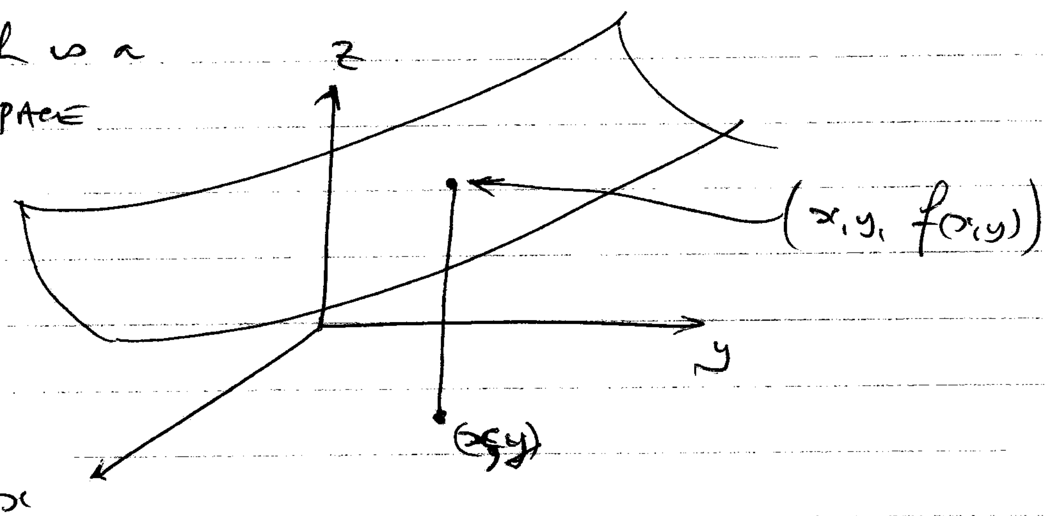
14.2 LIMITS + CONTINUITY

Recall if $z = f(x,y)$ we can sketch the GRAPH of f which is a SURFACE IN SPACE



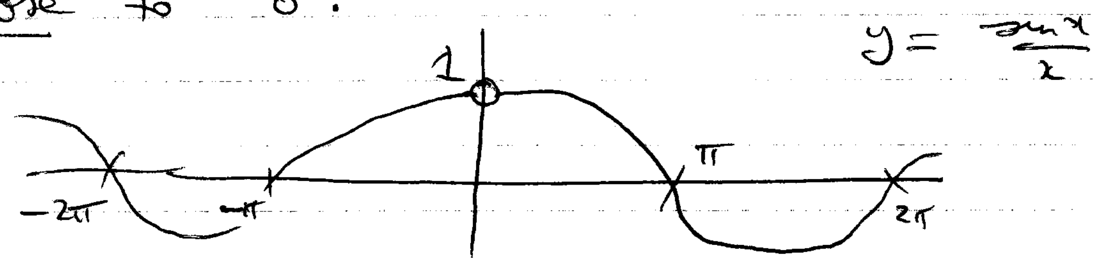
HEIGHT of SURFACE above (x,y) IS $z = f(x,y)$.

LIMITS IN CALCULUS 1

Suppose $y = f(x)$ is defined for all x except maybe at $x=0$.

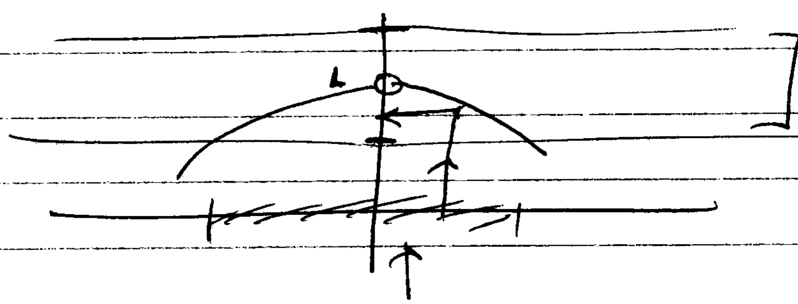
EG $\quad f(x) = \dfrac{\sin x}{x}$

QUESTION What happens to values of f as x gets close to 0?



We say $\lim_{x \to 0} f(x) = L$ EXISTS if

you can make values $f(x)$ as close as you like to L provided you choose x close enough to 0.



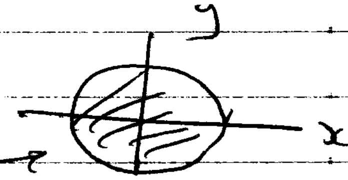
NEED x TO BE IN THIS SMALL INTERVAL ABOUT 0.

