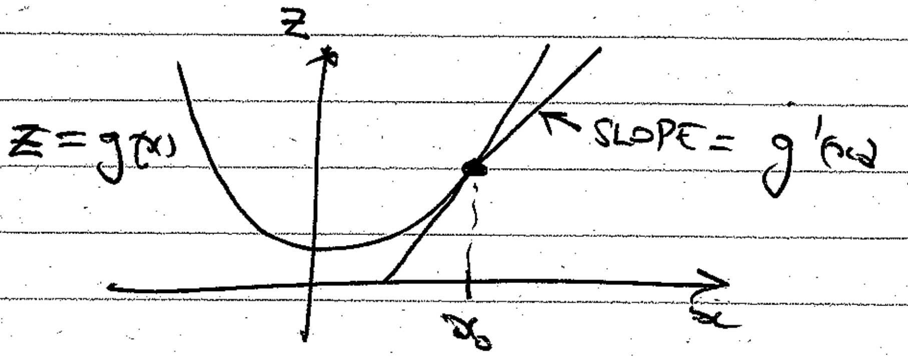
14.3 PARTIAL DERIVATIVES

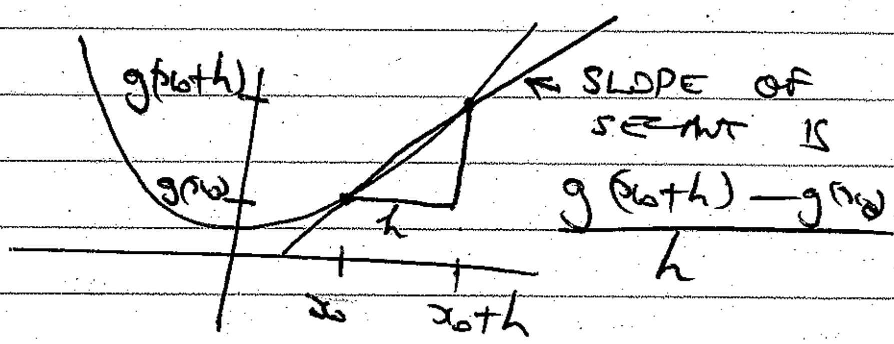
CALC I WARM UP



GIVEN $z = g(x)$
and a pt x_0,

The Rate of Change of g at x_0 is

$$g'(x_0) = \frac{dz}{dx}(x_0) \overset{\text{DEF}}{=} \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h}$$



