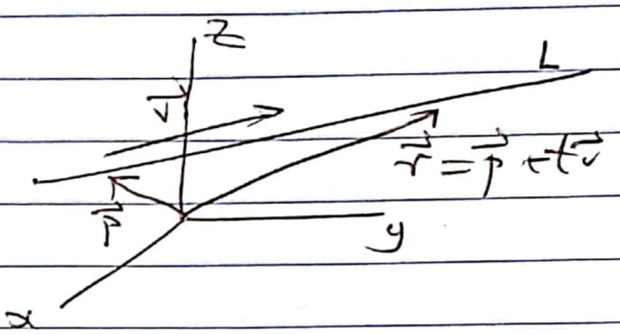
13.1 – 13.3 CURVES

RECALL from 12.5 that the LINE L thru $\vec{p}$ in direction of $\vec{v}$ is parametrized by

$$\vec{r}(t) = \vec{p} + t\,\vec{v}$$



OR
$$\begin{cases} x = p_1 + t v_1 \\ y = p_2 + t v_2 \\ z = p_3 + t v_3 \end{cases}$$

REGARD $\vec{r} : \mathbb{R} \to \mathbb{R}^3$ as a vector-valued function of (time) t.

DOMAIN of $\vec{r} = \mathbb{R}$
RANGE of $\vec{r} = \mathbb{R}^3$
IMAGE of $\vec{r}$ = LINE L.

A line is simplest ex of a curve

DEF A CURVE, C, is the IMAGE of a function
$$\vec{r} : \mathbb{R} \to \mathbb{R}^3$$

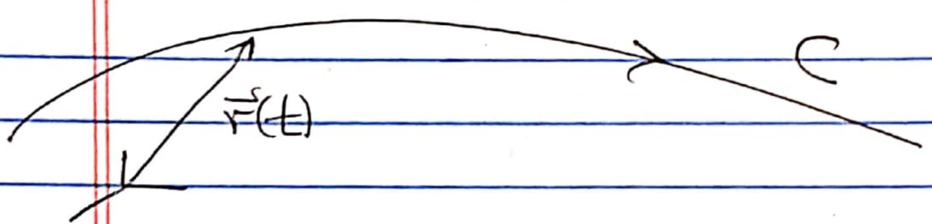
Write
$$\vec{r}(t) = (x(t), y(t), z(t)) \quad t \in \mathbb{R}$$
$$= x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

We call $\vec{r}$ a PARAMETRIZATION of C

THINK

t = TIME (a parameter along C)

$\vec{r}(t)$ = POSITION of particle at time t



As $t \uparrow$, $\vec{r}(t)$ traces out a curve C in $\mathbb{R}^3$.

IF $\vec{r}(t)$ = POSITION of aircraft at time t (MOVIE)

THEN C = Image of $\vec{r}$ = CONTRAIL (~~PHOTO~~)