CALCULUS III

The same idea works for $z = f(x,y)$:

We say $\lim_{(x,y) \to (0,0)} f(x,y) = L$ EXISTS if

you can make values $f(x,y)$ as close as you like to L provided you choose pt (x,y) close enough to $(0,0)$

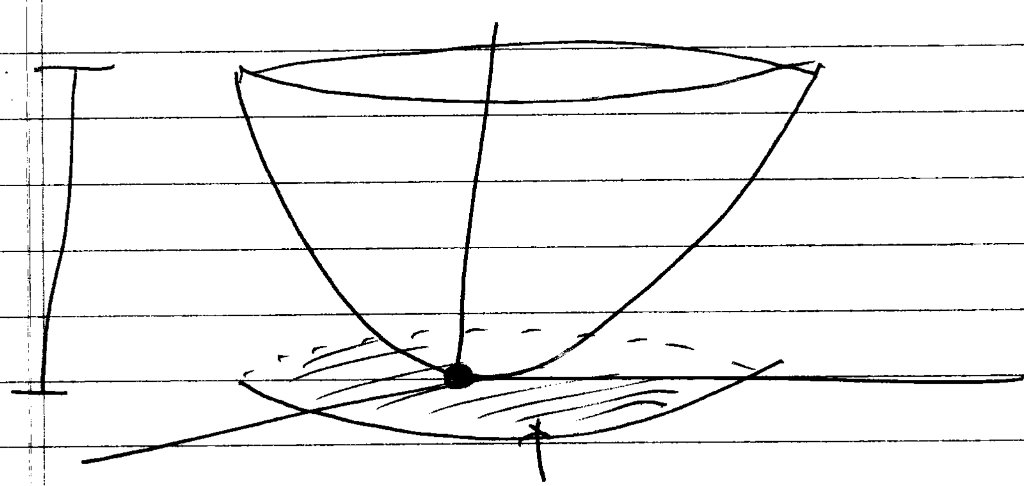


SMALL DISC

EX $z = f(x,y) = x^2 + y^2$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \quad \text{since}$$

IF WANT VALUES OF f CLOSE TO 0



THEN NEED TO PICK (x,y) IN THIS DISC.

The ~~smaller~~ CLOSER we want $f(x,y)$ to be to L
The smaller we need to pick This DISC

CALCULUS I

METHODS TO SHOW $\lim_{x\to 0} f(x) = L$ EXISTS

① Suppose $f(x) = \dfrac{g(x)}{h(x)}$ where g, h are CTS and $h(0) \neq 0$.
Then

$$\lim_{x\to 0} \frac{g(x)}{h(x)} = \frac{g(0)}{h(0)} \qquad \text{"PLUG IN."}$$

<u>WORKS IN CALC III TOO:</u>

EX $$\lim_{(x,y)\to(0,0)} \frac{x^2+3e^y}{\cos x + y^2+5} = \frac{0^2+3e^0}{1+0^2+5} = \frac{3}{6} = \frac{1}{2}.$$

(2) Suppose $f(x) = \frac{g(x)}{h(x)}$ and $f(0) = \text{"}\frac{0}{0}\text{"}$.

FIRST Try to <u>FACTOR + CANCEL</u>:

EX $$\lim_{x\to 0} \frac{3x+2x^2}{x} = \lim_{x\to 0} \frac{x(3+2x)}{x}$$ FACTOR

$$= \lim_{x\to 0} 3+2x$$ CANCEL

$$= 3+2\times 0 = 3$$ NOW PLUG IN

<u>WORKS IN CALC III TOO:</u>

EX $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$$

"$\frac{0}{0}$"

$$= \lim_{(x,y)\to(0,0)} \frac{(x-y)(x+y)}{x+y} = \lim_{(x,y)\to(0,0)} x-y = 0.$$

FACTOR CANCEL NOW PLUG IN

(3) SPECIAL LIMITS LIKE

$$\lim_{x \to 0} \frac{\sin x}{x} \overset{0/0}{=} 1.$$

WE CAN USE THIS IN CALC III TOO:

EX

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y)}{x} \overset{0/0}{=} \lim_{(x,y)\to(0,0)} \left[\frac{\sin(x^2y)}{x^2y} \frac{x^2y}{x}\right]$$

$$= \left[\lim_{(x,y)\to(0,0)} \frac{\sin(x^2y)}{x^2y}\right]\left[\lim_{(x,y)\to(0,0)} \frac{x^2y}{x}\right]$$

$$= \left[\lim_{u\to 0} \frac{\sin u}{u}\right]\left[\lim_{(x,y)\to(0,0)} xy\right]$$

