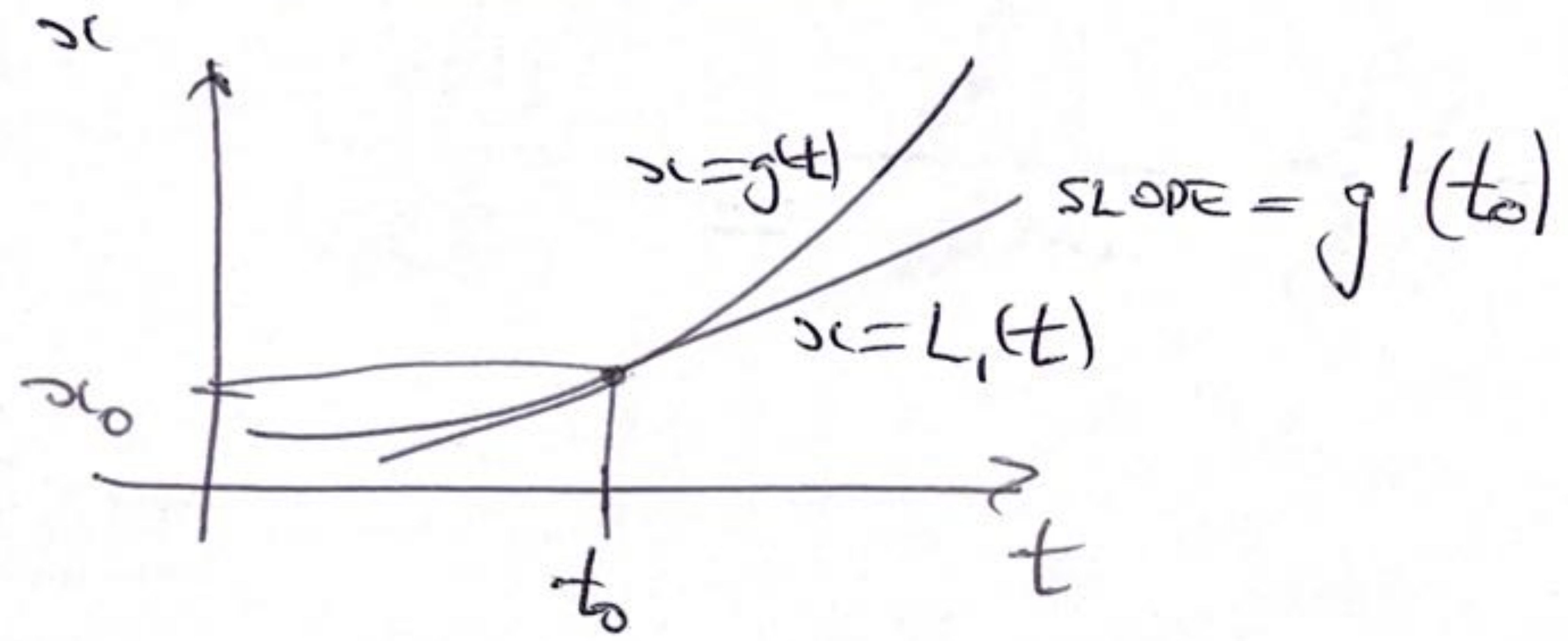
14.5 THE CHAIN RULE

CASE 0 CR IN CALC I

Given

$$\boxed{x = g(t)}$$

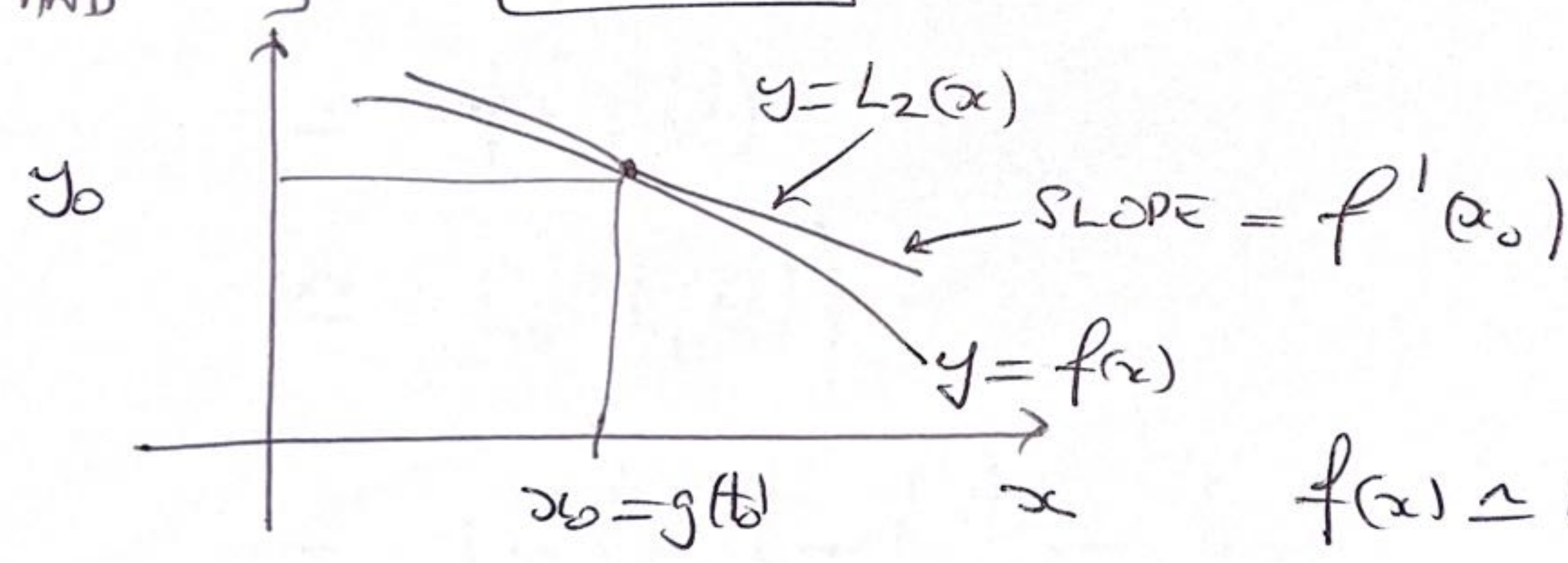


$g(t) \simeq L_1(t) = m_1 t + b_1$

for t near t_0.

