14.8 CONSTRAINED OPTIMIZATION + METHOD OF LAGRANGE MULTIPLIERS

Most real world optⁿ problems are CONSTRAINED.

SIMPLE CASE (ONLY one for us)

Find ABS MAX + MIN of

$$z = f(x,y) \qquad \text{OBJECTIVE FUNCTION}$$

subject to constraint that (x,y) lie on the curve C in plane given as a level curve of a 2nd f^n,

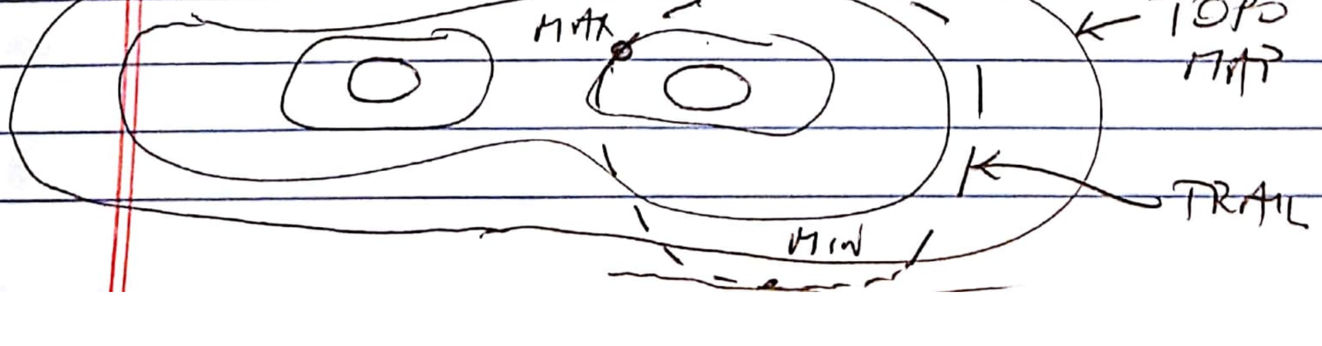
$$g(x,y) = k \qquad \text{CONSTRAINT EQN.}$$

THINK

$z = f(x,y) =$ ELEVATION @ Lake Tahoe

$g(x,y) = k$ is a SNOW-SHOE TRAIL

Find highest + lowest elevation on the trail



QUICK THEORY

Let $(x,y) = \vec{r}(t)$ parametrize constraint curve C (trail)

Then $h(t) = f(\vec{r}(t))$ is height along trail

GOAL find ABS MAX/MIN of h.

METHOD I (See 14.7B EX 2)

- Calculate formula for h
- Solve CASE I ABS MAX/MIN problem

METHOD II (LAGRANGE MULTIPLIERS)

Critical Points of h satisfy

$$0 = h'(t) \overset{CR}{=} \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

MUST BE NORMAL TO C. AT CPT.

TANGENT TO C

But by 14.6 we know ∇g is ALWAYS normal to level curve $g(x,y) = k$.

So at CPT $\nabla f \parallel \nabla g$

OR Level Curve of f tangent to Level Curve of g

HELPFUL EX [Redo ~~EX~~ 14.7B EX2]

