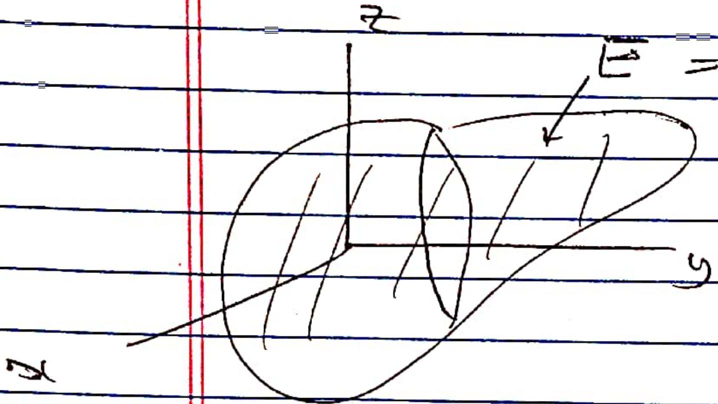
1

15.6, 15.7, 15.8 TRIPLE INTEGRALS

Let $E \subseteq \mathbb{R}^3$ be a 3D SOLID.

E = White Meat of Potato



LET $W = f(x,y,z)$

VOLUME (E) $= \iiint_E 1 \, dV$

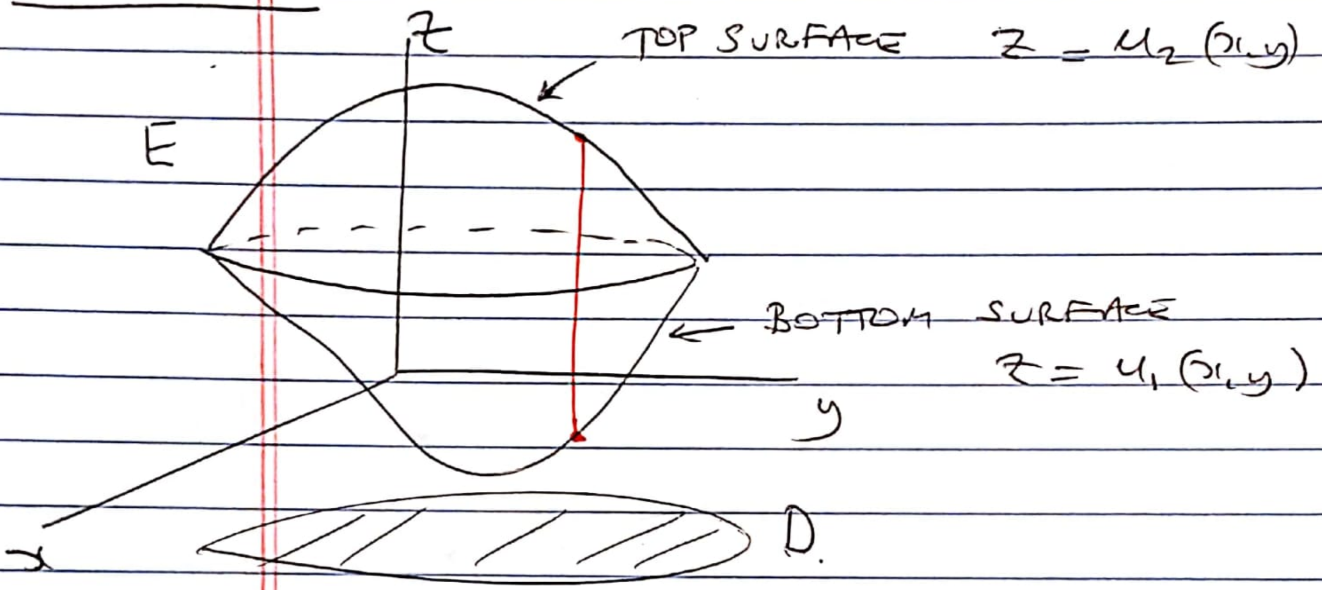
MASS (E) $= \iiint_E \rho(x,y,z)\, dV$ if ρ = DENSITY = MASS/VOL

VOL ELEMENT

$$
\begin{aligned}
dV &= dx\,dy\,dz && \text{RECT} \\
&= \boxed{r}\, dr\, d\theta\, dz && \text{CYL} \\
&= \boxed{\rho^2 \sin\phi}\, d\rho\, d\phi\, d\theta && \text{SPH}
\end{aligned}
$$

$\boxed{}$ = VOLUME STRETCHING FACTOR

RECT COORDS



FILL E WITH VERTICAL FRENCH FRIES WITH SHADOW D.

So E is

$$\begin{cases} u_1(x,y) \le z \le u_2(x,y) \\ (x,y) \in D. \end{cases}$$

Hope D is Type I or II.