$$\boxed{y = f(x)}$$

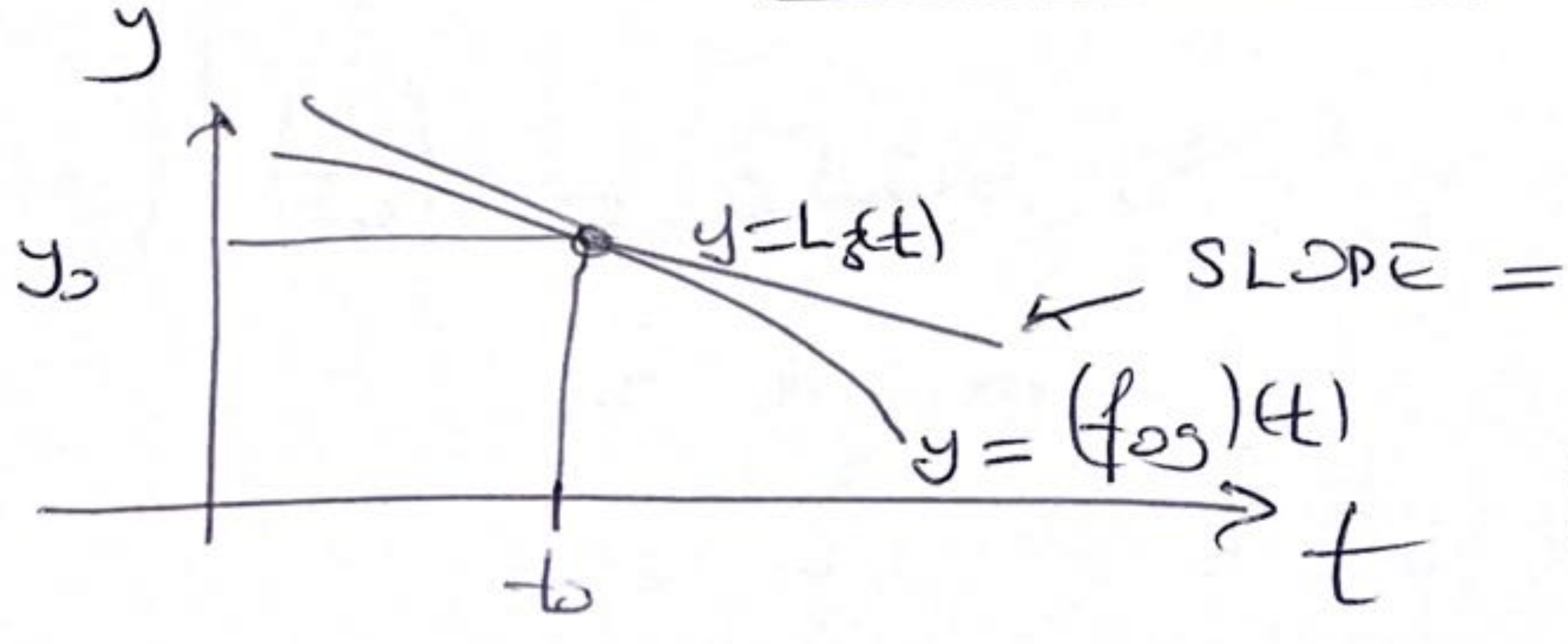
AND



$f(x) \simeq L_2(x) = m_2 x + b_2$

for x near x_0

FORM COMPOSITION $\boxed{y = (f \circ g)(t)}$



SLOPE $= (f \circ g)'(t_0)$

$(f \circ g)(t) \simeq L_3(t)$
for t near t_0

CHAIN RULE

(a) SLOPE OF L_3 = SLOPE OF L_2 × SLOPE OF L_1

(b) $(f \circ g)'(t_0) = f'(g(t_0))\, g'(t_0)$

(c) $$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

PROOF

$$L_3(t) \simeq (f \circ g)(t)$$
$$= f(g(t))$$
$$\simeq L_2(L_1(t))$$
$$= m_2(m_1 t + b_1) + b_2$$
$$= m_2 m_1 t + b_3$$

SO

$$(f \circ g)'(t_0) = \text{SLOPE OF } L_3$$
$$= m_2 m_1$$
$$= (\text{SLOPE OF } L_2) \times (\text{SLOPE OF } L_1)$$