$u = x^2y$ FACTOR

$$= 1 \times 0 = 0.$$

(4) $\boxed{\text{L'HOPITAL'S RULE}}$

- NEVER USE THIS IN CALCULUS III
- IT DOESN'T WORK

DEF We say $z = f(x,y)$ is **CONTINUOUS** at $(x,y) = (0,0)$ if

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

FUNCTIONS $z = f(x,y)$ built out of polys, trig f^{n}, exponentials involving x, y are CTS wherever they are defined.

EG $$z = \frac{\sin(x+y) + e^{x^2 - z}}{1 + x^2 + y^2}$$

is continuous at all (x,y) in $\mathbb{R}^2$.

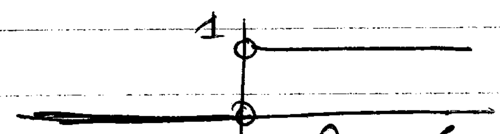
— o —

A METHOD TO SHOW LIMITS DNE

CALCULUS I

IF $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$ LEFT LIMIT $\neq$ RIGHT LIMIT

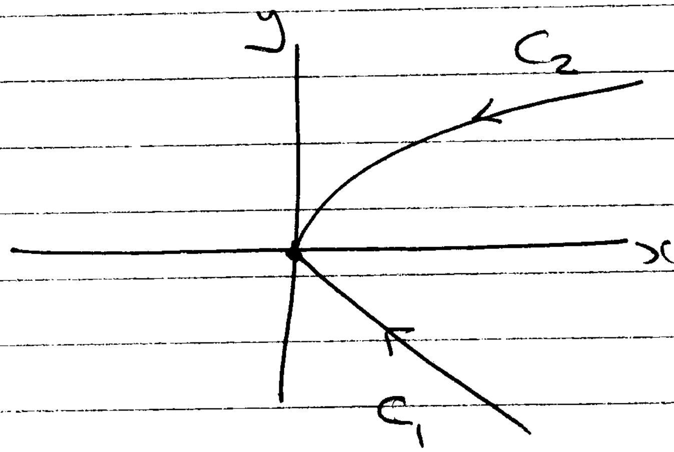
THEN $\lim_{x\to 0} f(x)$ DNE



EX $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$ $\lim_{x\to 0^-} f(x) = 0$ BUT $\lim_{x\to 0^+} f(x) = 1$.

CALCULUS II

$z = f(x,y)$



FOR

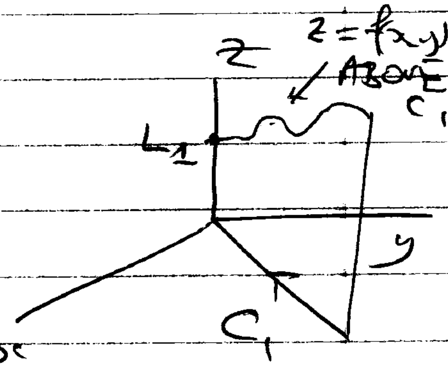
$$\lim_{(x,y)\to(0,0)} f(x,y)$$

There are ∞ # WAYS TO GO TO $(0,0)$.

- EVERY CURVE IN (x,y)-PLANE THAT GOES TO $(0,0)$ GIVES YOU A WAY.

SO IF you can find 2 curves C_1, C_2 so that

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{ALONG } C_1}} f(x,y) = L_1$$



$$\lim_{\substack{(x,y)\to(0,0)\\ \text{ALONG } C_2}} f(x,y) = L_2$$

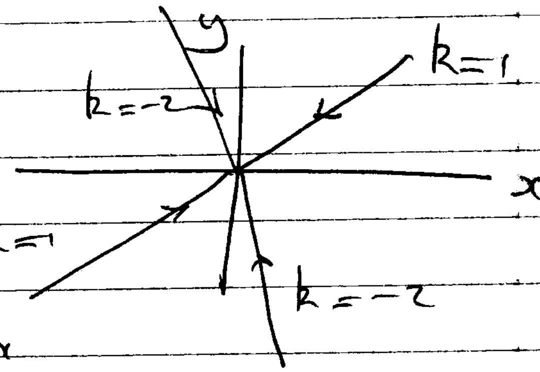
and $L_1 \neq L_2$ Then $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