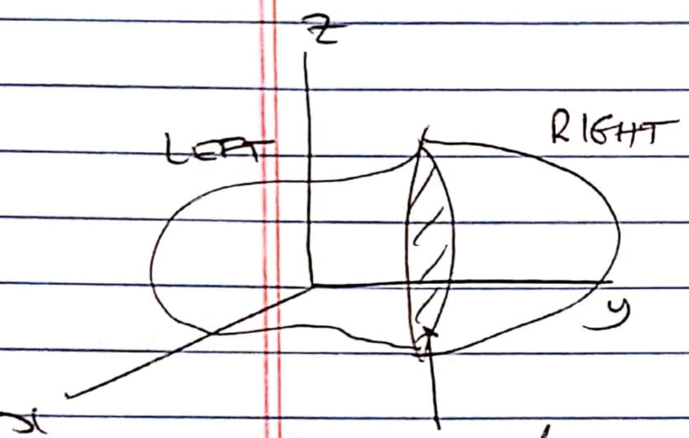
Then

$$\iiint_E f\,dV = \iint_D \left[\underbrace{\int_{z=u_1(x,y)}^{z=u_2(x,y)} f(x,y,z)\,dz}_{= G(x,y)} \right] dA$$

Then do Double Integral.

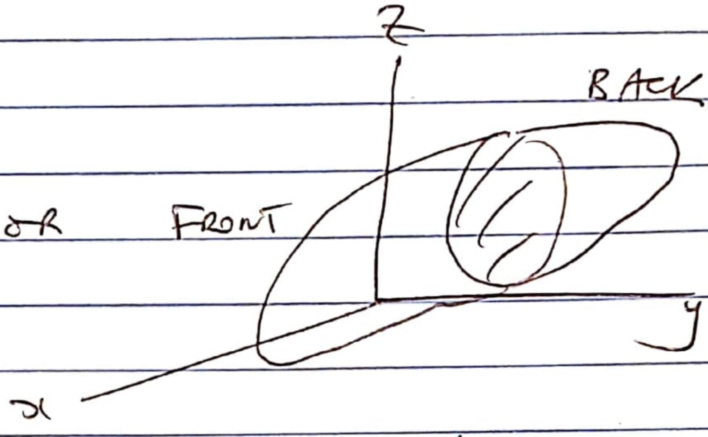
NOTE

Can also have



D in xz-plane

OR

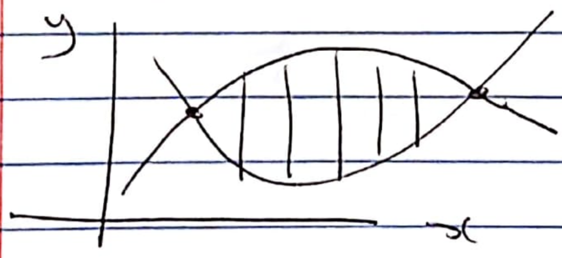


D in yz-plane

So $6 = 3 \times 2$ Types of Regions.

DOUBLE INTEGRAL ~~CALC I~~ REVIEW

IF D is region between a pair of graphs



Then to set up $\iint_D f(x,y)\, dA$

need to find intersection points of curves.

TRIPLE INTEGRAL METHOD

If E is SOLID bounded by a collection of surfaces

Then to set up $\iiint_E f(x,y,z)\,dV$ need to sketch curves obtained by intersecting pairs of these surfaces

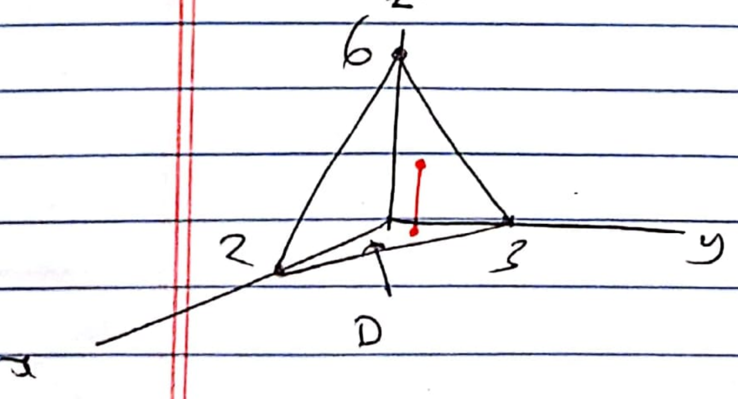
THESE CURVES are like EDGES of SOLID

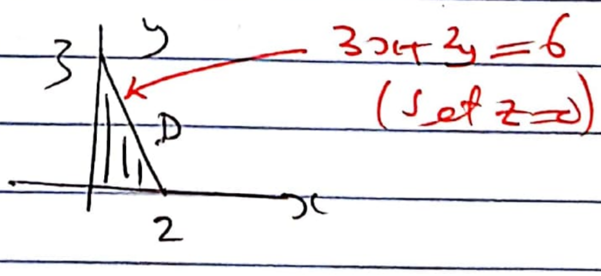
EXS

① FIND $\iiint_E x\,dV$

E is bounded by planes $x=0,\ y=0,\ z=0$
$3x+2y+z=6$

E = TETRAHEDRON





E is $\begin{cases} 0 \le x \le 2 \\ 0 \le y \le \dfrac{6-3x}{2} \\ 0 \le z \le 6-3x-2y \end{cases}$

$$\iiint_E x\,dV = \int_{x=0}^{2} \int_{y=0}^{3-\frac{3}{2}x} \int_{z=0}^{6-3x-2y} x \, dz\, dy\, dx$$

$$= \int_{x=0}^{2} x \int_{y=0}^{3-\frac{3}{2}x} (6-3x-2y)\, dy\, dx$$

$=$ TEDIOUS.

