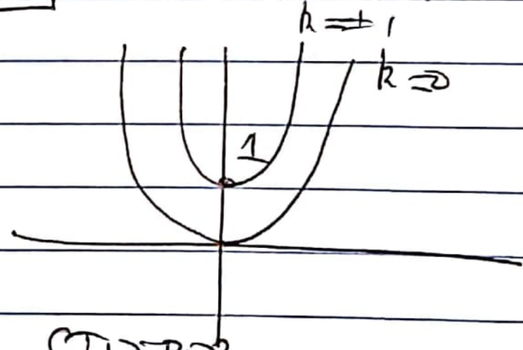
$k>0$



$\boxed{x=k}$ $z=4k^2+y^2$



$\boxed{y=k}$ $z=4x^2+k^2$

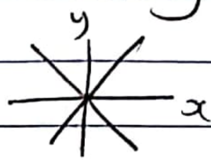


SHALLOWER

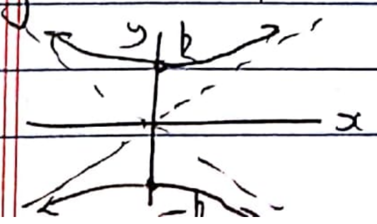
STEEPER

2 SADDLE SURFACE $z=y^2-x^2$

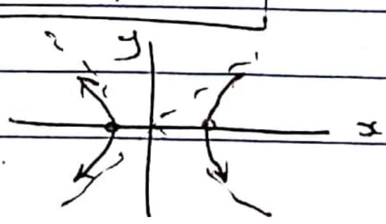
$\boxed{z=0}$ $y=\pm x$



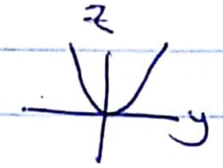
$\boxed{z=k^2>0}$ $y^2-x^2=k^2$



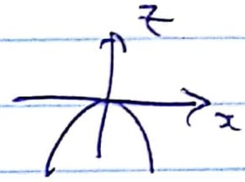
$\boxed{z=-k^2<0}$



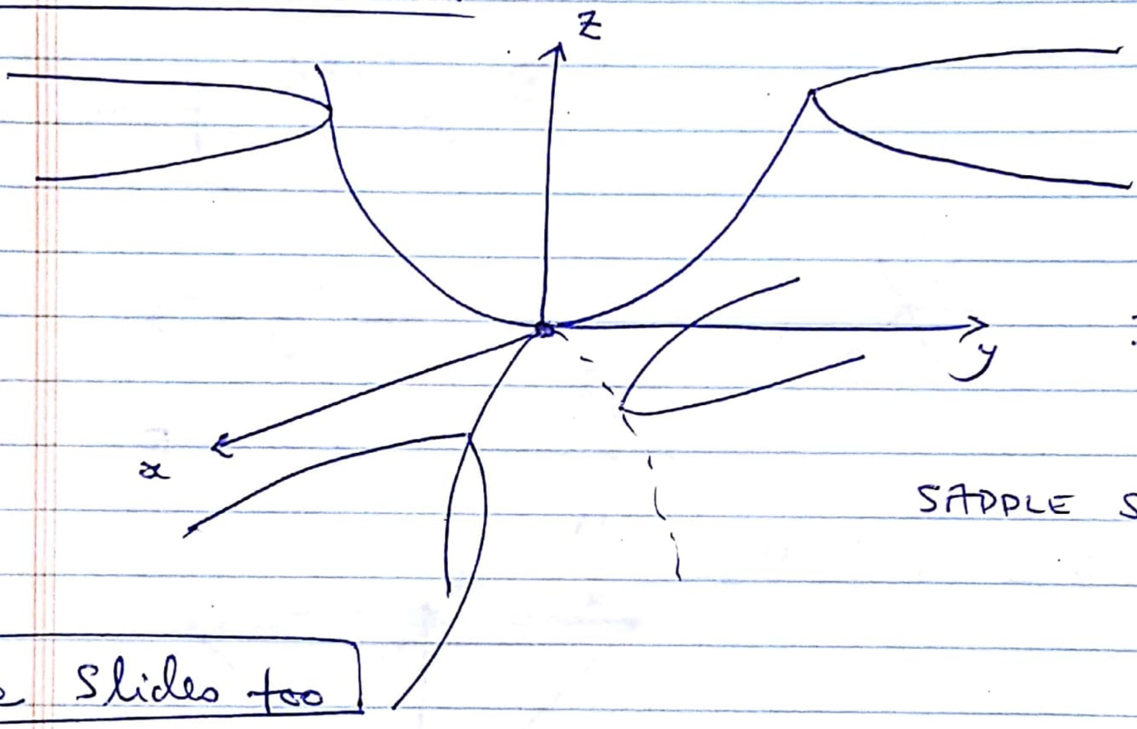
$x=0$ $z=y^2$



$y=0$
$z=-x^2$



PUTTING IT ALL TOGETHER



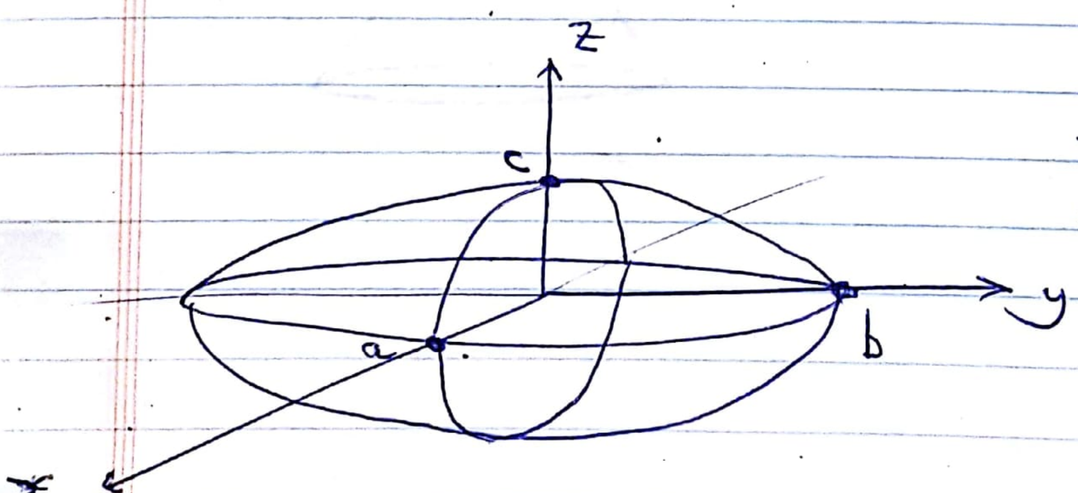
