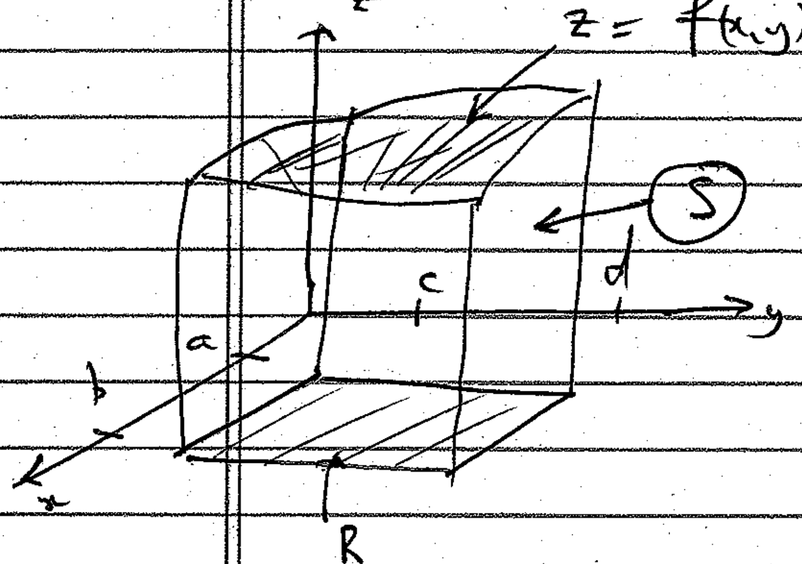
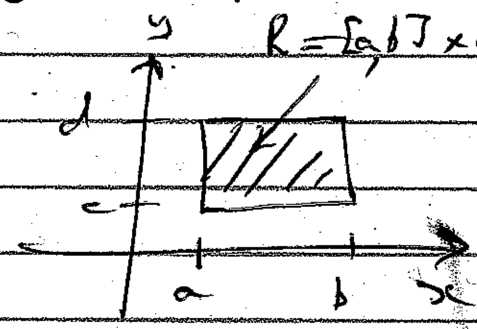
# 15.1 DOUBLE INTEGRALS OVER RECTANGLES

## VOLUMES OF SOLIDS

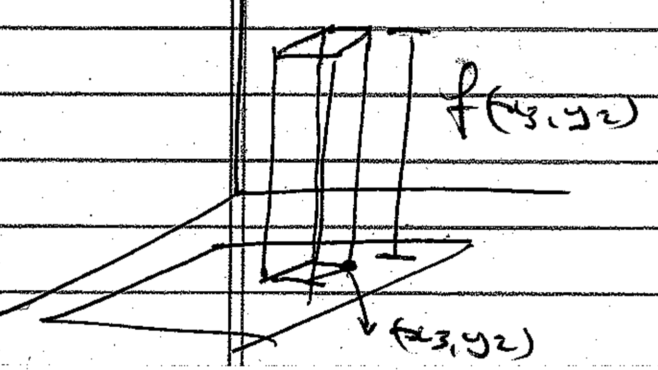
Find volume of solid region above rectangle $R = [a, b] \times [c, d]$ in $xy$-plane and below graph of a function $z = f(x,y)$ (where $f \geq 0$)

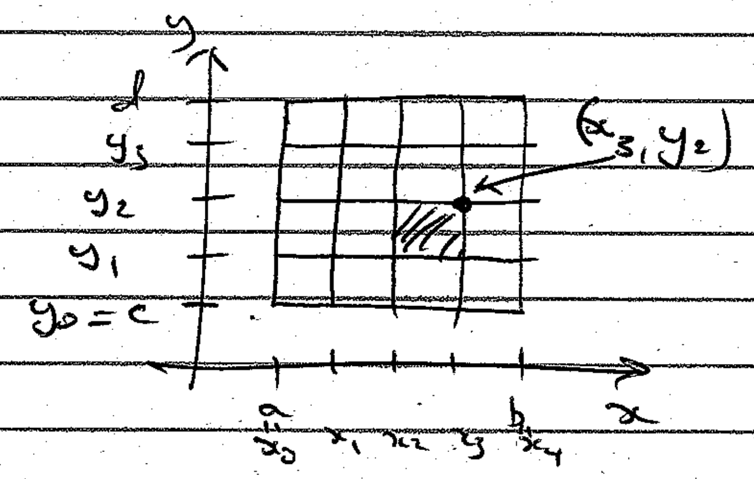



IDEA Approximate S as union of thin boxes with base area $\Delta A = \Delta x \, \Delta y$ and height given by value of $f$ at some point (eg top right corner) of ~~each~~ base of each box.

