14.6 DIRECTIONAL DERIVATIVES + THE GRADIENT

GIVEN $z = f(x,y)$ and a point (x_0, y_0)

RECALL $\dfrac{\partial f}{\partial x}(x_0, y_0) = \lim\limits_{h \to 0} \dfrac{f(x_0+h, y_0) - f(x_0)}{h}$

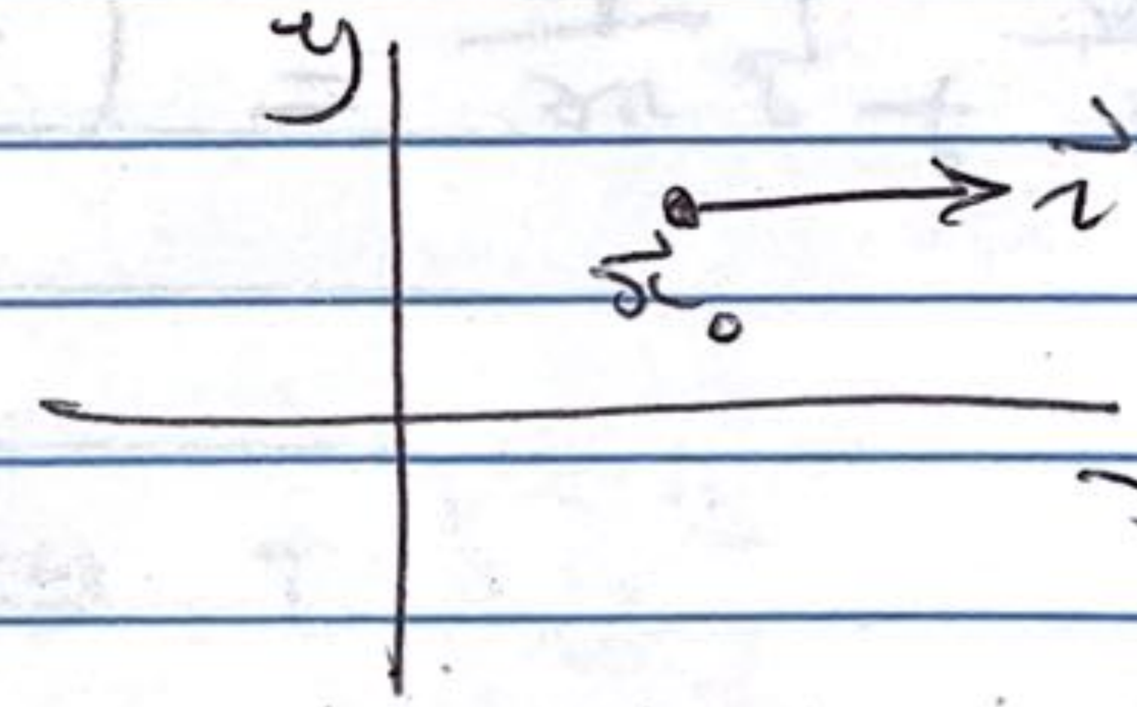
Recast using vectors:

$\vec{x} = (x,y), \quad \vec{x}_0 = (x_0, y_0)$

$(x_0 + h, y_0) = (x_0, y_0) + h(1,0) = \vec{x}_0 + h\vec{i}$

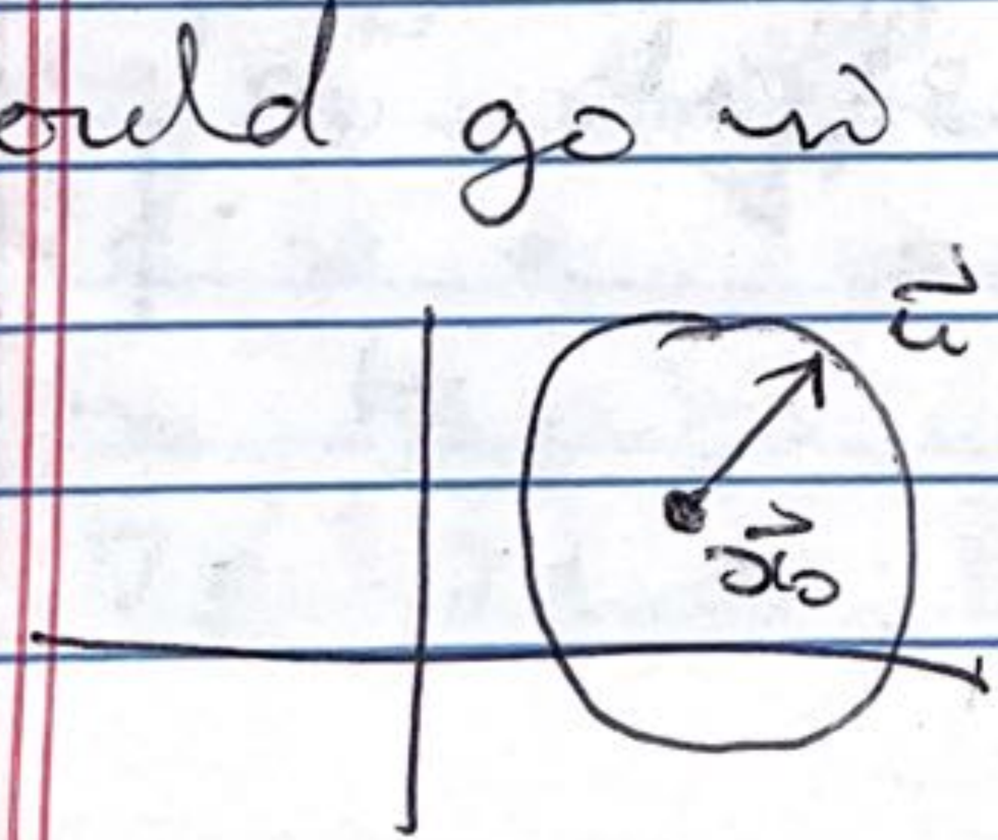
So $\dfrac{\partial f}{\partial x}(\vec{x}_0) = \lim\limits_{h \to 0} \dfrac{f(\vec{x}_0 + h\vec{i}) - f(\vec{x}_0)}{h}$

$\dfrac{\partial f}{\partial x}(\vec{x}_0) = $ RofC of f at $\vec{x}_0$ IN DIRECTION $\vec{i}$.



There is nothing special about the dir^n $\vec{i}$.

We could go in ANY direction $\vec{u}$ from $\vec{x}_0$



ALWAYS CHOOSE $|\vec{u}| = 1$

DEF

The DIRECTIONAL DERIVATIVE of f at $\vec{x}_0$ in direction $\vec{u}$ is

$$(D_{\vec{u}} f)(\vec{x}_0) := \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$$

Q How to compute $D_{\vec{u}} f(\vec{x}_0)$?

A Use the Gradient of f.

DEF The GRADIENT of $z = f(x,y)$ is the vector field

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}.$$

EX

$$z = f(x,y) = 3x^2 + 4xy + 5y^2$$

FIND ∇f:

$$\nabla f(x,y) = (6x + 4y,\ 4x + 10y)$$

– gives a different vector at each (x,y)

So ∇f is a "vector field"

If plug in $(x,y) = (1,2)$ get

$$\nabla f(1,2) = (6+8,\ 4+20) = (14, 24)$$

– A VECTOR AT $(1,2)$

THM $\quad (D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}$

[SCALAR $\qquad$ VECTOR $\cdot$ VECTOR]

EX $\quad z = f(x,y) = 3x^2 + 4xy + 5y^2.$

(a) $\vec{x}_0 = (1,2)$, $\quad \vec{u} = \dfrac{(2,3)}{|(2,3)|} = \dfrac{1}{\sqrt{13}}(2,3)$

UNIT VECTOR.

