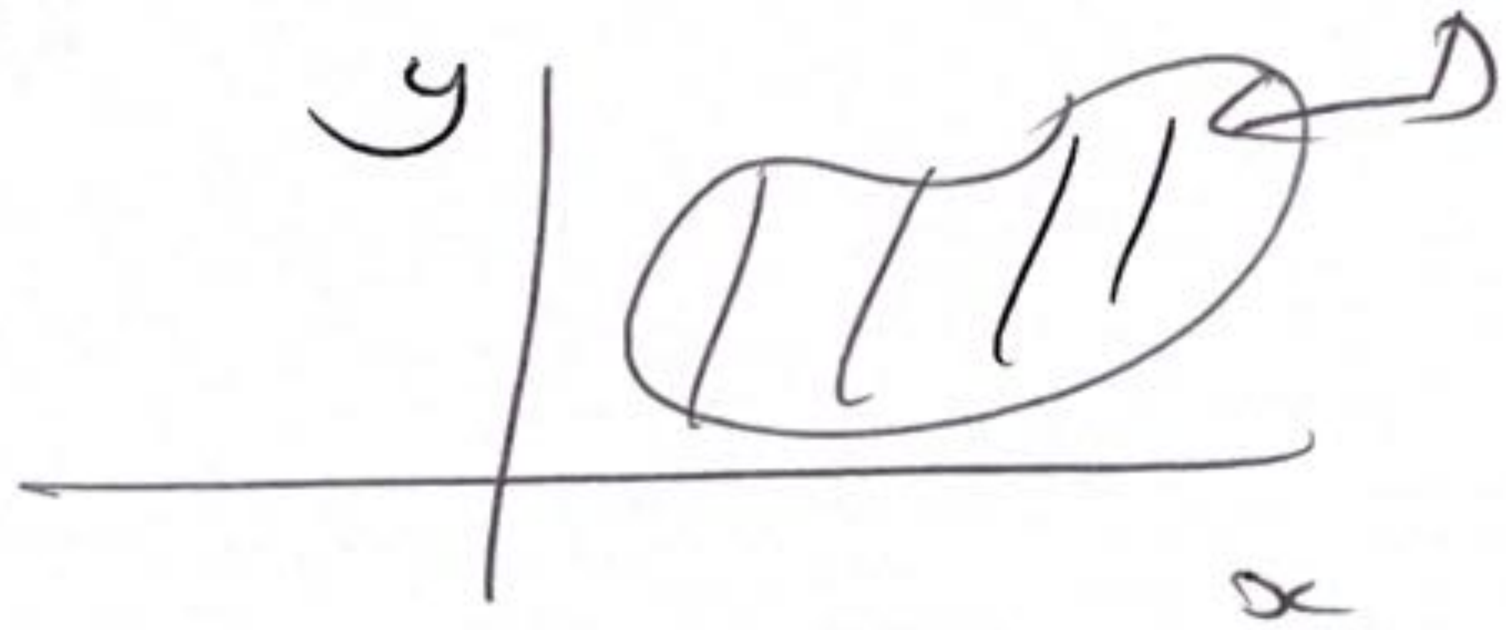
15.2 DOUBLE INTEGRALS OVER GENERAL REGIONS (1A)

