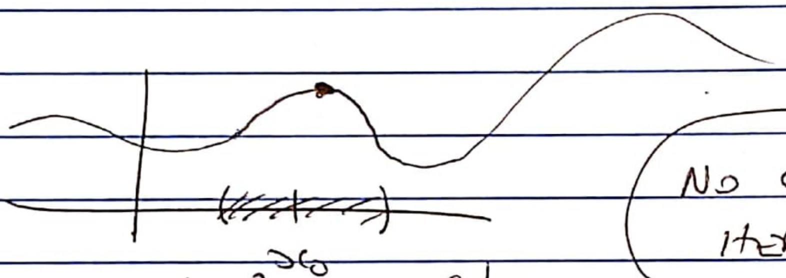
14.7A LOCAL OPTIMIZATION

CALC I REVIEW: $y = f(x)$

DEF $y = f(x)$ has a LOCAL MAX at $x = x_0$

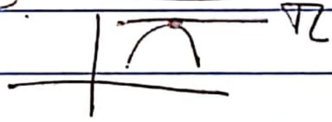
if

$$f(x_0) \geq f(x) \quad \text{for all } x \text{ near } x_0$$



NO CALCULUS HERE!!

THM x_0 LOCAL MAX of $f \Rightarrow f'(x_0) = 0$.



2ND DERIVATIVE TEST

Suppose x_0 is a CRITICAL POINT of f, $f'(x_0) = 0$

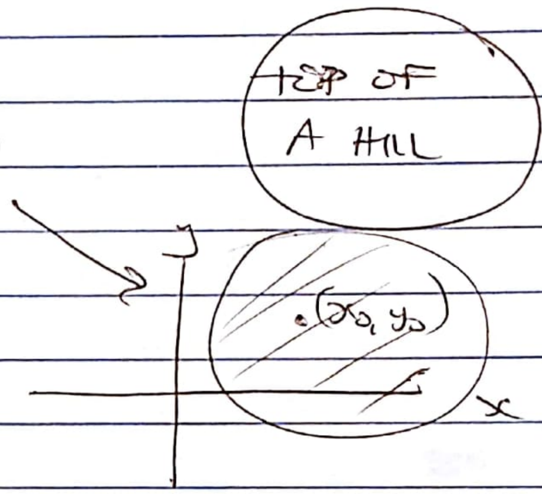
$f''(x_0)$	CLASSIFICATION	EXS
$+$	LOCAL MIN	$y = f(x) = x^2$, $f''(0) = 2$ 
$-$	LOCAL MAX	$y = f(x) = -x^2$, $f''(0) = -2$ 
0	ANYTHING GOES	TRY $f(x) = x^3$ or x^4 or $-x^4$

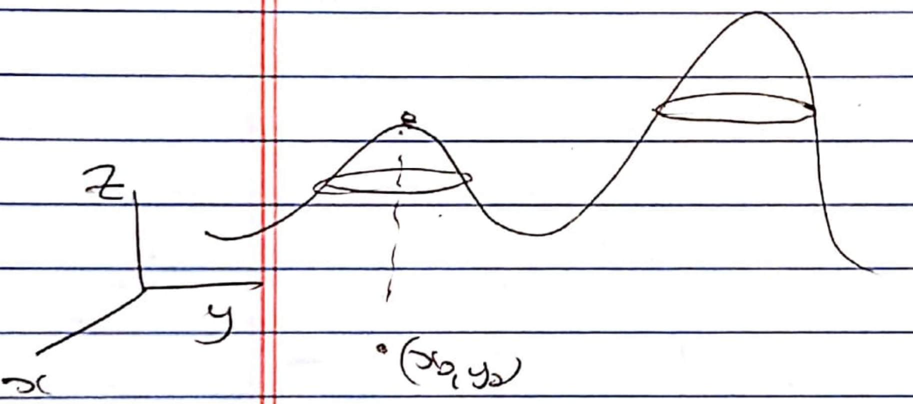
CALCULUS III $z = f(x,y)$

DEF $z = f(x,y)$ has LOCAL MAX at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x,y)$$

for all (x,y) near (x_0, y_0)





THM If (x_0, y_0) is LOCAL MAX of $z = f(x,y)$

Then tangent plane is horizontal at $(x_0, y_0) = \vec{x}_0$

SINCE normal to TP is

$$\vec{n} = -\frac{\partial f}{\partial x}\vec{i} - \frac{\partial f}{\partial y}\vec{j} + \vec{k}$$

we have

$$\nabla f(\vec{x}_0) = \left(\frac{\partial f}{\partial x}(\vec{x}_0), \frac{\partial f}{\partial y}(\vec{x}_0)\right) = (0,0)$$

— CRITICAL POINT CRITERION.