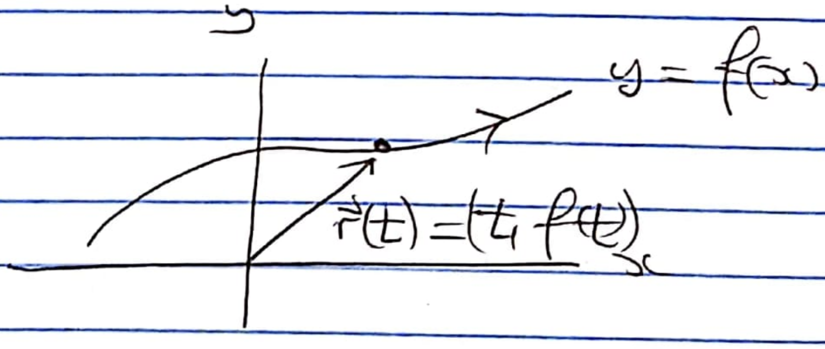
Ⓐ CURVES IN THE PLANE

① GRAPH OF $y = f(x)$

SET

$x = t$

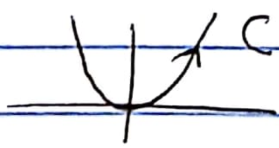
$y = f(t)$



EX PARABOLA $y = x^2$

$x = t$

$y = t^2$



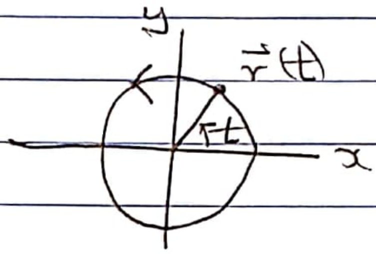
2 CIRCLE

a $\vec{r}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

$t =$ ANGLE

$\vec{r}(0) = (1, 0)$

$\vec{r}(\pi/2) = (0, 1)$



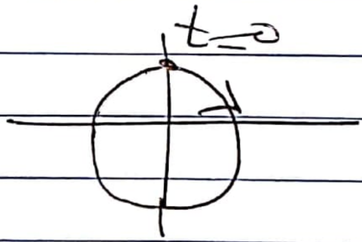
THE IMAGE of $\vec{r}$ IS THE CIRCLE $x^2 + y^2 = 1$
since

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

b $\vec{r}(t) = (\sin t, \cos t)$

$\vec{r}(0) = (0, 1)$

$\vec{r}(\pi/2) = (1, 0)$

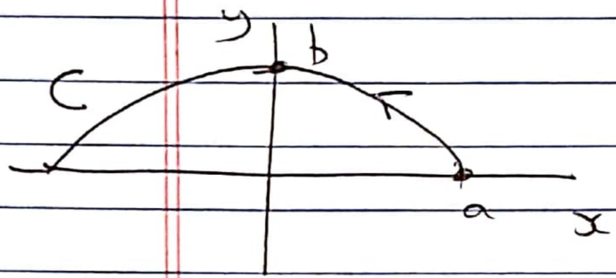


CLOCKWISE STARTING AT $(0, 1)$

a + b are 2 par^ns of same circle

* There are ∞ # par^ns of any given curve C

(3) HALF-ELLIPSE



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$y \geq 0.$$

Parametrize C

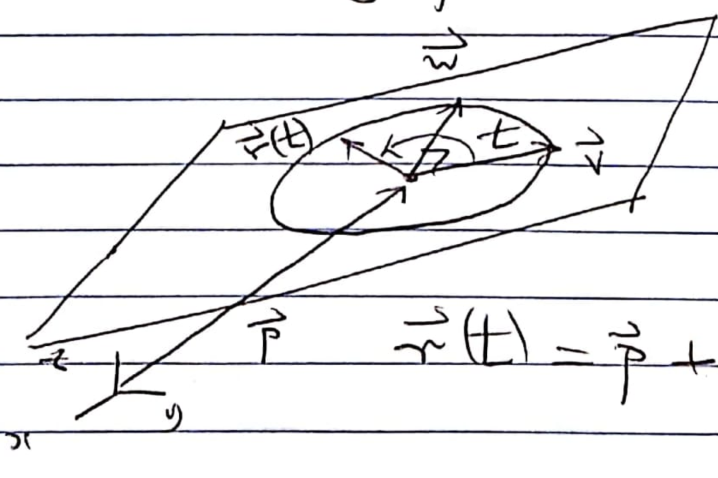
Set $x = a\cos t \qquad 0 \leq t \leq \pi$
$y = b\sin t$

CHECK $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \left(\frac{a\cos t}{a}\right)^2 + \left(\frac{b\sin t}{b}\right)^2$

$= \cos^2 t + \sin^2 t = 1$

So image of $\vec{r}$ is C.

(4) PARAMETRIZE CIRCLE CENTER $\vec{p}$, RADIUS R IN PLANE P containing $\vec{p}$



CHOOSE $\vec{v}, \vec{w}$ IN P WITH

$|\vec{v}| = |\vec{w}| = R$

$\vec{v} \perp \vec{w}$

$\vec{r}(t) = \vec{p} + (\cos t)\,\vec{v} + (\sin t)\,\vec{w}.$

(B) CURVES IN SPACE

(3) HELIX

$\vec{r}(t) = (\cos t, \sin t, t)$, $t \in \mathbb{R}$

$$x = \cos t$$
$$y = \sin t$$
$$z = t$$

What does image curve C look like?