HORSE GOES $\longrightarrow$ THIS WAY.

SADDLE SURFACE

$\boxed{\text{See Slides too}}$

(B) ELLIPSOID

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

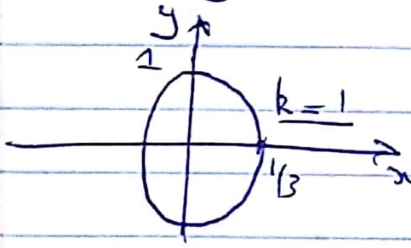


SLICES in $x=0$, $y=0$, $z=0$ are ellipses.

4

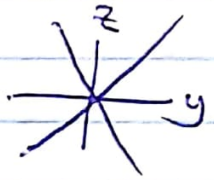
$z^2 = 9x^2 + y^2$ DOUBLE ELLIPTICAL CONE

$z = k$ $9x^2 + y^2 = k^2$ $\left(\frac{3x}{k}\right)^2 + \left(\frac{y}{k}\right)^2 = 1$

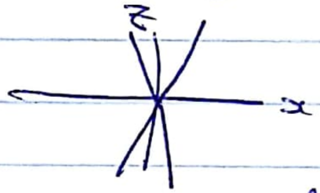


(*)

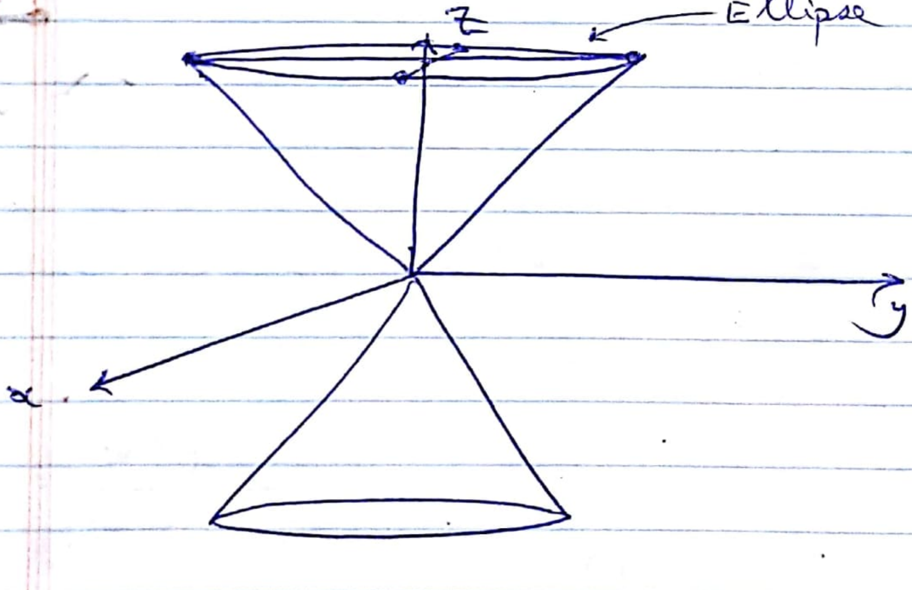
$x = 0$ $z^2 = y^2 \Rightarrow z = \pm y$