Then

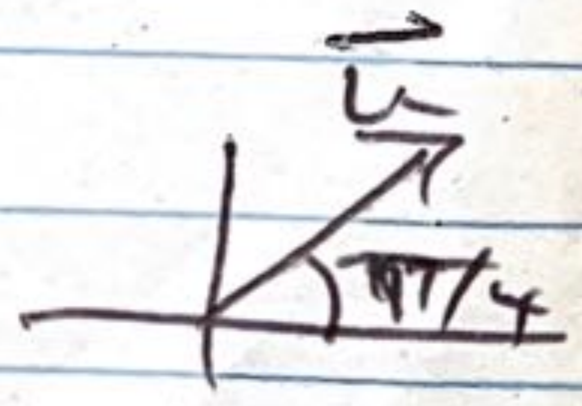
$$(D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}$$

$$= \nabla f(1,2) \cdot \frac{1}{\sqrt{13}}(2,3)$$

$$\overset{\text{ABOVE}}{=} \frac{1}{\sqrt{13}}(14,24) \cdot (2,3) = \frac{1}{\sqrt{13}}(100).$$

(b) Find Dir Der of f in dirn $\theta = \pi/4$ at $\vec{x}_0 = (1,2)$

USE $\quad \vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\quad$ UNIT LENGTH



$$D_{\vec{u}} f(\vec{x}_0) = (14,24) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} 38.$$

PROOF OF THM

Fix $\vec{x}_0 \in \mathbb{R}^2$ and a unit vector $\vec{u}$.

I walk in xy-plane along curve $(x,y) = \vec{r}(t)$
$= \vec{x}_0 + t\vec{u}$

My friend walks on surface $z = f(x,y)$ immediately above me.

So my friend's elevation at time t is

$$z = g(t) = (f \circ \vec{r})(t)$$

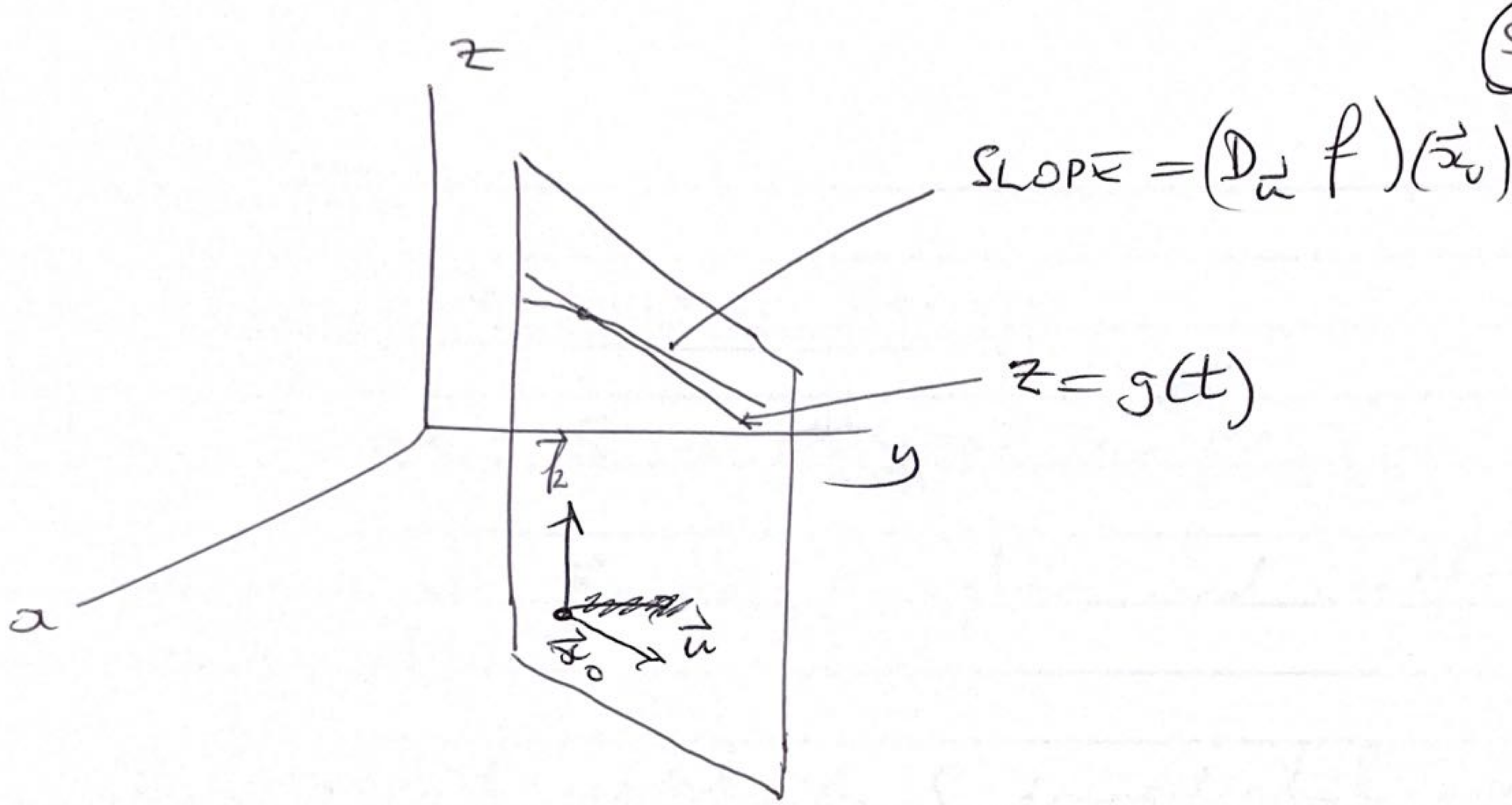
Notice $\vec{r}(0) = \vec{x}_0$, $\vec{r}\,'(0) = \vec{u}$