STRATEGY 1 Find a surface that C lies on.

Find an eqn relating x, y, z by eliminating t.

EX There are lots of possibilities. Here is one.
Use

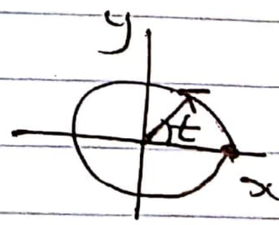
$$1 = \cos^2 t + \sin^2 t \quad \text{to get}$$
$$1 = x^2 + y^2 \quad \text{CYLINDER in } \mathbb{R}^3$$

So C lies on a cylinder

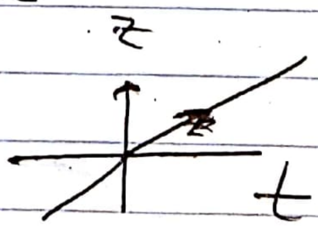
STRATEGY 2 Look at projections of C onto a coordinate plane or an axis.

EX xy-PLANE $x = \cos t$, $y = \sin t$

As $t \uparrow$ go round circle



z-AXIS $z = t$



As $t \uparrow$, $z \uparrow$.

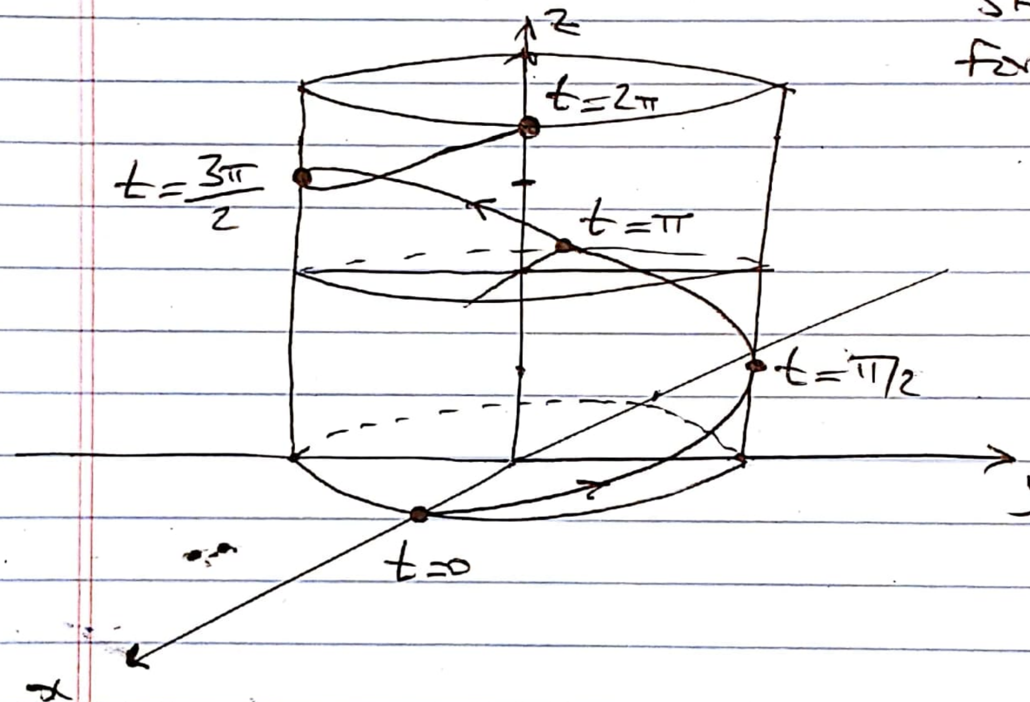
6

STRATEGY 3 Plot a few points

$\vec{r}(0) = (1,0,0)$, $\vec{r}(\pi/2) = (0,1,\pi/2)$, $\vec{r}(2\pi) = (1,0,2\pi)$

PUT IT ALL TOGETHER TO SKETCH C

SKETCH for $0 \le t \le 2\pi$.