GOAL Let D be a domain (region) in xy-plane and $z = f(x,y)$

Calculate $\iint_D f\,dA$



MEANINGS

(1) $\iint_D 1\,dA = \text{Area}(D)$

(2) If $f \geq 0$ on D Then

$\iint_D f\,dA =$ Vol of Solid under $z = f(x,y)$ and over D

(3) If $z = \rho(x,y)$ = Density of Thin plate D in kg/m^2

Then $\iint_D \rho\,dA = \text{MASS}(D)$

1B

(4) Center of Mass of plate is (x_c, y_c) where

$$x_c = \iint_D x\rho(x,y)\,dA$$

$$y_c = \iint_D y\,\rho(x,y)\,dA$$

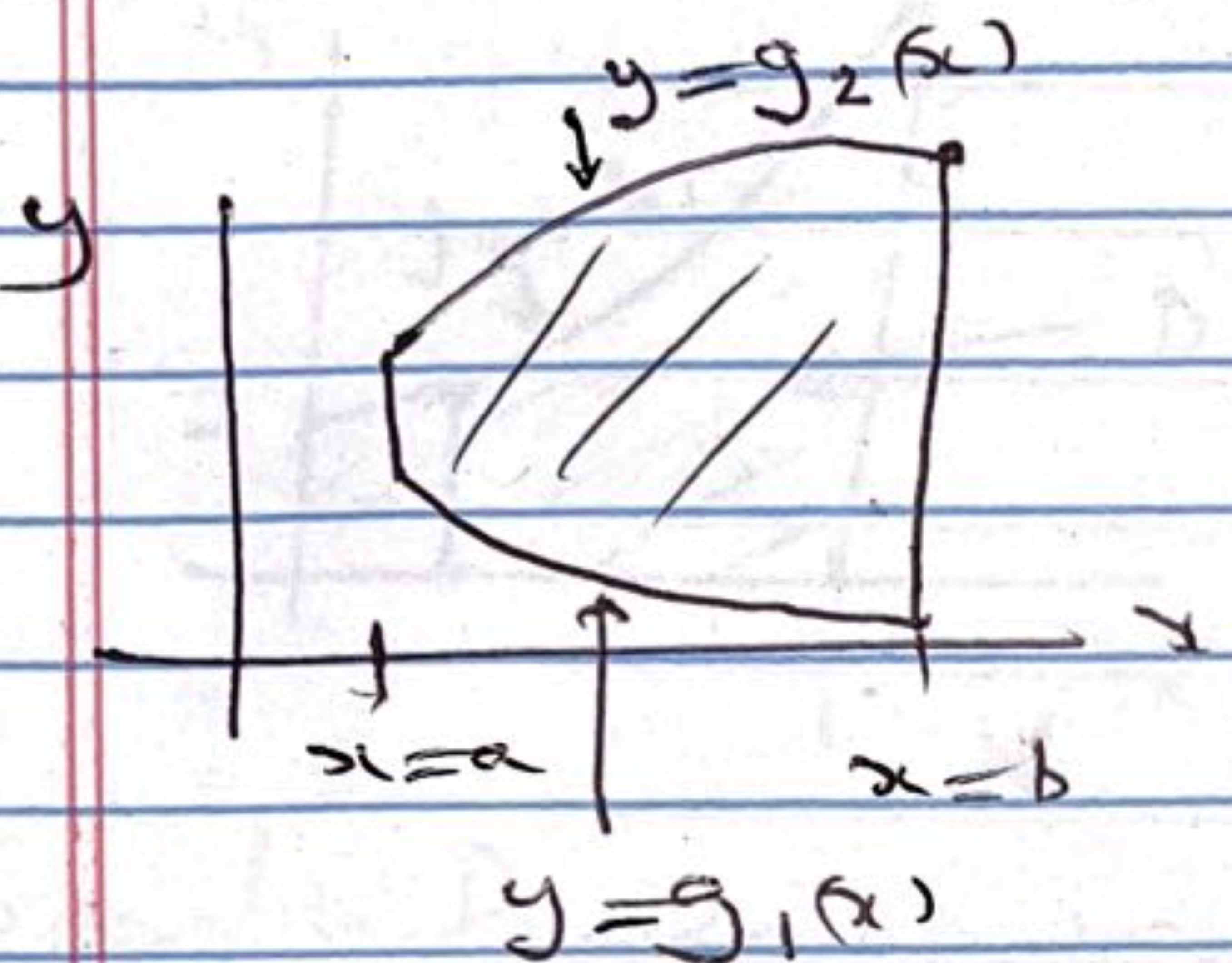
(5) Moment of Inertia (about origin)