CLAIM

$$(D_{\vec{u}} f)(\vec{x}_0) = g'(0)$$
$$= \text{R of C of friend's elevation @ } t = 0$$

Given Claim:

$$(D_{\vec{u}} f)(\vec{x}_0) = (f \circ \vec{r})'(0) = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0)$$
$$= \nabla f(\vec{x}_0) \cdot \vec{u}.$$



PF OF CLAIM

$$(D_{\vec{u}} f)(\vec{x}_0) = \lim_{h \to 0} \frac{f(\vec{x}_0 + h\vec{u}) - f(\vec{x}_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(\vec{r}(h)) - f(\vec{r}(0))}{h}$$

$$= \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= g'(0)$$

APPLICATIONS

(I) DIRECTION OF STEEPEST ASCENT

THINK SURFACE S = GRAPH of $z = f(x,y)$.

You are at $\vec{x}_0$ in xy-plane and walk in direction $\vec{u}$.

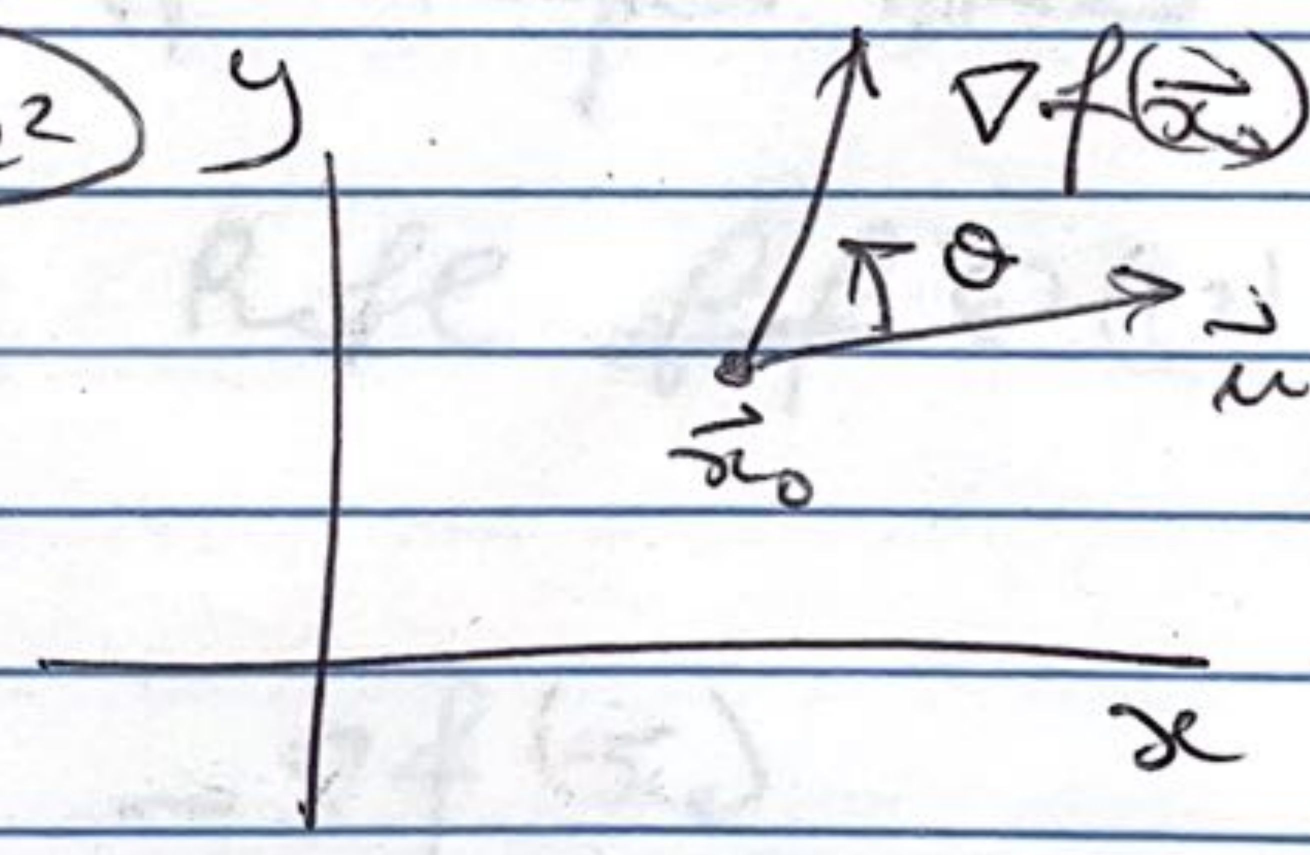
Your friend walks on S immediately above you.