(3) HELIX ON CONE

$x = t\cos t$
$y = t\sin t$
$z = t$

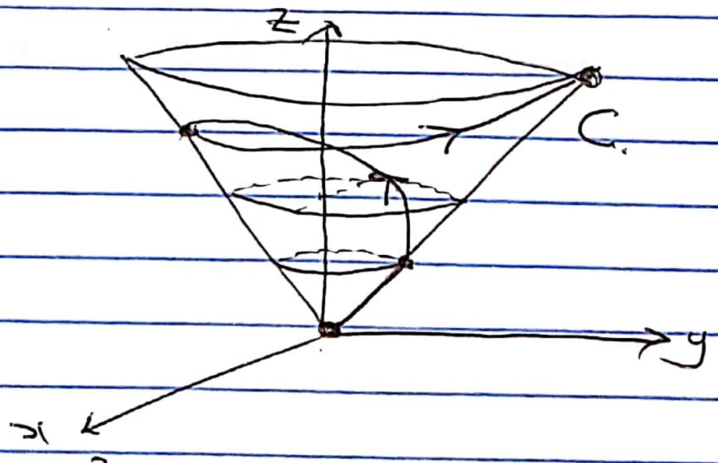


IMAGE OF $\vec{r}$:

$x^2+y^2 = (t\cos t)^2+(t\sin t)^2 = t^2 = z^2$

$z^2 = x^2+y^2$. SO C LIES ON A CONE

$z = r$

3 CURVE ~~AS~~ INTERSECTION OF 2 SURFACES

USUALLY 2 SURFACES INTERSECT IN A CURVE

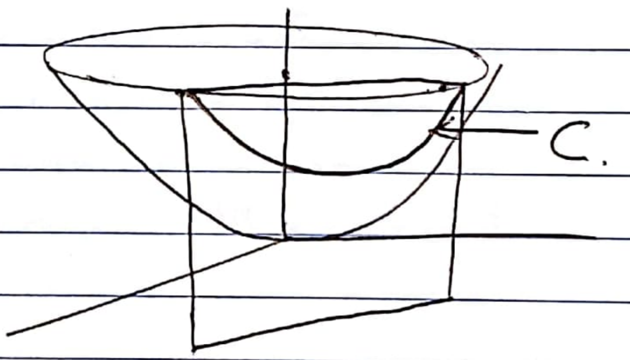
EX

$$y = f(x) = 1 - x \quad \text{PLANE}$$

$$z = g(x,y) = x^2 + y^2 \quad \text{PARABOLOID}$$

PARN

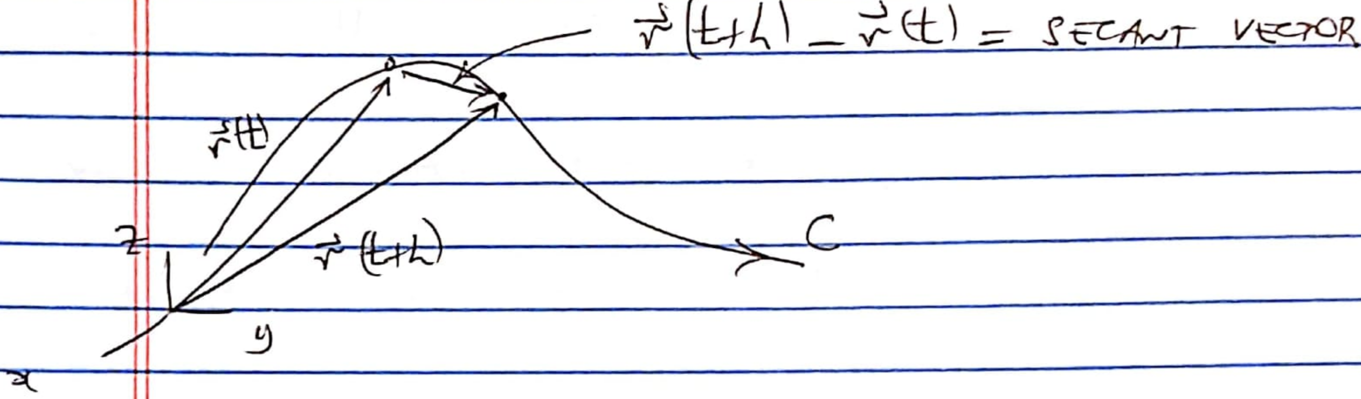
$$x = t$$
$$y = f(t) = 1 - t$$
$$z = g(t, f(t)) = t^2 + (1-t)^2 = 2t^2 - 2t + 1$$