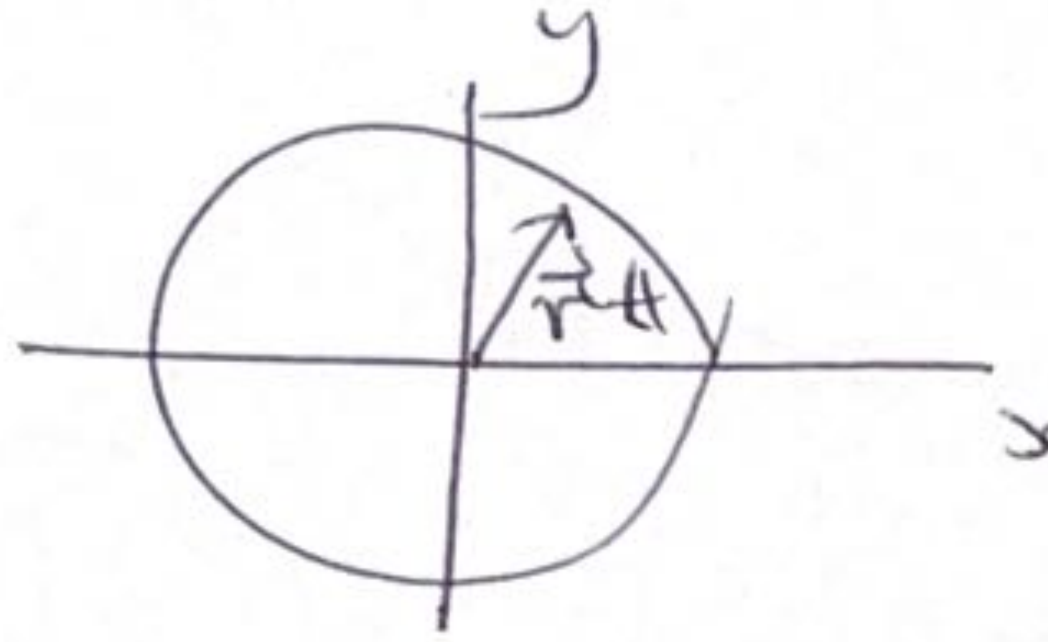
CASE 1 — CR FOR FUNCTIONS ON CURVES

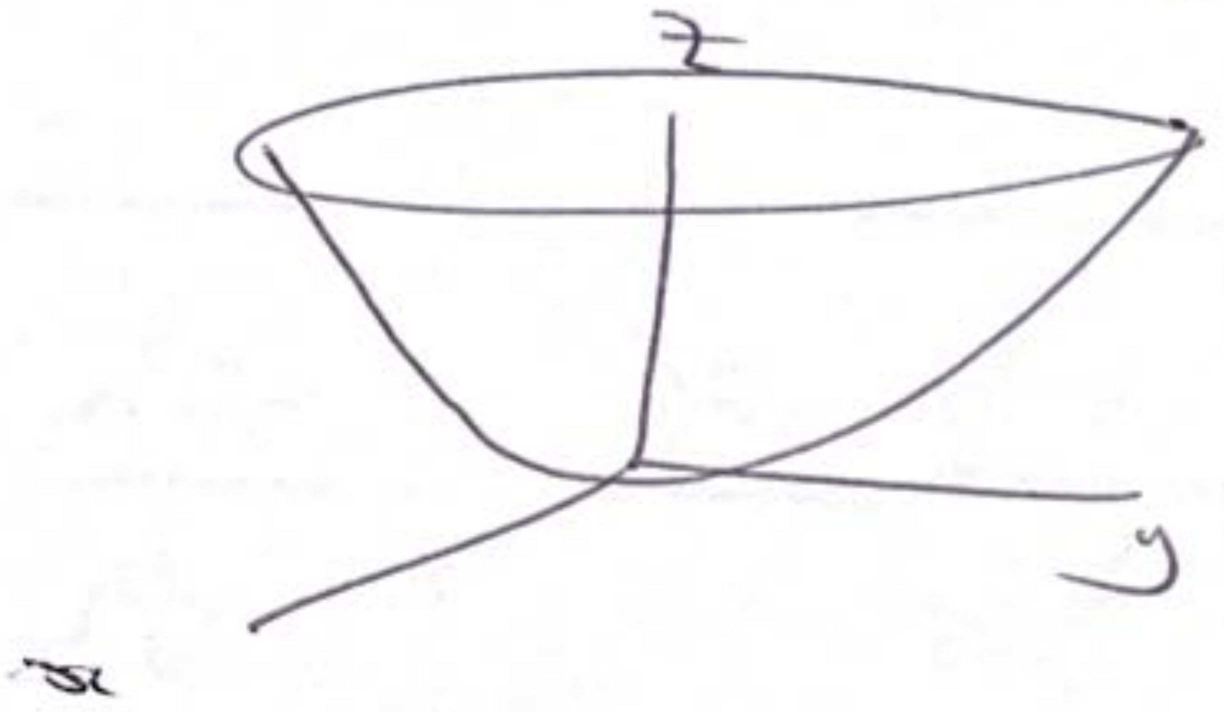
GIVEN

(1) $(x,y) = \vec{r}(t) \overset{EX}{=} (\cos t, \sin t)$ = POS^N OF ANT AT TIME t

where $\vec{r}$ is a curve in plane.

(2) $z = f(x,y) \overset{EX}{=} 3x^2 + 4y^2$ = TEMP AT PT (x,y) IN PLANE





How does temperature of ant change with time at $t = \pi/4$?

IE FIND $\dfrac{dz}{dt}$

METHOD I FORM THE COMPOSITION.

$$z(t) = (f \circ \vec{r})(t) = f(\vec{r}(t))$$ = RESTRICTION OF f TO CURVE $\vec{r}$.

$$= f(x(t), y(t))$$

$$\overset{EX}{=} 3\cos^2 t + 4\sin^2 t$$

$$= 3 + \sin^2 t$$

So

$$\frac{dz}{dt} = 2\sin t\cos t$$

$$\frac{dz}{dt}\left(\frac{\pi}{4}\right) = 2 \cdot \frac{1}{\sqrt{2}}\,\frac{1}{\sqrt{2}} = \underline{1}$$