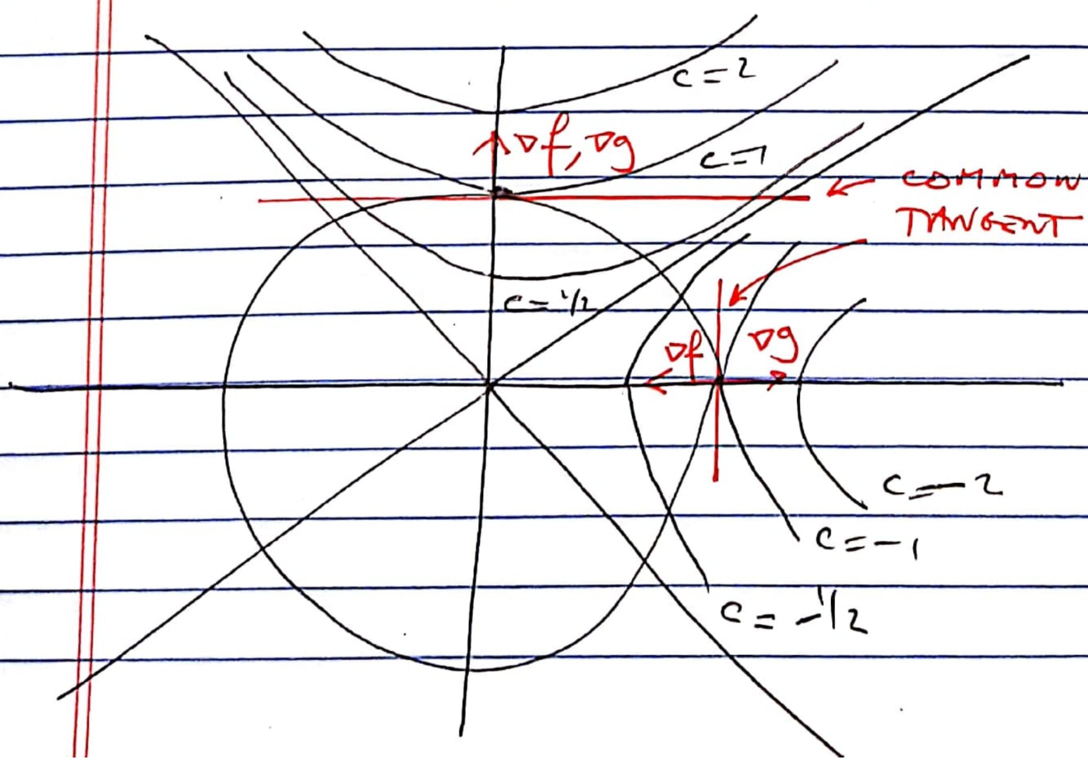
Find MAX + MIN of $z = f(x,y) = y^2 - x^2$

on curve $g(x,y) = x^2 + y^2 = 1$

STRATEGY

- Sketch level curves $f(x,y) = c$ of f.
- What is MAX value of c for which level curve of f intersects constraint curve $x^2 + y^2 = 1$.

(i.e. What is highest elevation, c, along snow-shoe trail $x^2 + y^2 = 1$).



As $c\uparrow$, last c-value for which level curves of f, g intersect is one for which

tangent line to $g=k$ agrees with tangent line to $f=c$.

Once Again: $\nabla f \parallel \nabla g$.

or $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$

λ = LAGRANGE MULTIPLIER "$\lambda = L$"

METHOD: Find value of f at all points (x_0, y_0) for which there is a λ so that

① $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

AND ② $g(x_0, y_0) = k$ (MUST BE ON CONSTRAINT).

EX CONT'D

$$\nabla g = (2x, 2y)$$
$$\nabla f = (-2x, 2y)$$

Get $(-2x, 2y) = \lambda (2x, 2y)$

OR

$$\begin{cases} -x = \lambda x & (1) \\ y = \lambda y & (2) \\ x^2 + y^2 = 1 & (3) \end{cases}$$