What dir^n $\vec{u}$ should you walk so that your friend goes up hill the steepest on S.

IE How do we choose $\vec{u}$ to MAXIMIZE $D_{\vec{u}} f(\vec{x}_0)$?

SOLN

VECTORS IN $\mathbb{R}^2$



$$(D_{\vec{u}} f)(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \vec{u}$$

$$= |\nabla f(\vec{x}_0)|\,|\vec{u}| \cos\theta$$

$$= |\nabla f(\vec{x}_0)| \cos\theta \qquad \text{as } |\vec{u}| = 1$$

$\leadsto$ biggest when $\theta = 0$.

So want $\vec{u} \parallel \nabla f(\vec{x}_0)$.

SUMMARY [PHYSICS INTERPRETATION OF VECTOR $\nabla f(\vec{x}_0)$]

DIRECTION of STEEPEST ASCENT is

$$\vec{u} = \frac{\nabla f(\vec{x}_0)}{|\nabla f(\vec{x}_0)|}$$

R of C of f in that dir^n = $|\nabla f(\vec{x}_0)|$ (as $\theta = 0$)

EX $z = f(x,y) = 3x^2 + 4xy + 5y^2$ @ $\vec{x}_0 = (1,2)$

$\nabla f(1,2) = (14, 24)$

So

$$\vec{u} = \frac{1}{\sqrt{14^2+24^2}}(14,24) = \text{Dir}^n \text{ of Steepest Ascent}$$

$|\nabla f(1,2)| = \sqrt{14^2+24^2}$ = MAX R of C of f @ (1,2)

SIMILARLY

DIR^n of Steepest DESCENT is $\vec{u} = \dfrac{-\nabla f(\vec{x}_0)}{|\nabla f(\vec{x}_0)|}$