ONE SUCH BOX



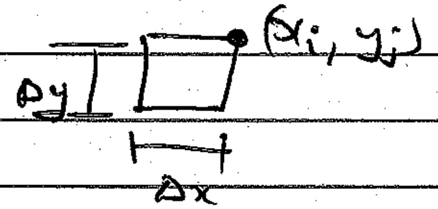



Use $N_x$ boxes in x-dim of width $\Delta x = \frac{b-a}{N_x}$.

— $N_y$ —— y-dim —— $\Delta y = \frac{d-c}{N_y}$

**DEF** The <u>DOUBLE INTEGRAL</u> of $z = f(x,y)$ over rectangle $R$ is

$$\iint_R f(x,y)\,dA = \lim_{\substack{N_x \to \infty \\ N_y \to \infty}} \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} f\underbrace{(x_i, y_j)}_{\substack{\text{TOP RIGHT} \\ \text{CORNER} \\ \text{OF LITTLE RECT}}} \Delta A$$




**THM** If $f$ is CTS this limit exists.

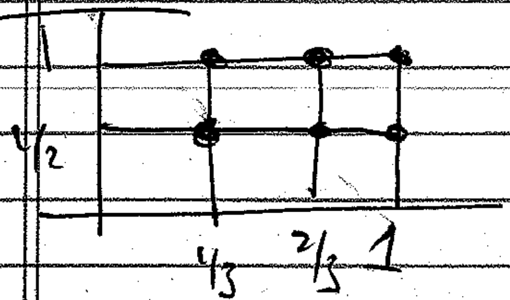
We can approximate $\iint_R f\,dA$ using finite values of $N_x, N_y$.

**EX** $R = [0,1] \times [0,1]$

$z = f(x,y) = 3x^2 + 4y^2$

$N_x = 3$, $N_y = 2$. ~~Top Right Corners.~~

TOP RIGHT CORNERS



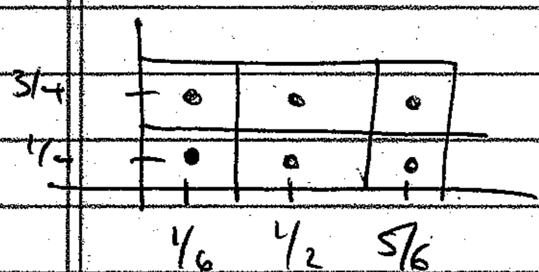

$\Delta x = 1/3$

$\Delta y = 1/2$

$$\iint_{[0,1]\times[0,1]} (3x^2+4y^2)\,dA \simeq \frac{1}{3}\times\frac{1}{2}\Big( f(\tfrac{1}{3},\tfrac{1}{2}) + f(\tfrac{2}{3},\tfrac{1}{2}) + f(1,\tfrac{1}{2}) + f(\tfrac{1}{3},1) + f(\tfrac{2}{3},1) + f(1,1) \Big)$$

$$= 4.0556$$

OR MIDPOINTS




$$\iint_{[0,1]\times[0,1]} (3x^2+4y^2)\,dA \simeq \frac{1}{3}\frac{1}{2}\Big( f(\tfrac{1}{6},\tfrac{1}{4}) + f(\tfrac{1}{2},\tfrac{1}{4}) + \dots + f(\tfrac{5}{6},\tfrac{3}{4}) \Big)$$

$$= 2.222.$$

LATER TRUE ANS IS $2\frac{1}{3}$

MEANINGS OF $\iint f\,dA$

(1) If $z = f(x,y)$ = density of rectangular plate (in kg/m²), Then

$$\iint_R f\,dA = \text{Total Mass of Plate}$$

$$[\text{kg}/\text{m}^2 \times \text{m}^2 = \text{kg}]$$

(2) AVERAGE of $f$ over $R$ $= \bar{f} = \dfrac{1}{\text{Area}(R)} \iint_R f\,dA$

If $f \geq 0$ Then

$$\bar{f} \cdot \text{Area}(R) = \iint_R f\,dA$$

$$\text{Height} \times \text{Area}(R) = \text{Volume of } S$$

So $\bar{f}$ is height of a BOX whose volume equals volume of solid $S$ under $z = f(x,y)$ and over $R$