$$I_0 = \iint_D (x^2+y^2)\,\rho(x,y)\,dA.$$

(6) Average of $f = \dfrac{1}{\text{AREA}(D)} \iint_D f\,dA$

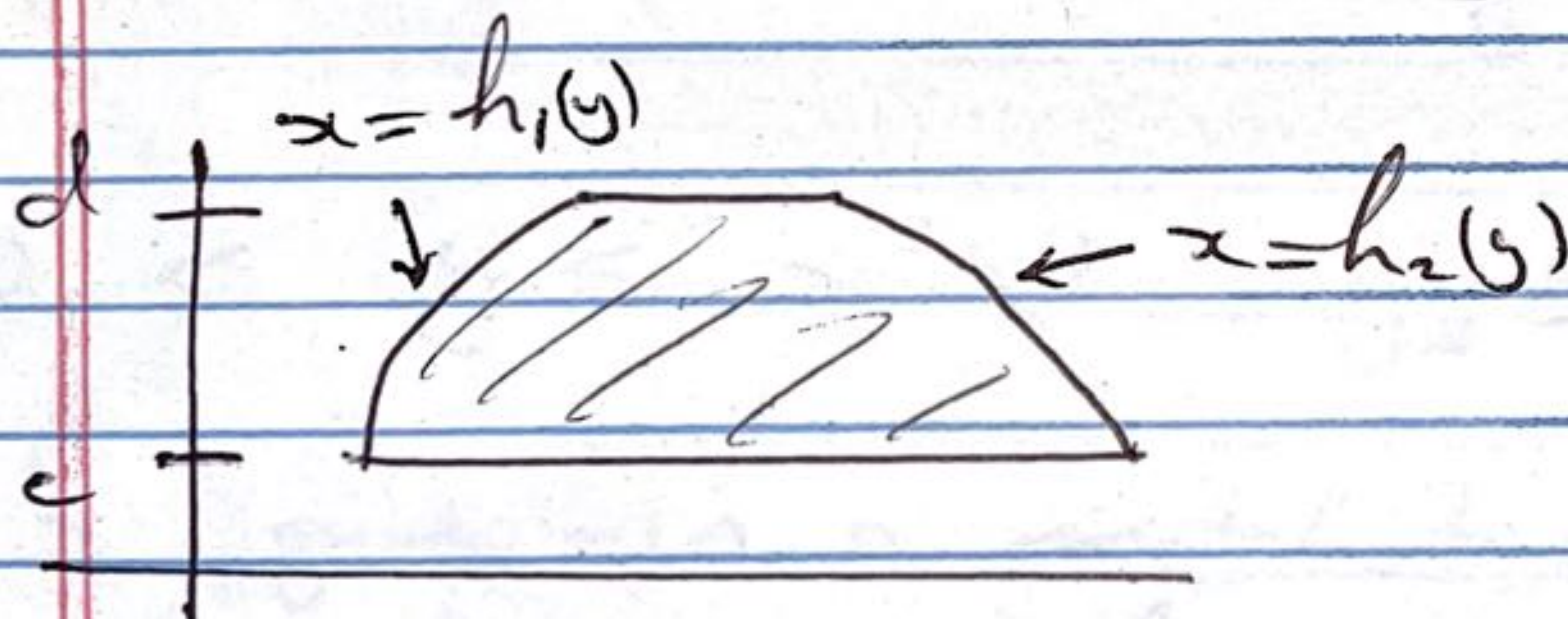
TWO SPECIAL DOMAINS

TYPE I (VERTICAL SIDES)



OR

TYPE II (HORIZONTAL TOP + BOTTOM)