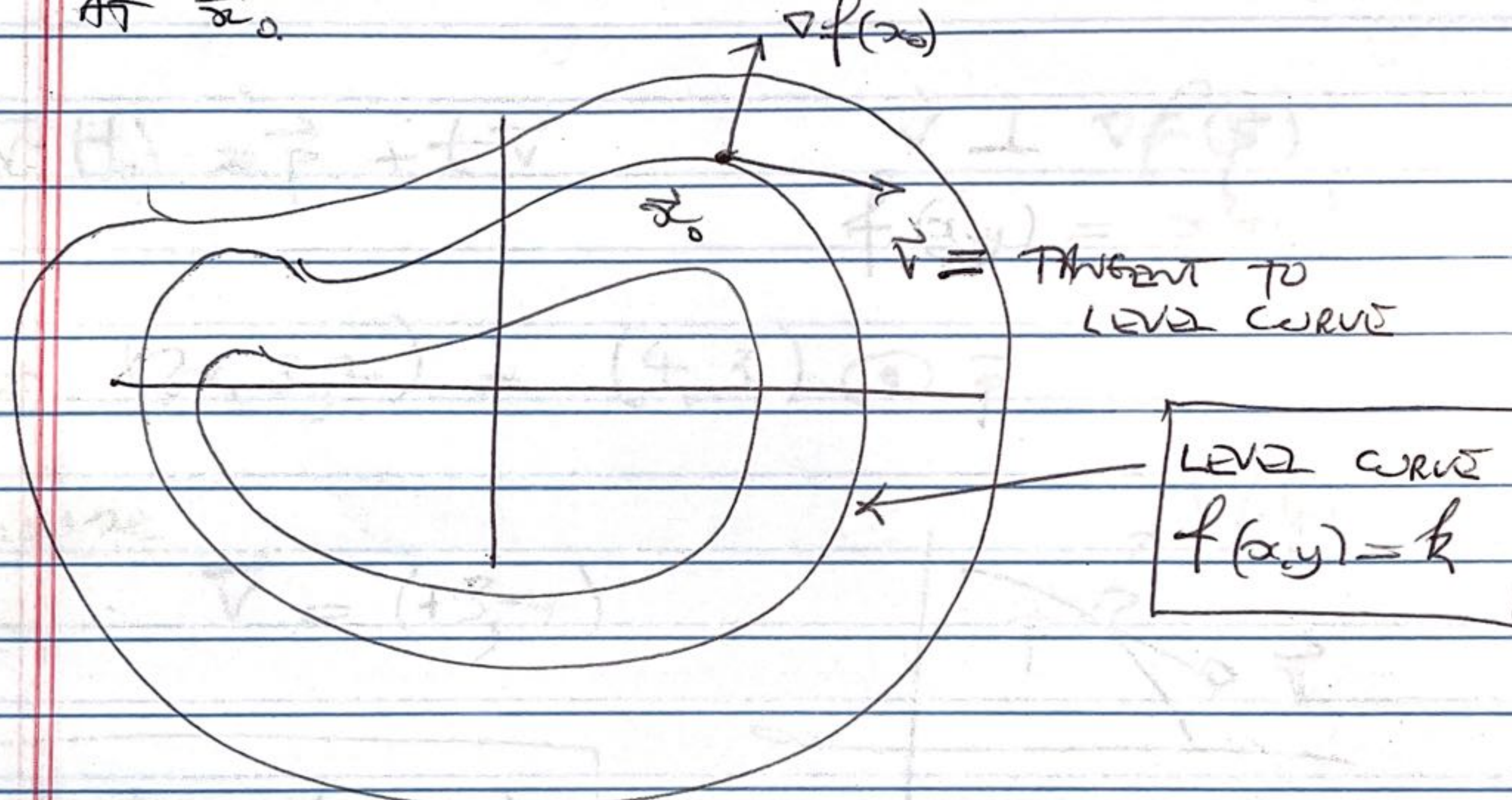
with

R of C $= -|\nabla f(\vec{x}_0)|$ (as $\theta = \pi$)

II GRADIENTS + LEVEL CURVES

IDEA For $z = f(x,y)$ @ $\vec{x}_0$

DIR^n OF STEEPEST ASCENT $\perp$ LEVEL CURVE THRU $\vec{x}_0$ AT $\vec{x}_0$.



Pf
Let $\vec{r}(t)$ parametrize the Level Curve of f thru $\vec{x}_0$

Choose $\vec{r}(0) = \vec{x}_0$.

We know $(f \circ \vec{r})(t) = k$

$\boxed{\text{RESTRICTION OF } f \text{ TO LEVEL CURVE IS CONSTANT FN}}$

So

$$0 = (f \circ \vec{r})'(0)$$

$$0 = \nabla f(\vec{r}(0)) \cdot \vec{r}\,'(0) = \nabla f(\vec{x}_0) \cdot \vec{v}$$

So $\nabla f(\vec{x}_0) \perp \vec{v}$.

