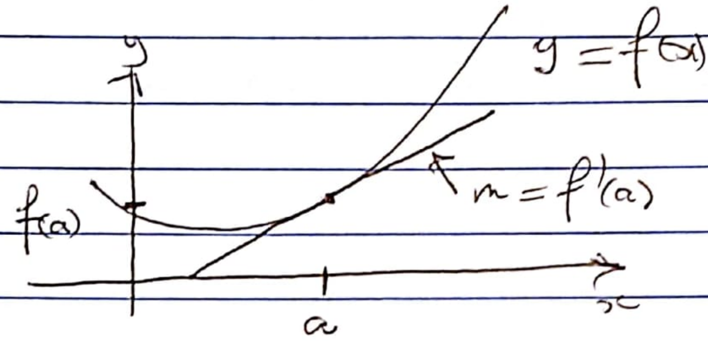
14.4 TANGENT PLANES + LINEAR APPROXIMATIONS

CALCULUS 1 REVIEW. $y = f(x)$



Eq^n of Tangent Line to $y = f(x)$ at $x = a$ is

$$y = L(x) = f(a) + f'(a)(x-a)$$

LINEAR APPROX^n to f AT $x=a$

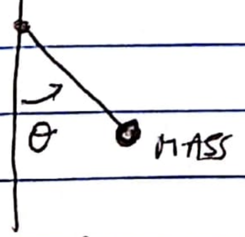
As you zoom in on the point $(x,y) = (a, f(a))$ the graph of f looks more + more like graph of L.

So $\quad f(x) \simeq L(x) \quad$ for x near a

This approximation is valid provided f is differentiable at $x = a$.

APPLICATION [SIMPLE PENDULUM]

FIND $\theta = \theta(t)$



Newton's Laws tell us

$$\boxed{\ddot{\theta} \overset{(*)}{=} -\sin\theta}$$

$\dot{} = \frac{d}{dt}$

We want a function $\theta = \theta(t)$ satisfying (*)

HARD!

But if we suppose $\theta \simeq 0$ is small
Then we can linearize:

$$f(\theta) = \sin\theta \simeq L(\theta) = f(0) + f'(0)(\theta - 0)$$
$$= \theta$$

to get

$$\boxed{\ddot{\theta} = -\theta}$$

Linear Differential Eqn
(MATH 2420)

Solution: $\theta(t) = A\cos t + B\sin t$.

Physicists and Engineers use Linear Approximations in this manner all the time.

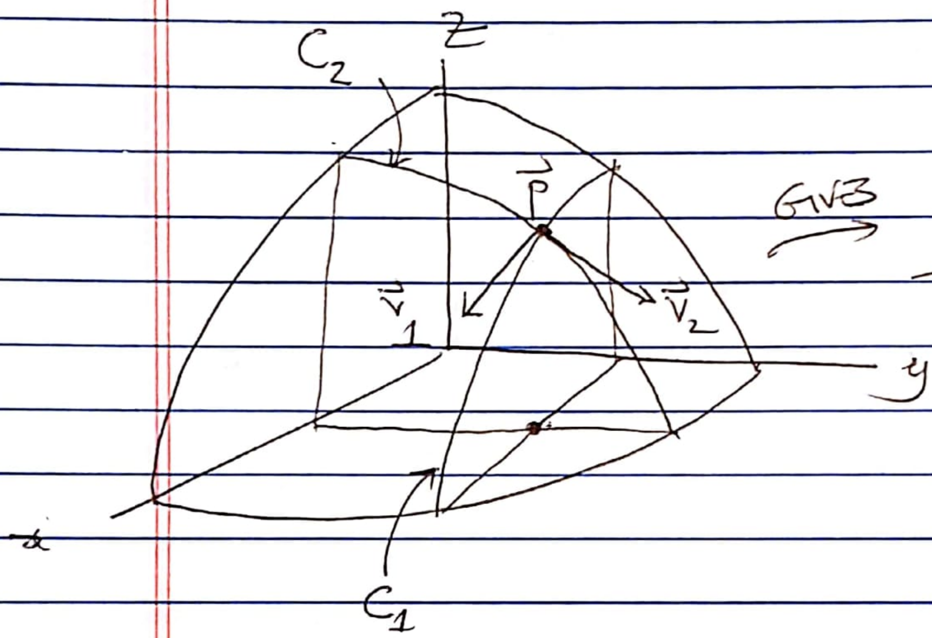
$\boxed{\text{CALCULUS III}}$ $z = f(x,y)$