$\boxed{\text{METHOD II}}$ CHAIN RULE FOR FUNCTIONS ON CURVES

$$\begin{tikzcd}
 & z \arrow[dl, dash, "\frac{\partial z}{\partial x}"'] \arrow[dr, dash, "\frac{\partial z}{\partial y}"] & \\
x \arrow[dr, dash, "\frac{dx}{dt}"'] & & y \arrow[dl, dash, "\frac{dy}{dt}"] \\
 & t &
\end{tikzcd}$$

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

<u>EX</u> $\dfrac{\partial z}{\partial x} = 6x \qquad \dfrac{\partial z}{\partial y} = 8y$

$\dfrac{dx}{dt} = -\sin t \qquad \dfrac{dy}{dt} = \cos t$

EVALUATE DERIVATIVES OF OUTER FUNCTION $z = f(x,y)$ AT VALUES OF INNER F^n $x = x(t)$, $y = y(t)$

So $$\frac{dz}{dt} = 6x(t)(-\sin t) + 8y(t)\cos t$$
$$= -6\cos t\sin t + 8\sin t\cos t$$
$$= 2\sin t\cos t \checkmark$$

EX Suppose $z = f(x,y)$
$(x,y) = \vec{r}(t)$

$\vec{r}(0) = (1,3)$ $\qquad f(1,3) = 4$

$\vec{r}\,'(0) = (2,5)$

$$\frac{\partial f}{\partial x}(1,3) = 6$$

$$\frac{\partial f}{\partial y}(1,3) = 7$$

Then

$$\frac{dz}{dt}(0) = \frac{\partial f}{\partial x}(\vec{r}(0))\,\frac{dx}{dt}(0) + \frac{\partial f}{\partial y}(\vec{r}(0))\,\frac{dy}{dt}(0)$$

$$= \frac{\partial f}{\partial x}(1,3)\,\frac{dx}{dt}(0) + \frac{\partial f}{\partial y}(1,3)\,\frac{dy}{dt}(0)$$

$$= 6 \times 2 + 7 \times 5 = 47$$

ALTERNATE FORM OF CR FOR FUNCTIONS ON CURVES

$\vec{r} : \mathbb{R}^1 \to \mathbb{R}^2$ $\qquad f : \mathbb{R}^2 \longrightarrow \mathbb{R}^1$

$\vec{r}(t) = (x(t),\ y(t))$ $\qquad z = f(x,y)$

$\vec{r}\,'(t) = (x'(t),\ y'(t))$ $\qquad \nabla f = \left[\frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y}\right]$

"GRADIENT OF f"

$$(f \circ \vec{r})'(t) = \frac{\partial f}{\partial x}(\vec{r}(t))\,\frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(\vec{r}(t))\,\frac{dy}{dt}(t)$$

$$= \left(\frac{\partial f}{\partial x}(\vec{r}(t)), \frac{\partial f}{\partial y}(\vec{r}(t))\right) \cdot \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right)$$