2ND DERIVATIVE TEST

Suppose $\vec{x}_0 = (x_0, y_0)$ is a CRITICAL PT of $z = f(x,y)$.
Let

$$D = \det \begin{bmatrix} f_{xx}(\vec{x}_0) & f_{xy}(\vec{x}_0) \\ f_{yx}(\vec{x}_0) & f_{yy}(\vec{x}_0) \end{bmatrix}$$

HESSIAN MATRIX of f

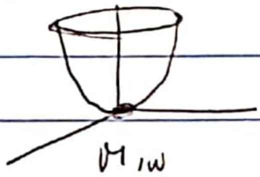
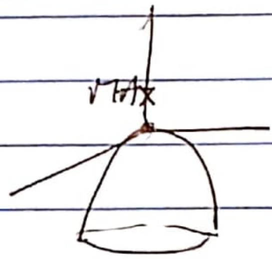
D	$f_{xx}(\vec{x}_0)$	CLASSIFICATION
$+$	$+$	LOCAL MIN
$+$	$-$	LOCAL MAX
$-$	ANYTHING	SADDLE POINT

Otherwise anything goes.

At a saddle point, f increases in some dirs and decreases in others.

HOW TO REMEMBER THE TEST

CRITICAL PTS @ $\vec{x}_0 = (0,0)$

$z = f(x,y)$	$z=3x^2+4y^2$	$z=-3x^2-4y^2$	$z=3x^2-4y^2$
GRAPH	 MIN	MAX 	SADDLE SURFACE
∇f	$(6x, 8y)$	$(-6x, -8y)$	$(6x, -8y)$
D	$\begin{vmatrix}6 & 0\\ 0 & 8\end{vmatrix} = 48 > 0$	$\begin{vmatrix}-6 & 0\\ 0 & -8\end{vmatrix} = 48 > 0$	$\begin{vmatrix}6 & 0\\ 0 & -8\end{vmatrix} = -48 < 0$
$f_{xx}(\vec{x}_0)$	$6 > 0$	$-6 < 0$	*
CLASSIFICATION	MIN	MAX	SADDLE

EXS

(I) $f(x,y) = 4 + x^3 + y^3 - 3xy$ See Lect 14.7 APlots for contour plot

CRITICAL POINTS

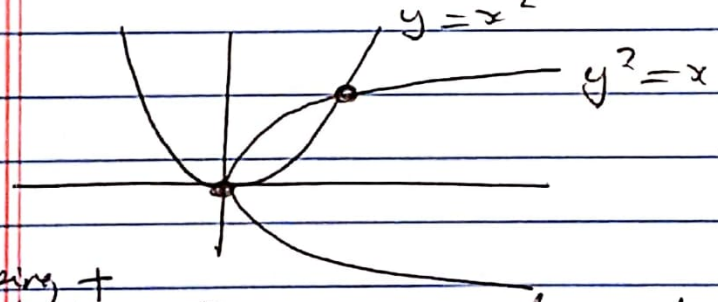
$\nabla f = (3x^2 - 3y,\ 3y^2 - 3x) = (0,0)$ where

① $y = x^2$ 2 NONLINEAR EQNS

and ② $y^2 = x$ IN 2 UNKNOWNS.

GEOMETRIC METHOD [OFTEN GIVES # AND APPROX LOCATION OF CRITICAL POINTS]

Need (x,y) which lie on both curves



$\Rightarrow$ 2 CRITICAL POINTS

By Guessing + Symmetry They are located at $(0,0)$

and $(1,1)$

ALGEBRAIC METHOD [CAN GIVE EXACT LOCATION OF CRITICAL POINTS]

$x \overset{②}{=} y^2 \overset{①}{=} x^4$ ELIMINATE y

So $x(x^3 - 1) = 0$ SET RHS $= 0$.

$\Rightarrow \quad x(x-1)(x^2+x+1)=0$ FACTOR

$\Rightarrow \quad x=0$ or $x=1$ SOLVE

NB Quadratic has no real roots.