3 NL EQNS IN
3 UNKNOWNS (x, y, λ)

(1) $(\lambda + 1)x = 0$ GET RHS $= 0$ AND FACTOR

(2) $(\lambda - 1)y = 0$

(3) $x^2 + y^2 = 1$

By (1) $\lambda = -1$ or $x = 0$

$\boxed{\lambda = -1}$ By (2) $-2y = 0 \Rightarrow y = 0$

By (3) $x = \pm 1$

Get $\boxed{(x, y, \lambda) = (\pm 1, 0, -1)}$ $f(\pm 1, 0) = -1$ MIN

OR

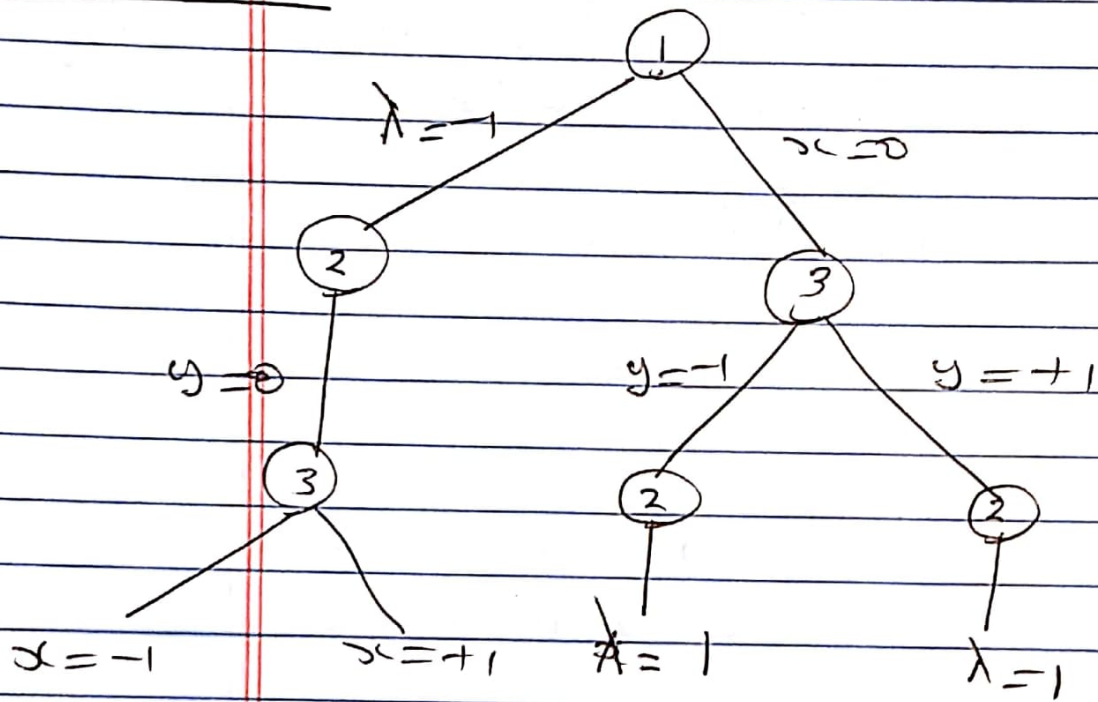
$\boxed{x = 0}$ By (3) $y = \pm 1$

By (2) $\lambda = 1$

$f(0, \pm 1) = 1$ MAX

Get $\boxed{(x, y, \lambda) = (0, \pm 1, 1)}$

TREE DIAGRAM



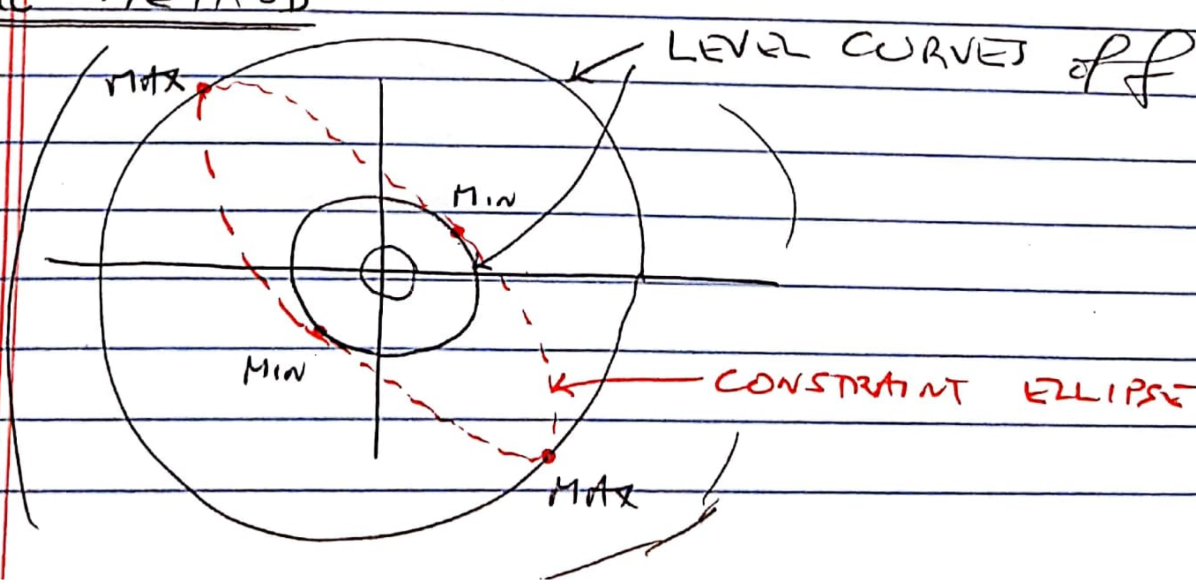
Trace All Possible Branches

$\boxed{\text{EX 2}}$

Find MAX + MIN of $f(x,y) = x^2+y^2$

on rotated ellipse $4(x+y)^2+(x-y)^2=1$

GEOMETRIC METHOD



EX2 ALGEBRAIC METHOD II