$$= \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

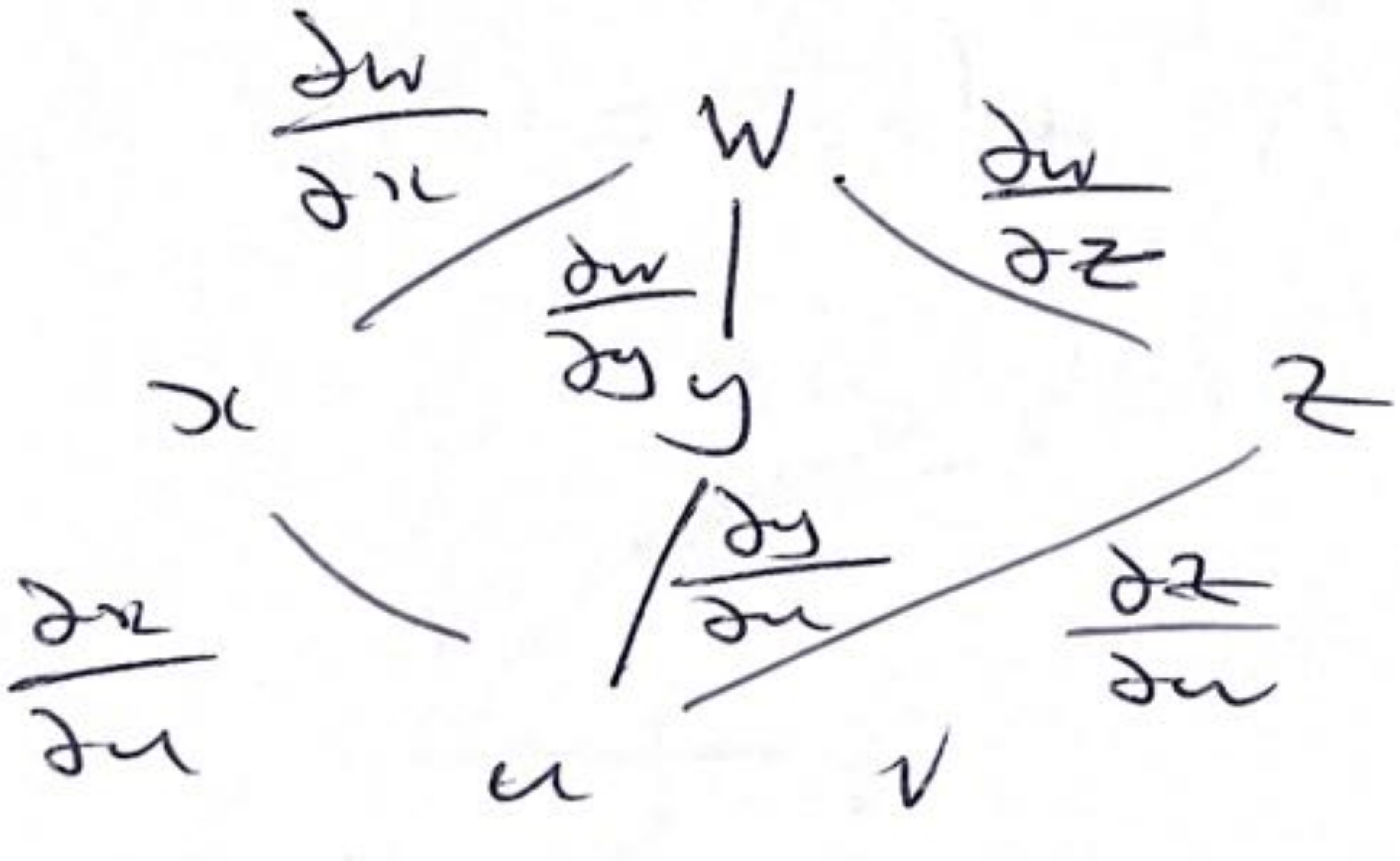
$$\boxed{(f \circ \vec{r})'(t) = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}$$

CASE 2

$$(x, y, z) = \vec{r}(u, v) \overset{EX}{=} (v\cos u,\ v\sin u,\ v)$$