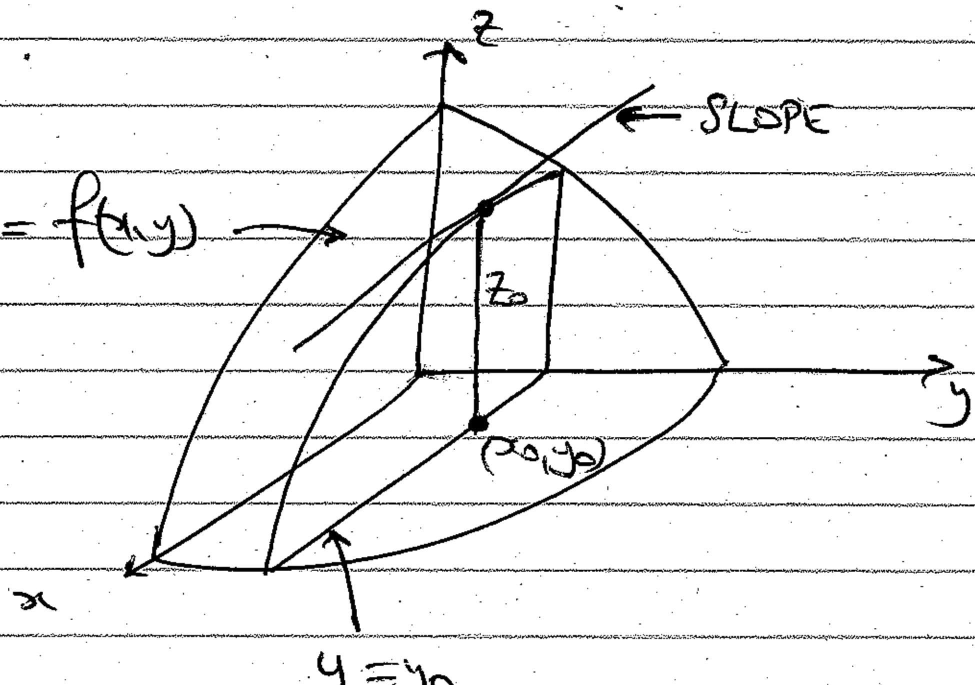
So

$g'(x_0)$ = SLOPE of Tangent Line to $z = g(x)$ at x_0

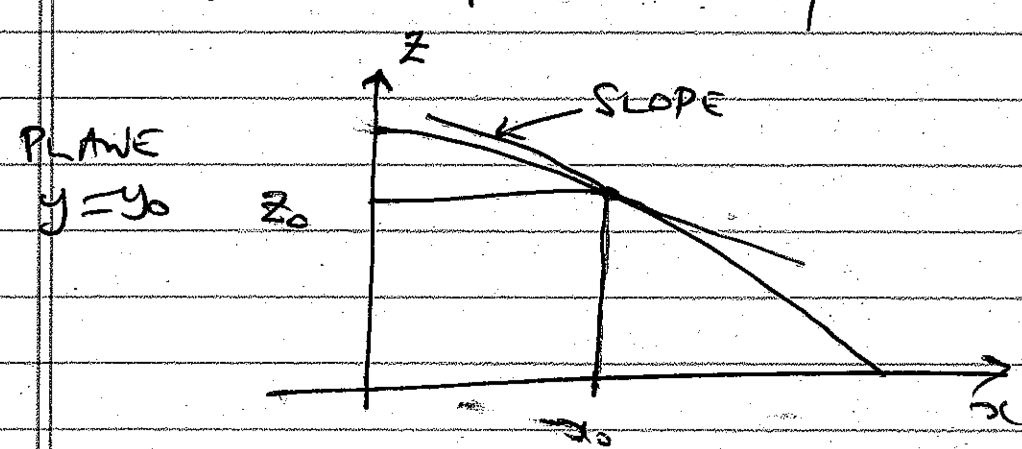
CALC III GIVEN $z = f(x,y)$ and a point (x_0, y_0)

The Rate of Change of f in x-direction

at (x_0, y_0) is defined as follows



SLICE graph $z = f(x,y)$ in plane $y = y_0$ to get



Set

$$z = g(x) = f(x, y_0)$$

NEW NOTATION!

$\frac{\partial f}{\partial x}(x_0, y_0)$ = R of C of f in x-direction at (x_0, y_0)

$\overset{DEF}{=} g'(x_0)$

= Slope at $x = x_0$ of TL to slice of f in plane $y = y_0$.

= PARTIAL DERIVATIVE of f wrt x at (x_0, y_0)

FORMULA

$$\frac{\partial f}{\partial x}(x_0, y_0) = g'(x_0)$$

$$\overset{CALC\ I}{=} \lim_{h \to 0} \frac{g(x_0 + h) - g(x_0)}{h}$$

So

$$\boxed{\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}}$$

EX $z = f(x, y) = 16 - 4x^2 - y^2 - 7xy^3$

$(x_0, y_0) = (1, 2)$

FIND $\frac{\partial f}{\partial x}(1, 2)$

METHOD I $g(x) = f(x, y_0) = f(x, 2)$

$$= 16 - 4x^2 - 2^2 - 7x \cdot 2^3$$

$$= 12 - 4x^2 - 56x$$

So $g'(x) = -8x - 56$

So $\frac{\partial f}{\partial x}(1,2) = g'(1) = -8-56 = -64.$

METHOD II THINK of y as being a constant ($y=y_0$) and differentiate f with respect to (wrt) x:

$$\frac{\partial f}{\partial x} = 0 - 8x - 0 - 7y^3 = -8x - 7y^3.$$

So $\frac{\partial f}{\partial x}(1,2) = -8\times 1 - 7\times 2^3 = -64.$

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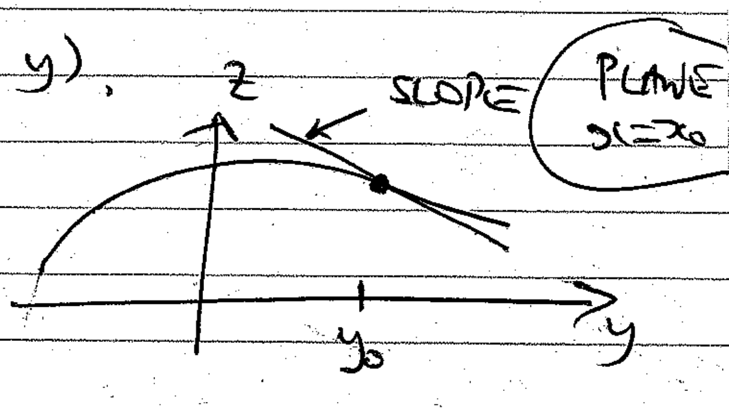
SIMILARLY

$$\frac{\partial f}{\partial y}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

MEANING SLICE $z=f(x,y)$ in plane $x=x_0$ to get $h(y) = f(x_0,y)$.

Then

$\frac{\partial f}{\partial y}(x_0,y_0)$ = SLOPE of h at $y=y_0$



EX CONT'D

$$f(x,y) = 16 - 4x^2 - y^2 - 7xy^3$$

$$\frac{\partial f}{\partial y} = 0 - 0 - 2y - 7x.3y^2 = -2y - 21xy^2$$

So $\frac{\partial f}{\partial y}(1,2) = -4 - 21 \times 4 = -88.$

ALT NOTATION $\frac{df}{dx} = f'$ CALC I

$$\frac{\partial f}{\partial x} = f_x \qquad \text{CALC III}$$

$$\frac{\partial f}{\partial y} = f_y.$$

2ND PARTIAL DERIVATIVES

$$f(x,y) = 16 - 4x^2 - y^2 - 7xy^3$$

$$\frac{\partial f}{\partial x} = -8x - 7y^3$$

$\cancel{\frac{\partial f}{\partial y}} =$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \stackrel{\text{DEF}}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(-8x - 7y^3\right) = -8.$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \stackrel{\text{DEF}}{=} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(-8x - 7y^3\right) = -21y^2. \quad *$$

AND

$$f_y = \frac{\partial f}{\partial y} = -2y - 21xy^2$$

So

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-2y - 21xy^2\right)$$

$$= -2 - 42xy$$

AND

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} \stackrel{\text{DEF}}{=} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(-2y - 21xy^2\right)$$

$$= -21y^2 \quad *$$

$*$'s are EQUAL !!

THM IF $z = f(x,y)$ THEN

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

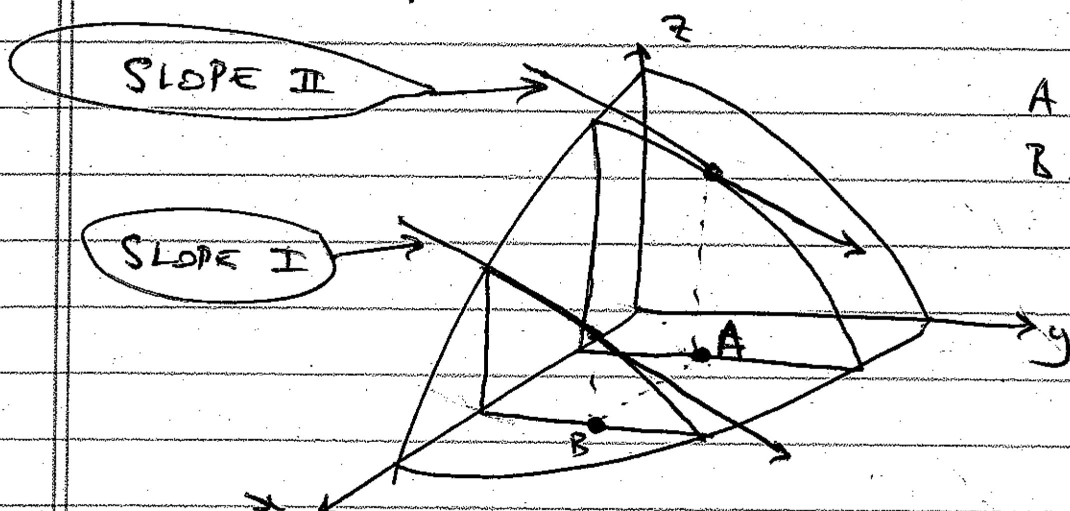
(provided f is "nice" enough).