FIND MAX/MIN OF $z = f(x,y) = x^2+y^2$ ⑦*

ON $g(x,y) = 4(x+y)^2 + (x-y)^2 = 1$

$= 5x^2 + 5y^2 + 6xy = 1$

THIS IS NICER THAN THE METHOD ON ⑦, ⑧

$f_x = \lambda g_x$: $2x = \lambda(10x+6y)$ ①

$f_y = \lambda g_y$: $2y = \lambda(10y+6x)$ ②

$g = 1$ $5x^2+5y^2+6xy = 1$ ③

① − ②: $2(x-y) = \lambda[10(x-y) + 6(y-x)]$

OR $(x-y)[10\lambda - 6\lambda - 2] = 0$

$(x-y)(4\lambda - 2) = 0$ ④

By ④ $y = x$ or $\lambda = \frac{1}{2}$

$\boxed{y = x}$ By ③ $16x^2 = 1 \Rightarrow x = \pm\frac{1}{4}$

So get $(x,y) = \left(\pm\frac{1}{4}, \pm\frac{1}{4}\right)$ $\boxed{f = \frac{1}{8}}$ MIN

By ① $\pm\frac{1}{2} = \pm\lambda\left(\frac{10}{4} + \frac{6}{4}\right) \Rightarrow \lambda = \frac{1}{8}$.

VERIFY ② HOLDS with these (x, y, λ)