$$w = f(x, y, z) \overset{EX}{=} 3x^2 + 4y^2 + 6z^2.$$

FIND $\frac{\partial w}{\partial u}$ AT $(u_0, v_0) = (0, 2)$



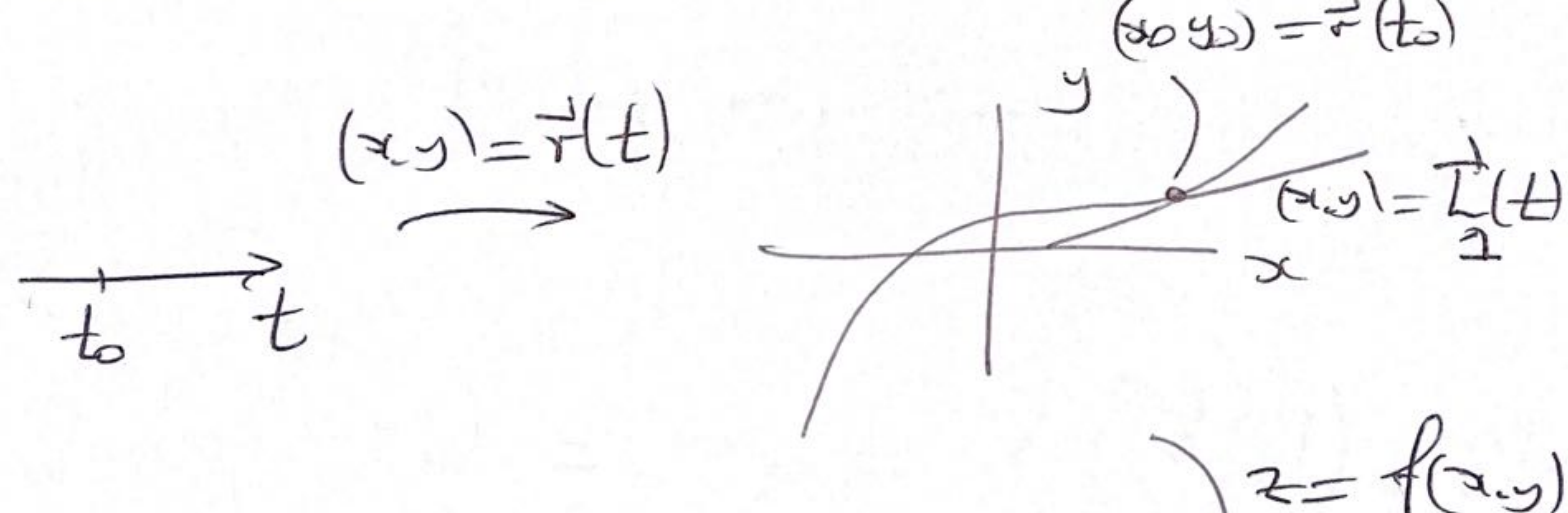
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= 6x(u,v)\left(-v\sin u\right) + 8y(u,v)\left(v\cos u\right) + 10z(u,v)\cdot 0$$