EX Parametrize tangent line to curve

$$x^2+y^3=5 \quad \text{in } \mathbb{R}^2$$

@ $(x,y)=(2,1)=\vec{p}$

Well

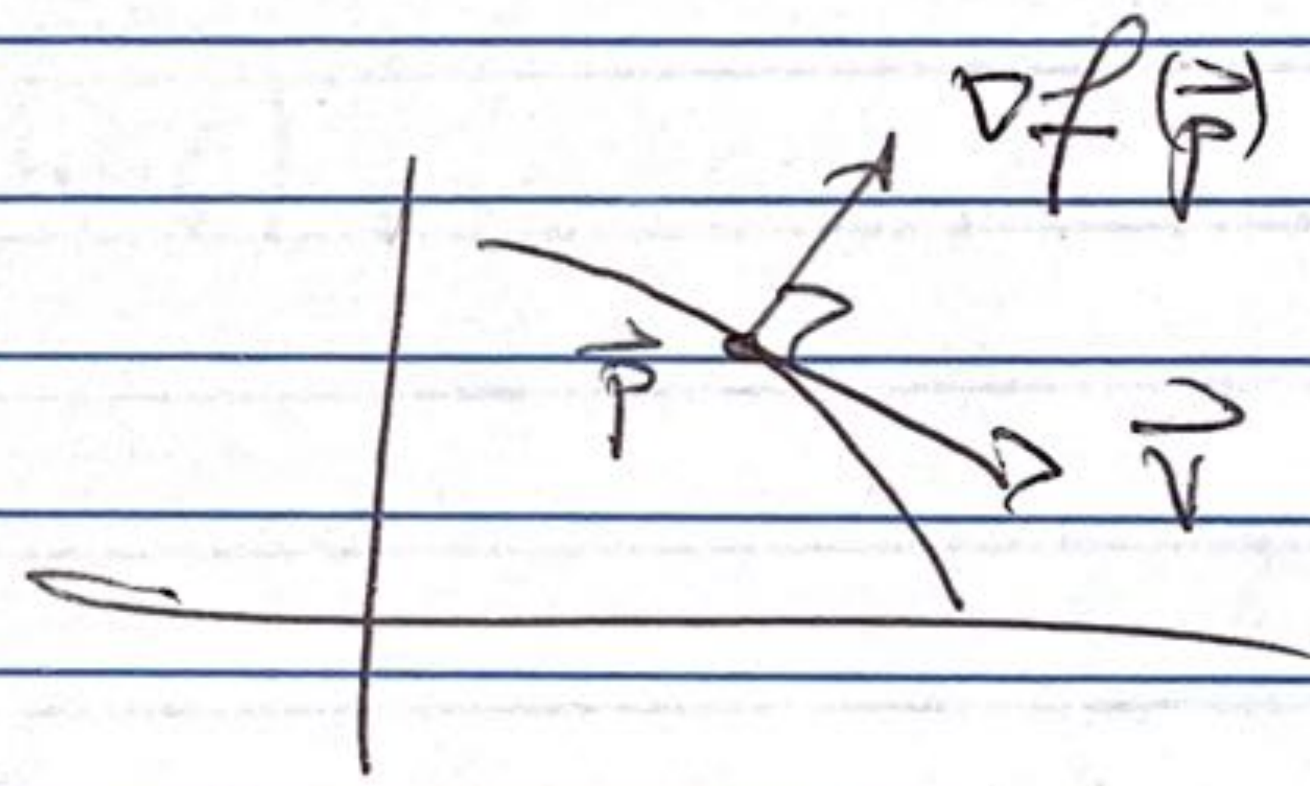
$\vec{r}(t)=\vec{p}+t\vec{v}$ $\qquad \vec{v}\perp \nabla f(\vec{p})$