So get $(x,y)=(0,0)$ and $(x,y)=(1,1)$.

from $y=x^2$

2ND DER TEST

$$D = \det\begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix} = 36xy-9 = 9(4xy-1)$$

CRITICAL PT	D	f_{xx}	CLASSIFICATION
$(0,0)$	$-9<0$	*	Saddle
$(1,1)$	$27>0$	$6>0$	Local Min

II $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 60$

$\nabla f = (6xy - 12x,\ 3y^2 + 3x^2 - 12y) = (0,0)$

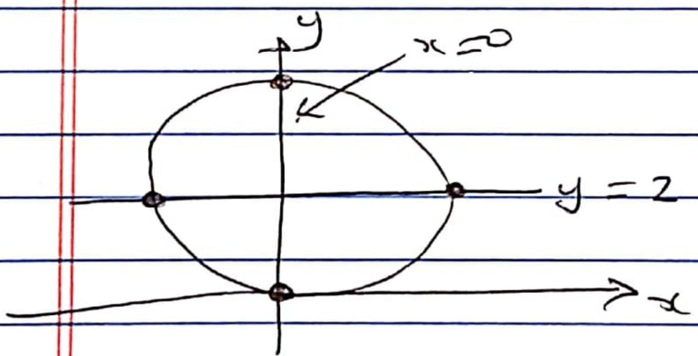
at

① $xy - 2x = 0 \Rightarrow x(y-2) = 0$ PAIR OF LINES

$\Rightarrow x = 0$ or $y = 2$

② $x^2 + y^2 = 4y \Rightarrow x^2 + (y-2)^2 = 2^2$ CIRCLE

GEOMETRIC METHOD



4 CRITICAL POINTS @

(0,0)

(0,4)

(−2,2)

(2,2).

ALGEBRAIC METHOD

By ① $x = 0$ or $y = 2$

$\boxed{x=0}$ By ② $y^2 = 4y$

$\Rightarrow y(y-4) = 0$

$\Rightarrow y = 0$ or $y = 4$

Get $(0,0)$ and $(0,4)$

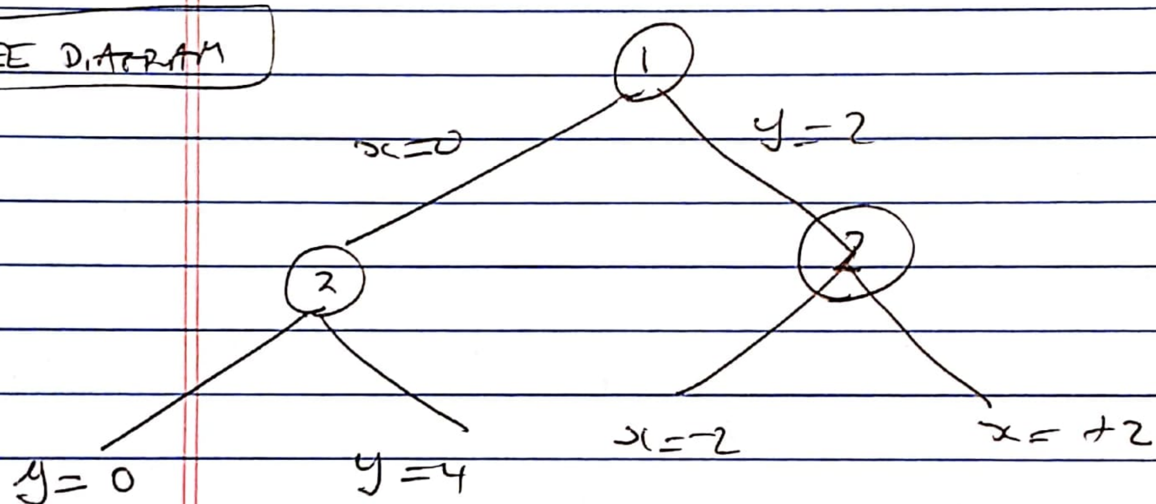
$\boxed{y=2}$ By (3) $x^2 + 4 = 8$

$$x^2 = 4$$

$$x = \pm 2$$

Get $(-2,2)$ and $(2,2)$

TREE DIAGRAM



2ND DER TEST

$$D = \det\begin{bmatrix} 6y-12 & 6x \\ 6x & 6y-12 \end{bmatrix} = 36\left[(y-2)^2 - x^2\right]$$

CRITICAL PT	D	f_{xx}	CLASSIFICATION
$(0,0)$	$+$	$-$	MAX
$(0,4)$	$+$	$+$	MIN
$(-2,2)$	$-$	*	SADDLE
$(2,2)$	$-$	*	SADDLE

See Lect 14.7A plots for Contour Plot.