$$= 6\cdot 2\cdot\left(-2\cdot 0\right) + 8\cdot 0\cdot ? + 0$$

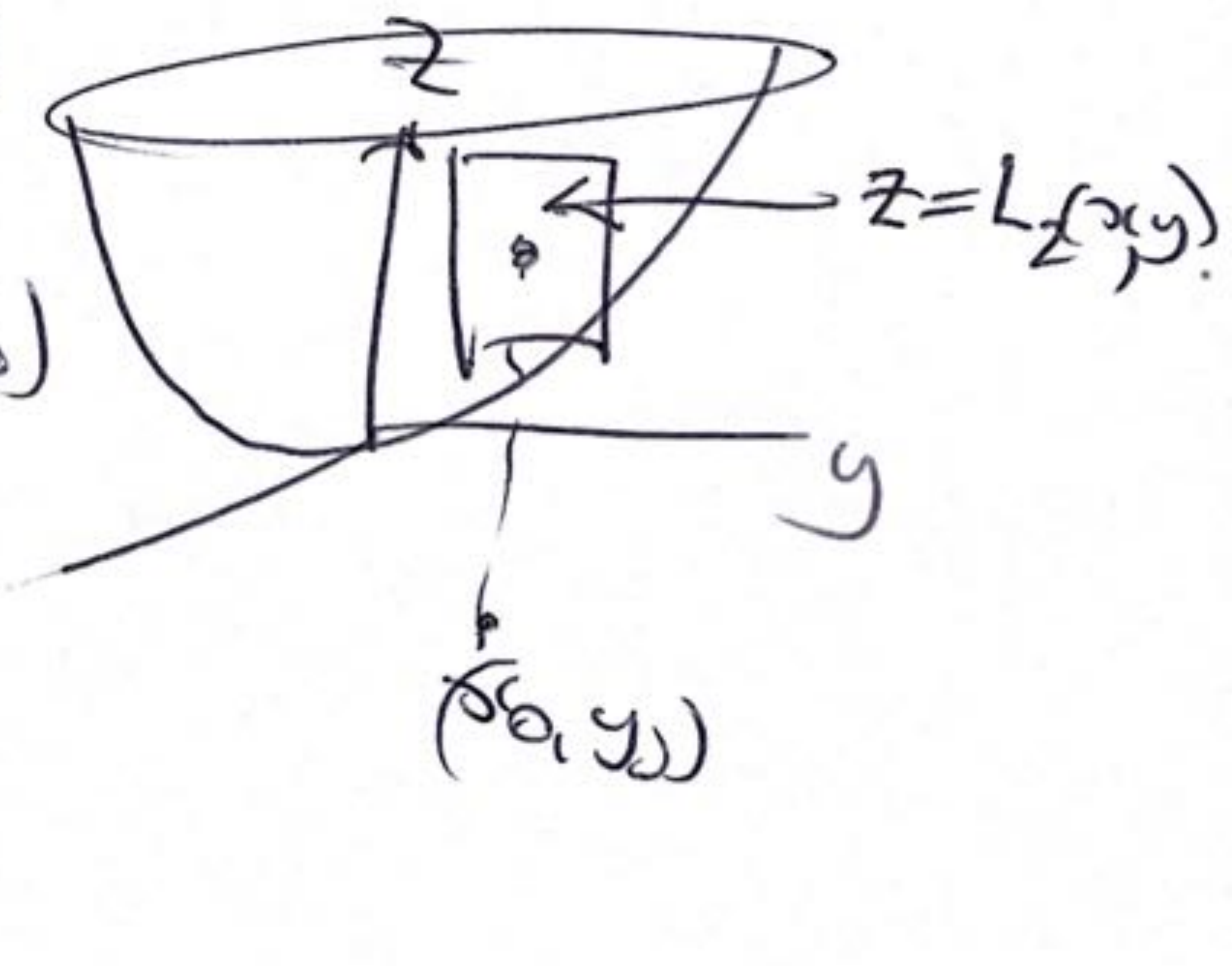
$$= 0.$$

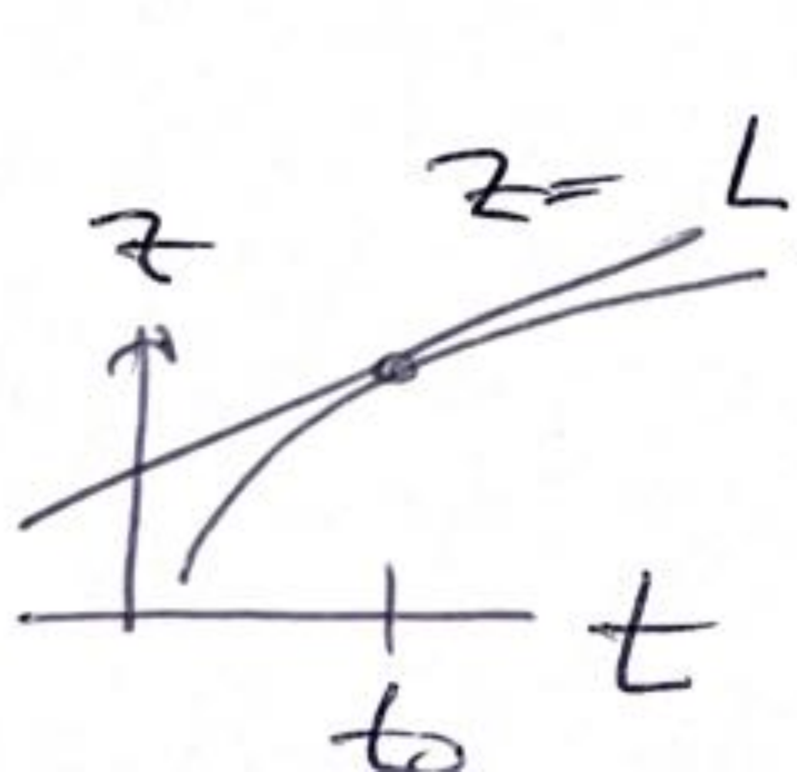
PROOF OF CR FOR FUNCTIONS ON CURVES.



$$\vec{r}(t) \simeq L_1(t) = \vec{r}(t_0) + (t-t_0)\,\vec{r}\,'(t_0)$$

$$f(\vec{x}) \simeq L_2(\vec{x}) = f(\vec{x}_0) + \nabla f(\vec{x}_0)\cdot(\vec{x}-\vec{x}_0)$$





$L_3(t) \simeq f(\vec{r}(t))$ for t near t_0 8

$$\simeq f(\underbrace{\vec{r}(t_0) + (t-t_0)\vec{r}\,'(t_0)}_{\vec{x}}) \qquad \vec{x}_0 = \vec{r}(t_0)$$

$$\simeq f(\vec{x}_0) + \nabla f(\vec{x}_0) \cdot \underbrace{(t-t_0)\vec{r}\,'(t_0)}_{\vec{x}-\vec{x}_0.}$$

$$= (f \circ \vec{r})(t_0) + \nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0)\,(t-t_0)$$

$$= b_3 + m_3 (t-t_0)$$

So

$$(f \circ \vec{r})'(t_0) = m_3 = \nabla f(\vec{r}(t_0)) \cdot \vec{r}\,'(t_0)$$