See Virtual Model on Course Web Page

DEF The DERIVATIVE of a parametrized curve, C, $\vec{r} : \mathbb{R} \to \mathbb{R}^3$ is

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

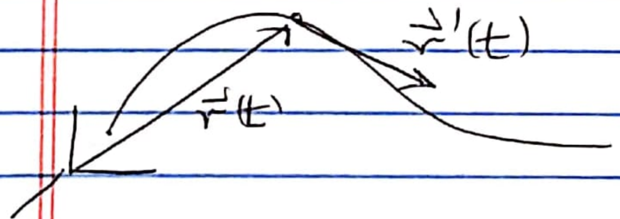


$$\frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \text{SCALED SECANT VECTOR}$$

As $h \to 0$, SECANT $\to$ TANGENT

So

$\vec{r}\,'(t)$ = TANGENT VECTOR TO C AT $\vec{r}(t)$



IF $\vec{r}(t)$ = POSITION at time t

THEN $\vec{r}\,'(t)$ = VELOCITY VECTOR

$|\vec{r}\,'(t)|$ = SPEED

$\vec{r}\,''(t)$ = ACCELERATION VECTOR

<u>PROPN</u> IF $\vec{r}(t) = (x(t), y(t), z(t))$

THEN $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$.

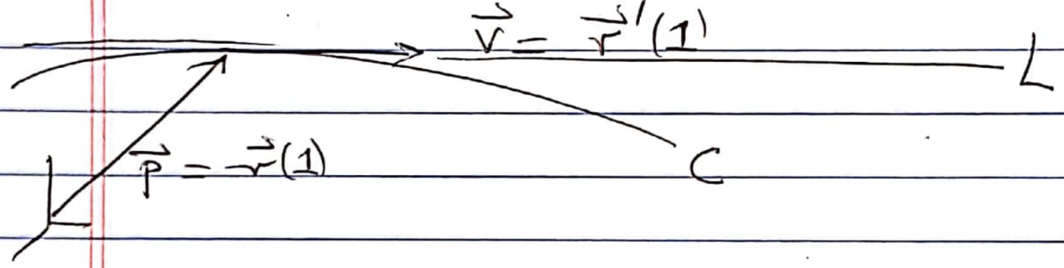
EX $\vec{r}(t) = (t, t^2, t^3)$ @ $t = 1$

(a) VELOCITY VECTOR at $t = 1$

$$\vec{r}\,'(t) = (1, 2t, 3t^2)$$

$$\vec{r}\,'(1) = (1, 2, 3)$$

(b)



Parametrize Tangent Line L to C at $t=1$.

$$\vec{l}(s) = \vec{p} + (s-1)\vec{v}$$

with $\vec{p} = \vec{r}(1) = (1, 1, 1)$
$\vec{v} = \vec{r}\,'(1) = (1, 2, 3)$

So $\vec{l}(s) = (1, 1, 1) + (s-1)(1, 2, 3)$

NOTICE $\vec{l}(1) = \vec{r}(1)$

$$\vec{l}\,'(1) = \vec{r}\,'(1)$$

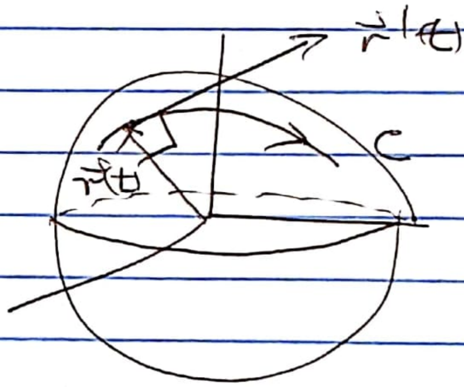
AS ZOOM IN ON $t=1$, MOTION OF $\vec{r} \simeq$ MOTION OF $\vec{l}$.