$\boxed{\lambda = \frac{1}{2}}$ By ① $\cancel{4}x = 10x + 6y \Rightarrow y = -x$

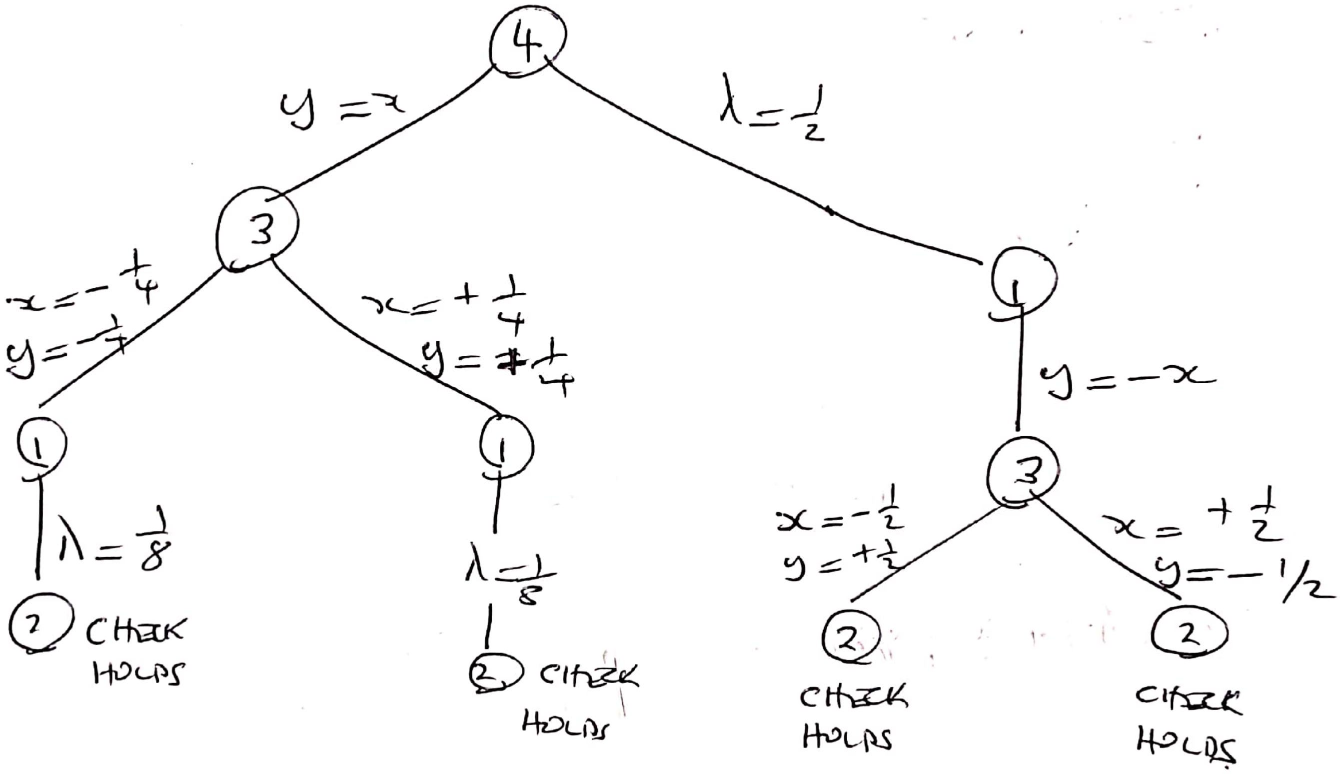
By ③ $4x^2 = 1 \Rightarrow x = \pm\frac{1}{2}$

So we get

$$(x, y, \lambda) = \begin{cases} \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \\ \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \end{cases} \qquad \boxed{f = \frac{1}{2}} \quad \text{MAX}$$

CHECK 2 HOLDS.

TREE DIAGRAM



ALGEBRAIC METHOD

$$g(x,y) = 4(x+y)^2 + (x-y)^2$$
$$= 5x^2 + 5y^2 + 6xy$$

$f_x = \lambda g_x$: $\quad 2x = \lambda(10x + 6y) \quad$ ①

$f_y = \lambda g_y$: $\quad 2y = \lambda(10y + 6x) \quad$ ②

① : $\quad (2 - 10\lambda)x = 6\lambda y \quad$ ③

② $\quad (2 - 10\lambda)y = 6\lambda x \quad$ ④

③ × ④ : $\quad (2-10\lambda)^2 xy = 36\lambda^2 xy$

$$xy\left((2-10\lambda)^2 - 36\lambda^2\right) = 0$$

$$\boxed{4xy\,(16\lambda^2 - 10\lambda + 1) = 0}$$

GIVES $\quad x = 0 \quad$ or $\quad y = 0 \quad$ or $\quad 16\lambda^2 - 10\lambda + 1 = 0$

$\boxed{x = 0}$ $\quad$ By ③ $\quad \lambda = 0 \quad$ or $\quad y = 0.$

- $\boxed{\lambda = 0}\,\boxed{x = 0} \Rightarrow y = 0. \quad (0,0)$ NOT ON CONSTRAINT CURVE

- $\boxed{y = 0,\ x = 0}$ $\quad$ NOT ON CURVE

$\boxed{y=0}$ By 4 $\lambda = 0$ or $x = 0$.