FACT

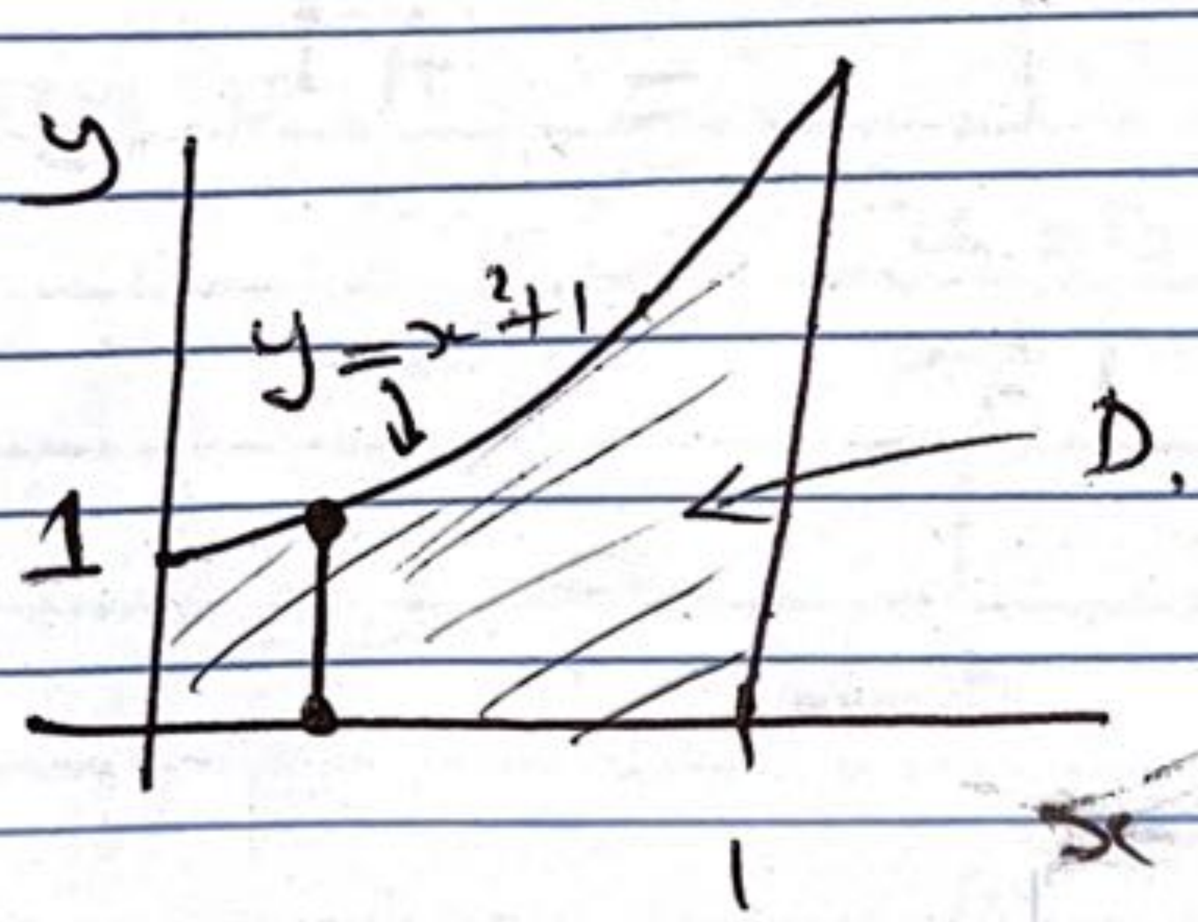
A general domain D can be cut up into subdomains of Types I, II.

EXS

① $\iint\limits_D x \cos y \, dA$

D bounded by $x = 0$, $y = 0$, $y = x^2 + 1$, $x = 1$

Ⓐ Draw Picture of D. Is it Type I, II or both?



Type I.
FILL UP WITH VERTICAL MATCHSTICKS of different heights.

Ⓑ Describe D using inequalities

$0 \le x \le 1$

For Type I:
Endpts for x const.

$g_1(x) = 0 \le y \le x^2+1 = g_2(x)$

EACH x gives us a matchstick.
Matchstick at x goes from $y=0$ to $y=x^2+1$.

Ⓒ Set up Iterated Integral

$$\iint_D x\cos y \, dA = \int_{x=0}^{x=1} \int_{y=0}^{y=x^2+1} (x\cos y)\, dy\, dx$$

OUTER INT: MUST HAVE CONSTS FOR ENDPTS

1. Evaluate Iterated Integral.

$$\iint_D x\cos y \, dA = \int_{x=0}^{x=1} \left[x \sin y \right]_{y=0}^{y=x^2+1} dx$$