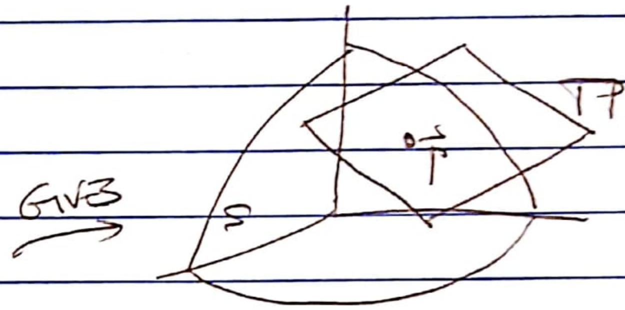
IDEA We say a function $z = f(x,y)$ is differentiable at $(x_0, y_0, f(x_0,y_0))$ if

as you zoom in on this point the graph of f looking more and more like a plane.

This plane is called the TANGENT PLANE to graph of f at that point

GOAL Find the level set equation of this tangent plane.





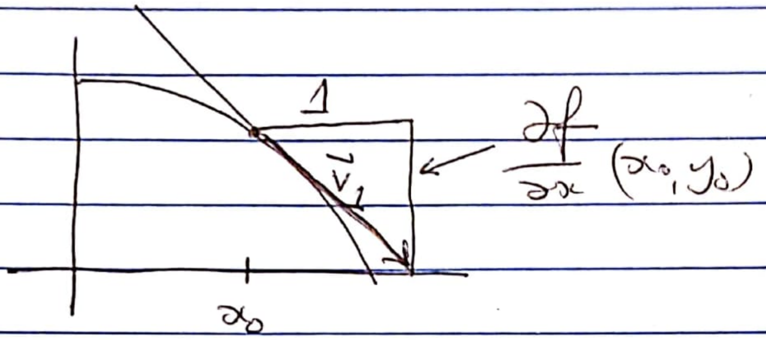
IDEA Tangent Plane is plane through $\vec{p} = (x_0, y_0, f(x_0, y_0))$ containing vectors $\vec{v}_1$ and $\vec{v}_2$.

To get $\vec{v}_1$: Slice graph of $z = f(x,y)$ in plane $y = y_0$ to get curve C_1. Let $\vec{v}_1$ be tangent vector to C at $\vec{p}$.

Similarly for $\vec{v}_2$.

FORMULA FOR $\vec{v}_1$

$\boxed{y = y_0}$



So

$$\vec{v}_1 = \left(1,\ 0,\ \frac{\partial f}{\partial x}(x_0, y_0)\right)$$

SIMILARLY $\vec{v}_2 = \left(0,\ 1,\ \frac{\partial f}{\partial x}(x_0, y_0)\right)$

<u>NORMAL TO TP</u>

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix}$$

$$= \left(-f_x,\ -f_y,\ 1\right)$$

<u>EQN OF TP</u> $(\vec{x} - \vec{p}) \cdot \vec{n} = 0$

OR $0 = \left(x - x_0,\ y - y_0,\ z - f(x_0, y_0)\right) \cdot \left(-\frac{\partial f}{\partial x}(x_0, y_0),\ -\frac{\partial f}{\partial y}(x_0, y_0),\ 1\right)$

Gives

$$\boxed{z = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)}$$

Analogous to $z = f(x_0) + f'(x_0)(x - x_0)$ for $z = f(x)$

So if (x,y) is near (x_0,y_0) then

$$f(x,y) \simeq L(x,y)$$

where

$$L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$= A + Bx + Cy$$

is the <u>linearization of</u> or <u>linear approx^n to f</u>

at (x_0,y_0).

Ex

① $z = f(x,y) = y^2 - x^2$ at $(x_0,y_0) = (-4,5)$

$\frac{\partial f}{\partial x} = -2x = 8$ @ $(-4,5)$ $\quad f(-4,5) = 9$

$\frac{\partial f}{\partial y} = 2y = 10$ @ $(-4,5)$

So eqn of TP is

$$z = 9 + 8(x+4) + 10(y-5)$$
$$= 8x + 10y - 9$$

2 $f(x,y) = \sin(x+y^2)$ @ $(x,y) = (3,2)$

FIND LINEAR^n of f

$L(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2)$

$= \sin(7) + \cos(7)(x-3) + 4\cos(7)(y-2)$