Similarly no sol^{n}?

$$\boxed{16\lambda^2 - 10\lambda + 1 = 0}$$

$\lambda = \frac{1}{2}$ or $\lambda = \frac{1}{8}$ by Quad Formula

$\boxed{\lambda = \frac{1}{2}}$ By 3 $-3x = 3y \Rightarrow y = -x$

From $5x^2 + 5y^2 + 6xy = 1$ 5 CONSTRAINT EQN

Get $4x^2 = 1$

$x = \pm \frac{1}{2}$

Get $(x, y, \lambda) = \left(\pm\frac{1}{2}, \mp\frac{1}{2}, \frac{1}{2}\right)$ $f = \frac{1}{2}$ MAX

$\boxed{\lambda = \frac{1}{8}}$ By 3 $\frac{3}{4}x = \frac{3}{4}y \Rightarrow y = x$

By 5 $16x^2 = 1$

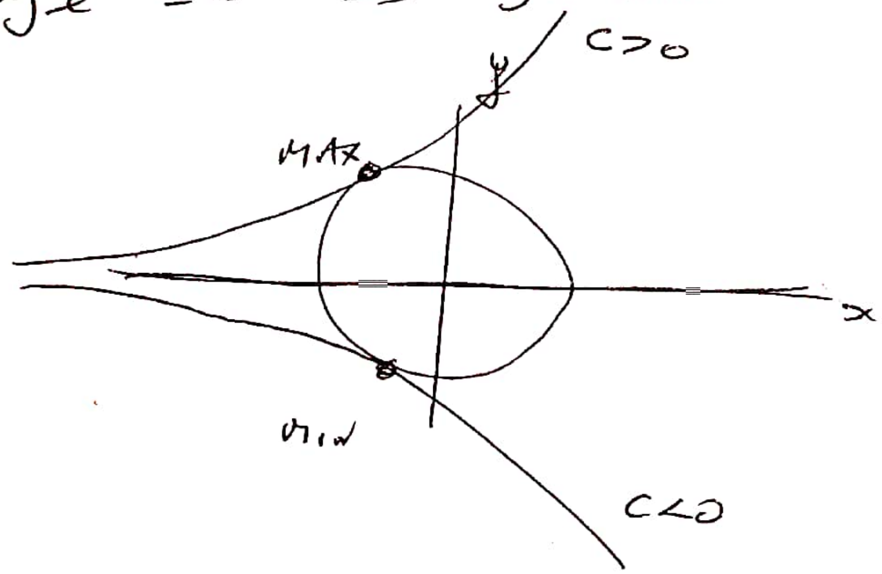
$x = \pm\frac{1}{4}$

Get $(x, y, \lambda) = \left(\pm\frac{1}{4}, \pm\frac{1}{4}, \frac{1}{8}\right)$, $f = \frac{1}{8}$ MIN

EX 3 Find MAX + MIN of $f(x,y) = y e^{-x}$ on circle $x^2 + y^2 = 2$

GEOMETRIC METHOD

$y e^{-x} = c \Rightarrow y = c e^{x}$



ALGEBRAIC METHOD

$-y e^{-x} = 2x\lambda$ (1)

$e^{-x} = 2y\lambda$ (2)

$x^2 + y^2 = 2$ (3)

(2) INTO (1) :

$-2y^2\lambda = 2x\lambda \Rightarrow \boxed{\lambda(x + y^2) = 0}$ (4)

(4) gives $\lambda = 0$ or $x = -y^2$

$\boxed{\lambda = 0}$ By (2) get $e^{-x} = 0$ NO SOLNS

$\boxed{x = -y^2}$ By (3) $y^4 + y^2 - 2 = 0$

$(y^2 + 2)(y^2 - 1) = 0$

$y^2 = -2$ or $y^2 = +1$

NO SOLNS $\boxed{y = \pm 1}$

$x = -y^2 = -1$

Get $(x, y) = (-1, \pm 1)$

$\lambda = \dfrac{e^{-x}}{2y} = \pm \dfrac{e}{2}$

$$\boxed{(x, y, \lambda) = \left(-1, \pm 1, \pm \frac{e}{2}\right)}$$

$f(-1, \pm 1) = \pm 1 . e^{+1} = \boxed{\pm e}$