$$= \int_{x=0}^{x=1} x \sin(x^2+1)\, dx$$

SUB

$u = x^2+1$

$du = 2x\, dx$

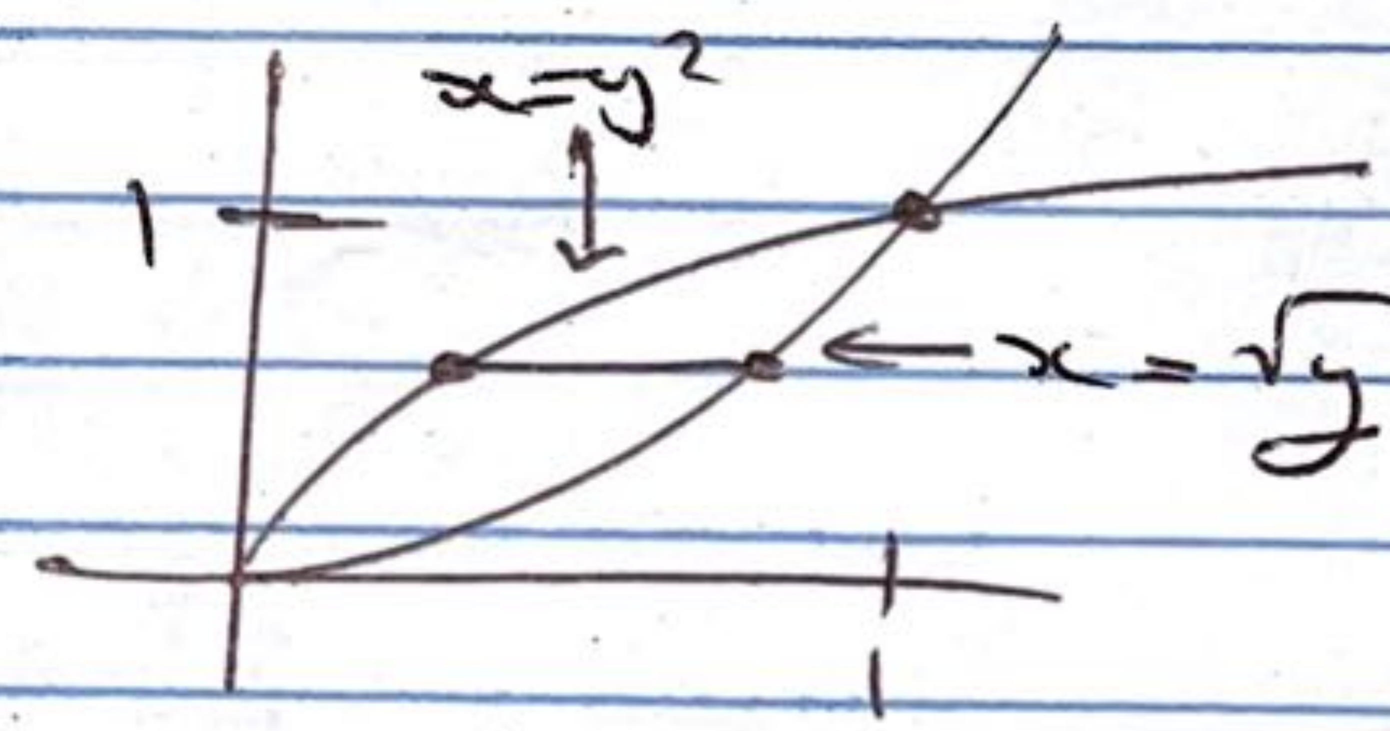
$$= \int_{u=1}^{u=2} \frac{1}{2} \sin u \, du$$

$$= \frac{1}{2}\left[\cos(1) - \cos(2)\right]$$

VOL OF SOLID bounded by $z=0$, $z=x+y$, $x=y^2$, $y=x^2$

2. $\iint_D (x+y)\, dA$ D bounded by $y=\sqrt{x}$, $y=x^2$

Both Type I and II



Type II

$0 \le y \le 1$ HORIZ MATCHSTICKS

LEFT ENDPT $= y^2 \le x \le \sqrt{y} =$ RIGHT ENDPT.