PRODUCT RULE

$$\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \frac{d\vec{u}(t)}{dt} \cdot \vec{v}(t) + \vec{u}(t) \cdot \frac{d\vec{v}(t)}{dt}$$

APPLICATION Suppose $\vec{r}$ is a parametrizn of a curve C on sphere radius R, center $\vec{O}$.

Then VELOCITY $\perp$ POSITION for all t



PF $|\vec{r}(t)| = R$ holds.

SHOW $\vec{r}'(t) \perp \vec{r}(t)$.

$$R^2 = |\vec{r}(t)|^2 = \vec{r}(t) \cdot \vec{r}(t)$$

$$\frac{d}{dt}: \quad 0 = \frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) \overset{PR}{=} \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t)$$

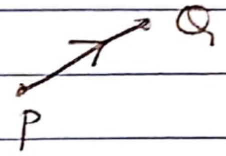
$$\Rightarrow \quad 0 = 2\,\vec{r}'(t) \cdot \vec{r}(t)$$

$$\Rightarrow \quad \vec{r}'(t) \perp \vec{r}(t)$$

ARC LENGTH

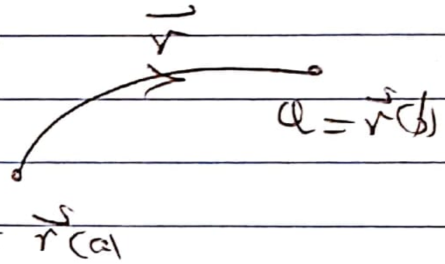
FOR STRAIGHT LINE MOTION AT CONSTANT SPEED

DISTANCE TRAVELLED = SPEED × TIME TAKEN



FOR General Parametrized Curve, C

$$\vec{r} : [a, b] \to \mathbb{R}^3$$



$$\text{LENGTH}(C) = \int_{t=a}^{t=b} |\vec{r}\,'(t)|\, dt$$

EX

① $\vec{r}(t) = (3\sin t,\ 4t,\ 3\cos t)$ HELIX

$$\vec{r}\,'(t) = (3\cos t,\ 4,\ -3\sin t)$$

$$|\vec{r}\,'(t)| = \sqrt{(3\cos t)^2 + 4^2 + (-3\sin t)^2} = 5$$

LENGTH of HELIX FROM $t=0$ to $t=2\pi$ is

$$L = \int_0^{2\pi} 5\, dt = 10\pi$$

(2) $\vec{r}(t) = (1, t^2, t^3)$ $\qquad 0 \le t \le 1$

$\vec{r}\,'(t) = (0, 2t, 3t^2)$

$|\vec{r}\,'(t)| = \sqrt{4t^2 + 9t^4} = t\sqrt{4+9t^2}$

So

$$L = \int_0^1 \sqrt{4+9t^2}\, t\, dt \qquad \begin{aligned} u &= 4+9t^2 \\ du &= 18t\,dt \end{aligned}$$

$$= \frac{1}{18}\int_4^{13} \sqrt{u}\, du$$

$$= \frac{1}{18}\,\frac{2}{3}\left[u^{3/2}\right]_4^{13} = \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right)$$

__NOTE__ It is usually IMPOSSIBLE to calculate integral for L analytically

__Ex__ $\vec{r}(t) = (\cos t, e^t)$

$$L = \int_a^b \sqrt{\sin^2 t + e^{2t}}\, dt \qquad \boxed{\text{YUK}}$$

BUT once we have "set this integral up"

we can use NUMERICAL INTEGRATION (eg SIMPSON'S RULE) to ~~estimate it~~ approximate L.