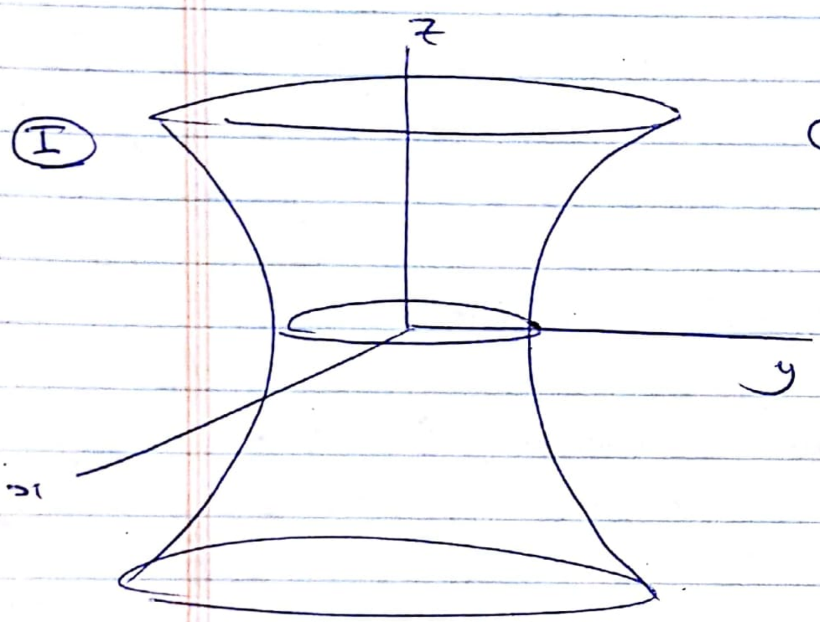
$y = 0$ $z = \pm 3x$

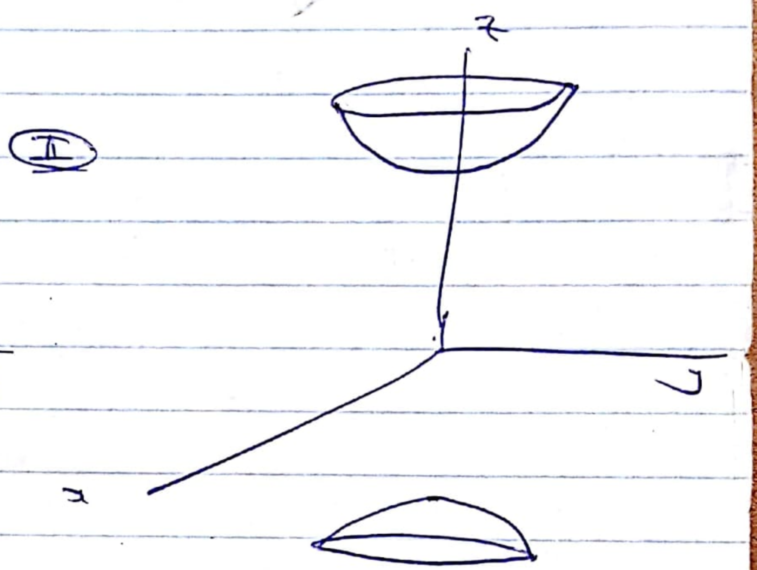


Ellipse as in (*)



WHICH IS WHICH?





(A) $x^2+y^2-z^2=1$

$x^2+y^2=1+z^2$

has $z=k$ nonempty for all z.

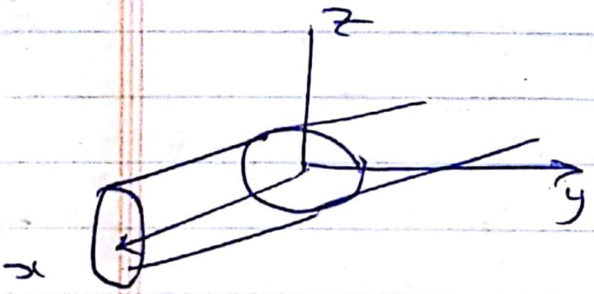
(B) $x^2+y^2-z^2=-1$

$x^2+y^2=z^2-1$

has empty $z=k$ trace for $k^2-1<0$, $\underline{|k|<1}$

FINALLY CYLINDER

$y^2+z^2=1$



Slice in $x=k$ and get circle $y^2+z^2=1$