So

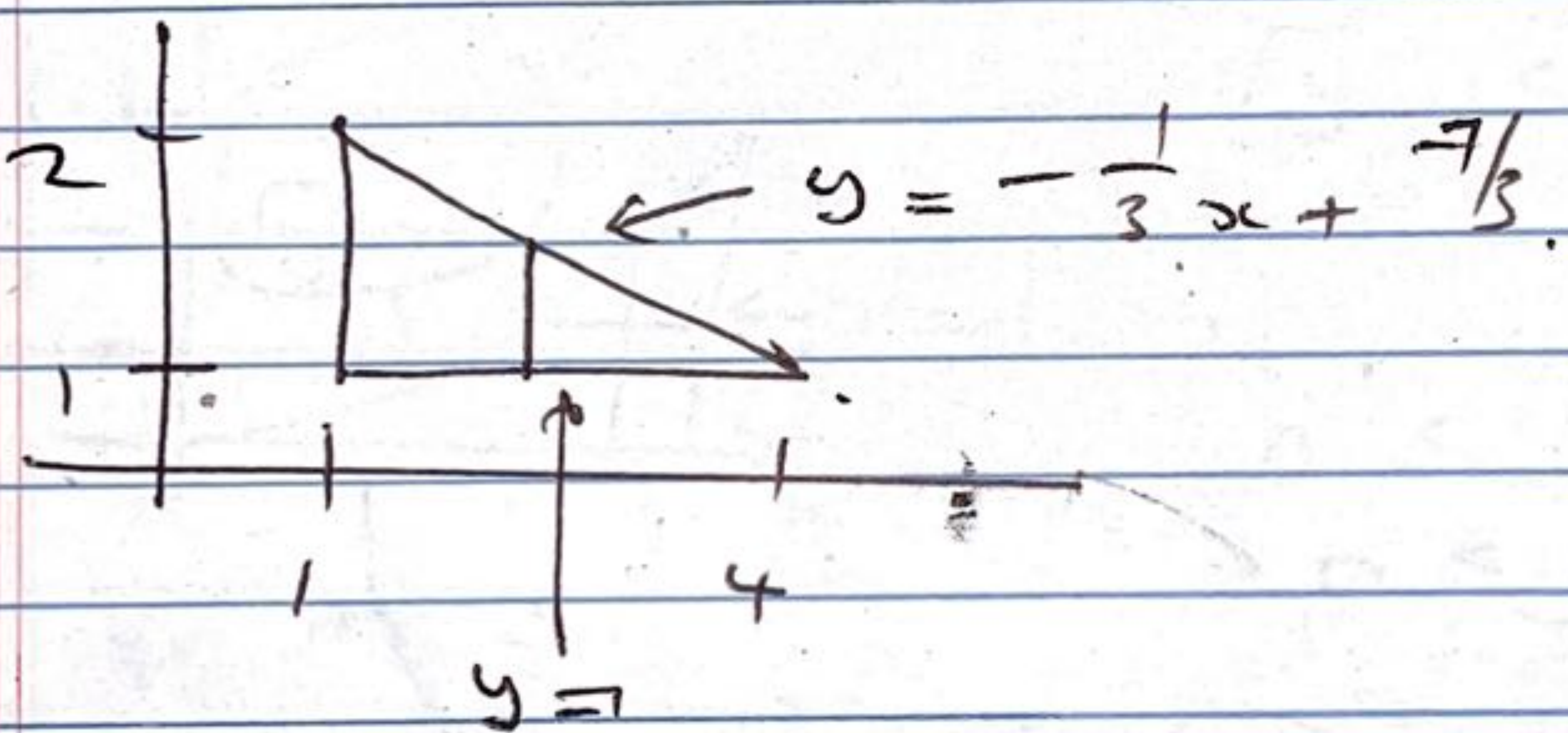
$$\iint_D (x+y)\, dA = \int_{y=0}^{y=1} \left[\int_{x=y^2}^{x=\sqrt{y}} (x+y)\, dx \right] dy$$

$$= \int_{y=0}^{y=1} \left[\frac{x^2}{2} + xy\right]_{x=y^2}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{y}{2} + y^{3/2}\right) - \left(\frac{y^4}{2} + y^3\right) dy$$

$$= 3/10.$$

(3) Find Vol under $z = xy$ above triangle with vertices $(1,1)$, $(4,1)$, $(1,2)$.