$\boxed{\text{MEANING OF } \frac{\partial^2 f}{\partial x \partial y}}$

$$\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \lim_{h \to 0} \frac{\overbrace{\frac{\partial f}{\partial y}(x_0+h, y_0)}^{\text{SLOPE I}} - \overbrace{\frac{\partial f}{\partial y}(x_0, y_0)}^{\text{SLOPE II}}}{h}$$

= RofC$_h$ wrt x of SLOPE of SLICE of Graph of f in planes x = CONSTANT



$A = (x_0, y_0)$
$B = (x_0+h, y_0)$

DIFFERENTIAL EQNS. (DEs)

DEs provide a mathematical formulation of physical laws or biological phenomena.

They are ubiquitous in engineering + science.

ORDINARY DEs (ODEs)

- Eqn involving an unknown function $u = u(t)$ of 1 variable t and its derivatives.

EXS ① EXPONENTIAL GROWTH

$$\frac{du}{dt} = 3u$$

PARTICULAR SOLN $u(t) = e^{3t}$

GENERAL SOLN $u(t) = Ae^{3t}$ for an arbitrary constant A.

② SIMPLE HARMONIC MOTION

$$\frac{d^2u}{dt^2} + u = 0$$

GENERAL SOLN

$$u(t) = A\cos t + B\sin t$$

for constants A, B.

PARTIAL DEs (PDEs)

- Eqns involving an UNKNOWN function of 2^+ variables an its partial derivatives.

EXS ① THE WAVE EQN

If $u = u(t, x)$ = Amplitude of a wave a position x. and time t and c is the wave speed

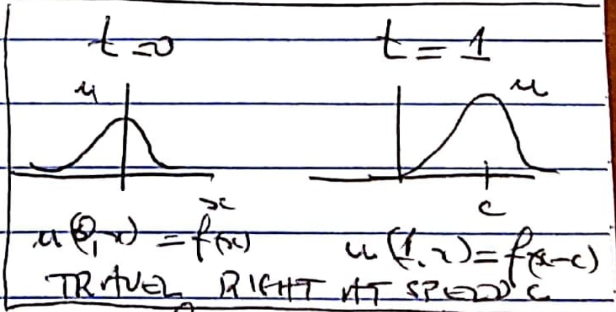
Then physics tells us

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

PARTICULAR SOLN : $u = \sin(x - ct)$

CHECK
$u_t = -c\cos(x - ct)$
$u_{tt} = -c^2 \sin(x - ct)$
$u_x = \cos(x - ct)$
$u_{xx} = -\sin(x - ct)$

So $u_{tt} = c^2 u_{xx}$.



MORE GENERAL SOLN : $u(t, x) = f(x - ct)$ for an arbitrary f^n $f: \mathbb{R} \to \mathbb{R}$

CHECK

$u_x = f'(x - ct)$ $\quad$ $u_t = -c f'(x - ct)$

$u_{xx} = f''(x - ct)$ $\quad$ $u_{tt} = (-c)^2 f''(x - ct)$

$= c^2 u_{xx}$

(2) LAPLACE'S EQN — time-independent

- Arises when modeling distributions of heat and electric charge

- Unknown function $u = u(x,y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

PARTICULAR SOLN: $u(x,y) = e^x \sin y$

$u_{xx} = e^x \sin y$
$u_{yy} = -e^x \sin y$ So $u_{xx} + u_{yy} = 0.$