BUT IF $L_1 = L_2$ That does NOT mean limit EXISTS

- Maybe there is another curve That gives different limit

EXS

(1) $$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$$



APPROACH $(0,0)$ ALONG LINES $y=kx$
FOR DIFFERENT k

SET $y=kx$

1 VARIABLE LIMIT

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{ALONG } y=kx}} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2x(kx)}{x^2+(kx)^2}$$

$$= \lim_{x\to 0} \frac{2kx^2}{x^2(1+k^2)} = \lim_{x\to 0} \frac{2k}{1+k^2}$$

FACTOR

CANCEL

$$= \frac{2k}{1+k^2}.$$

So different lines give different limits.
(OVERALL LIMIT DNE

(2) WHAT ABOUT

$$\cancel{\lim_{(x,y)\to(0,0)} \frac{2x^2y}{\ }}$$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} \quad ?$$

WHAT DOES GRAPH OF f LOOK LIKE?
I HAVE A MODEL OF IT!

TRY $\underline{y = kx}$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{ALONG } y=kx}} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2kx^3}{x^4+k^2x^2}$$

$$= \lim_{x\to 0} \frac{2kx^3}{x^2(x^2+k^2)} = \lim_{x\to 0} \frac{2kx}{x^2+k^2}$$

$$\overset{\text{PLUG}}{\underset{\text{IN}}{=}} \frac{2k\,0}{0^2+k^2} = \frac{0}{k^2} = 0 \quad \text{provided } k \neq 0.$$

AND IF $k=0$ Then

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{ALONG } y=0}} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{0}{x^4} = 0$$

TOO.

SO NO MATTER WHICH <u>LINE</u> YOU COME INTO $(0,0)$ ALONG <u>ALWAYS</u> GET TO HEIGHT 0.

Q DOES THIS TELL US LIMIT IS 0?

A **No!** Maybe if we come in along another curve we will end up at a different height.

In fact:

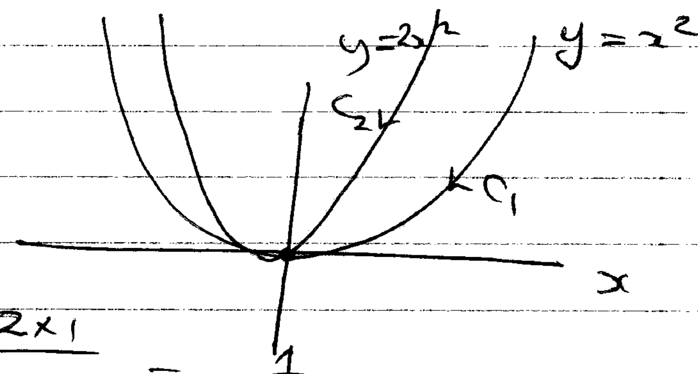
$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{2x^2\,kx^2}{x^4+k^2x^4}$$

Along $y=kx^2$

$$= \lim_{x\to 0} \frac{2k}{1+k^2} = \frac{2k}{1+k^2}$$

Depends on k.

So LIMIT DNE.



ALONG C_1, $L_1 = \frac{2\times 1}{1+1^2} = 1$

BUT ALONG C_2, $L_2 = \frac{2\times 2}{1+2^2} = \frac{4}{5}$.

Since $L_1 \neq L_2$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} \text{ DNE.}$$

WHAT DOES GRAPH OF f LOOK LIKE?

I HAVE A MODEL OF IT!

ONE FINAL METHOD CONVERT TO POLAR COORDS

$$x = r\cos\theta$$
$$y = r\sin\theta$$

NO MATTER HOW $(x,y) \to (0,0)$ WE KNOW
$r = \sqrt{x^2+y^2} \to 0$ HOLDS.

EX
$$z = f(x,y) = \frac{2x^2y}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{2r^3\cos^2\theta\sin\theta}{r^2}$$

$$= \lim_{r\to 0} 2r\cos^2\theta\sin\theta = 0$$

as $\quad -2r \le 2r\cos^2\theta\sin\theta \le 2r$

$$\underset{0}{\downarrow} \qquad\qquad\qquad \underset{0}{\downarrow}$$

So apply Sandwich Theorem

BUT : $\displaystyle \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{r\to 0} \frac{2r^2\cos\theta\sin\theta}{r^2}$

$= \displaystyle\lim_{r\to 0} (\sin 2\theta)$ DEPENDS ON θ AND SO LIMIT DNE.