Do as Type I

$$1 \le x \le 4$$
$$1 \le y \le -\frac{1}{3}x + 7/3$$

$$\text{VOL} = \int_{x=1}^{x=4} \int_{y=1}^{y=-\frac{1}{3}x + 7/3} xy \, dy \, dx$$

$$= \int_{x=1}^{x=4} \left[\frac{xy^2}{2}\right]_{y=1}^{y=-\frac{1}{3}x + 7/3} dx$$

ETC

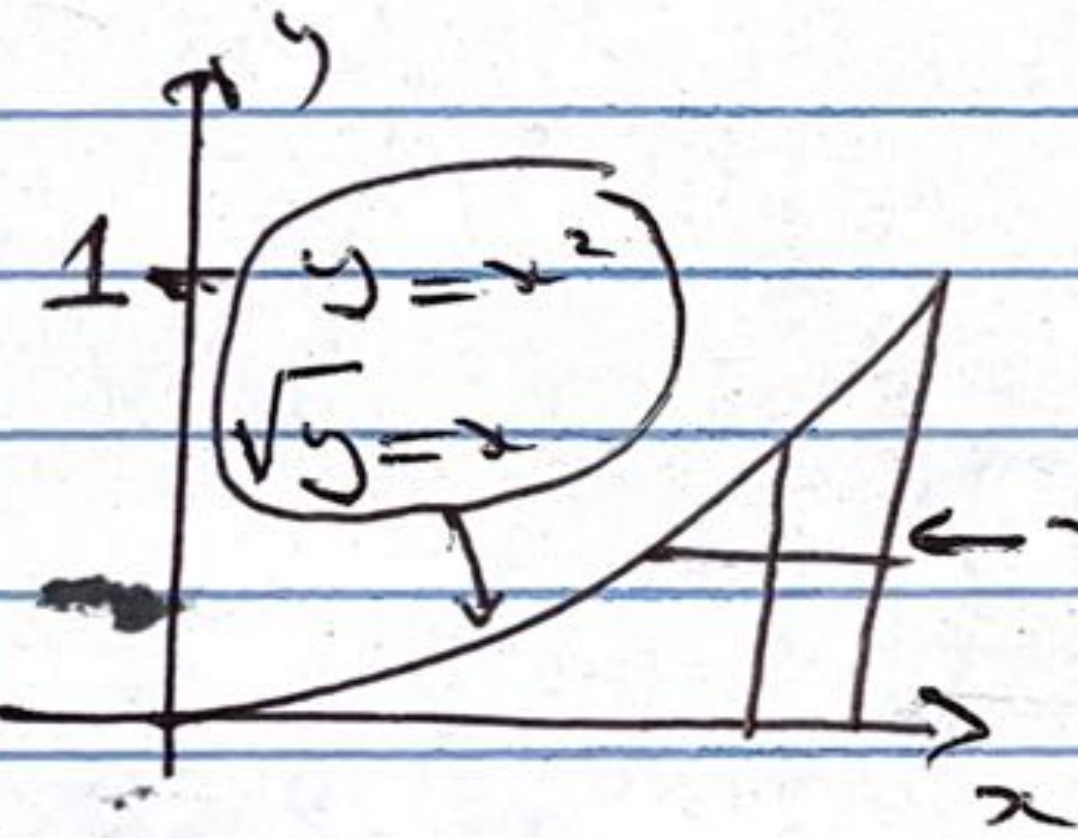
4 REVERSING ORDER OF INTEGRATION

$$I = \int_{y=0}^{y=1}\int_{x=\sqrt{y}}^{x=1} \sqrt{x^3+1}\,dx\,dy$$

- x-ANTIDER HORRID.
- But Integrand const in y
- Try Switching Order

WARNING

To SWITCH ORDER for general D
MUST Draw Picture



$0 \le y \le 1$ TYPE II
$\sqrt{y} \le x \le 1$

$0 \le x \le 1$ TYPE I
$0 \le y \le x^2$

So

$$I = \int_{x=0}^{x=1}\int_{y=0}^{y=x^2} \sqrt{x^3+1}\,dy\,dx$$

$$= \int_0^1 x^2\sqrt{x^3+1}\,dx$$

$$= \int_1^2 \frac{1}{3}u^{1/2}\,du = \frac{2}{9}\left(2^{3/2}-1\right)$$

$u = x^3+1$
$du = 3x^2\,dx$