$f(x,y)=x^2+y^3$

$\nabla f=(2x,3y^2)=(4,3)$ @ $\vec{p}$

So choose

$\vec{V}=(+3,-4)$



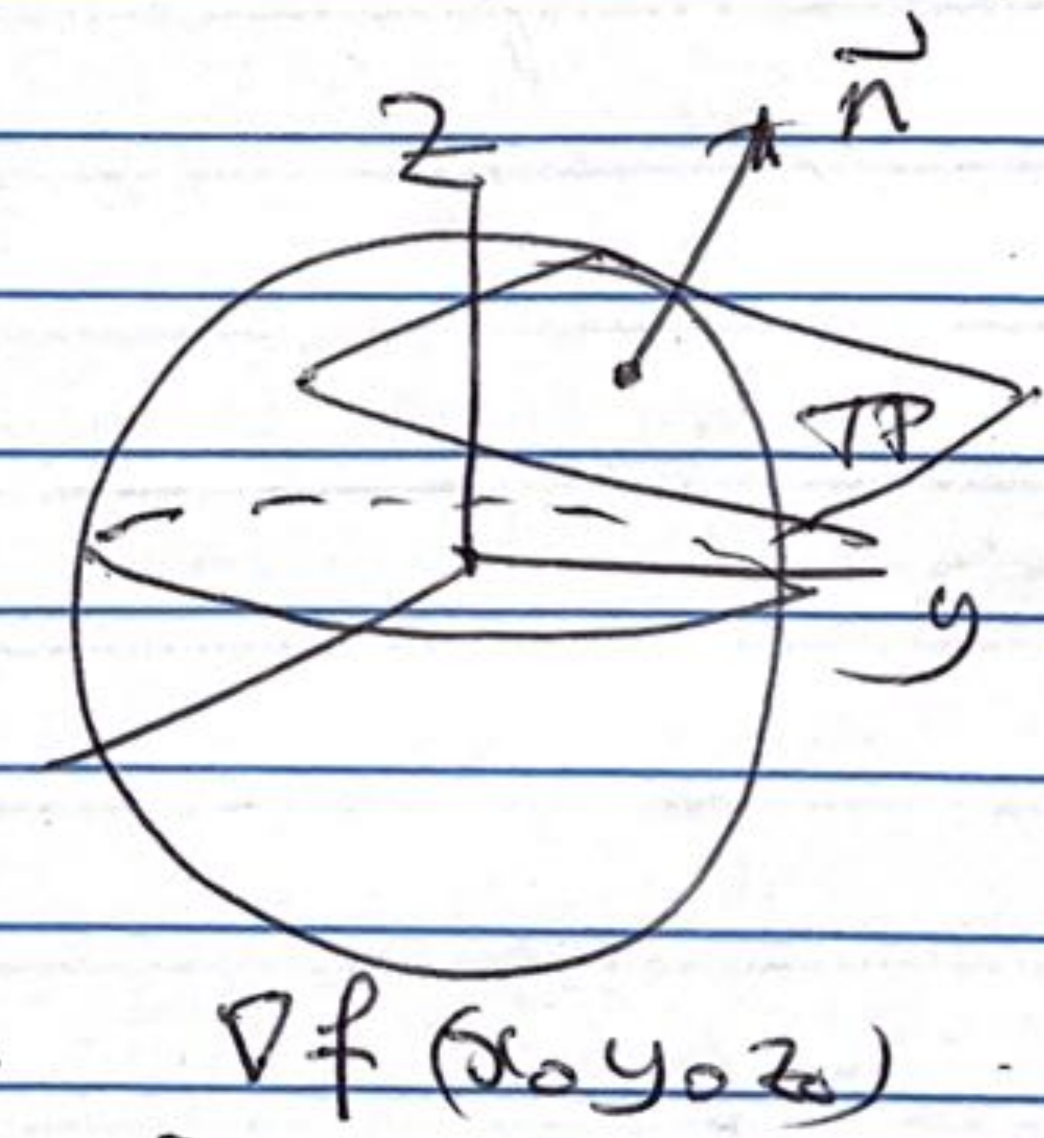
$$\boxed{\vec{r}(t)=(2,1)+t(3,-4)}$$

III NORMAL TO LEVEL SURFACE

IF S is Level-Surface of F,

ie $F(x,y,z)=c$ on S

and $(x_0,y_0,z_0)\in S$ Then



UNIT NORMAL to S at (x_0,y_0,z_0) $=\vec{n}=\dfrac{\nabla f(x_0y_0z_0)}{|\nabla f(x_0y_0z_0)|}$

Ex Suppose S is level surface

$$F(x,y,z) = x^2+y^2+z^2 = 14.$$

$\vec{p} = (1,2,3) \in S.$

Find unit normal at $\vec{p}$.

Well

$$\nabla F = (2x, 2y, 2z) = (2,4,6) \text{ @ } \vec{p}$$

So

$$\vec{n} = \frac{1}{\sqrt{4+16+36}}(2,4,6)$$