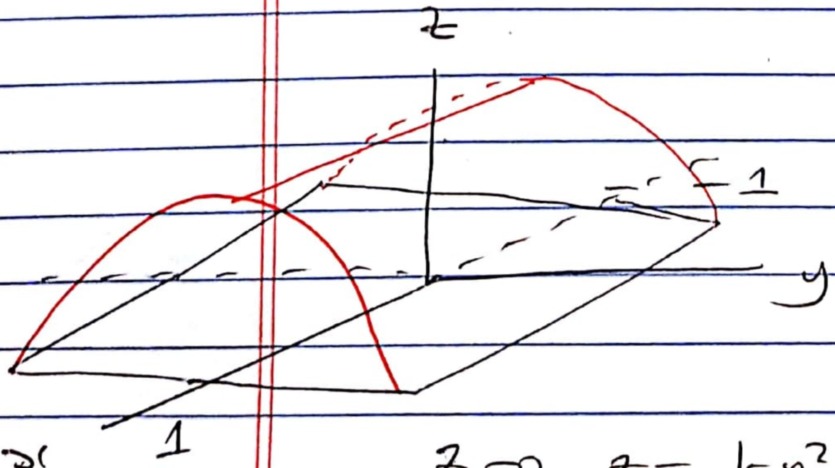
(2) $I = \iiint_E x^2 y^2 \, dV$

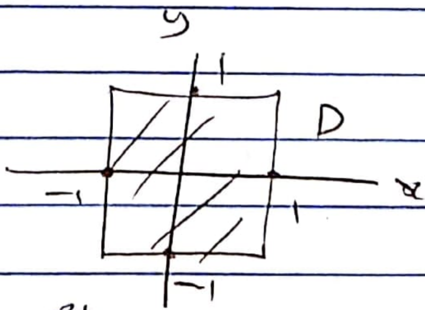
E is SOLID bounded by $z = 1-y^2$

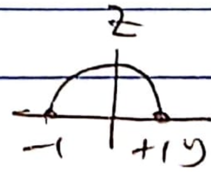
$z=0$

$x=1$

$x=-1$







$z=0$, $z = 1-y^2$ meet at $y = \pm 1$

$$I = \int_{x=-1}^{1} \int_{y=-1}^{1} \int_{z=0}^{1-y^2} x^2 y^2 \, dz \, dy \, dx$$

$$= \left(\int_{-1}^{1} x^2 dx \right) \left[\int_{-1}^{1} y^2 \left(\int_{0}^{1-y^2} dz \right) dy \right]$$

$$= \left[\frac{x^3}{3} \right]_{-1}^{1} \int_{-1}^{1} y^2 (1-y^2) \, dy$$

$$= \frac{4}{3} \left(\frac{1}{3} - \frac{1}{5} \right)$$

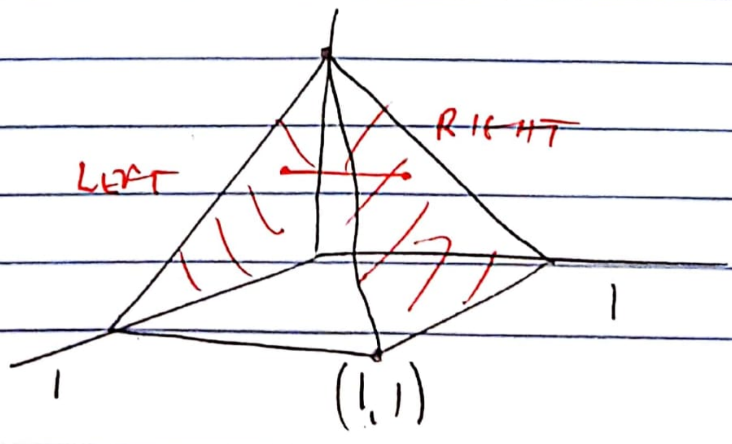
3 $I = \iiint_E z \, dV$

E bounded by planes $x=0$, $y=0$, $z=0$
$y+z=1$, $x+z=1$

The planes $y+z=1$, $x+z=1$ meet in line

$x = 1-t$
$y = 1-t$
$z = t$

joining $(0,0,1)$ to $(1,1,0)$



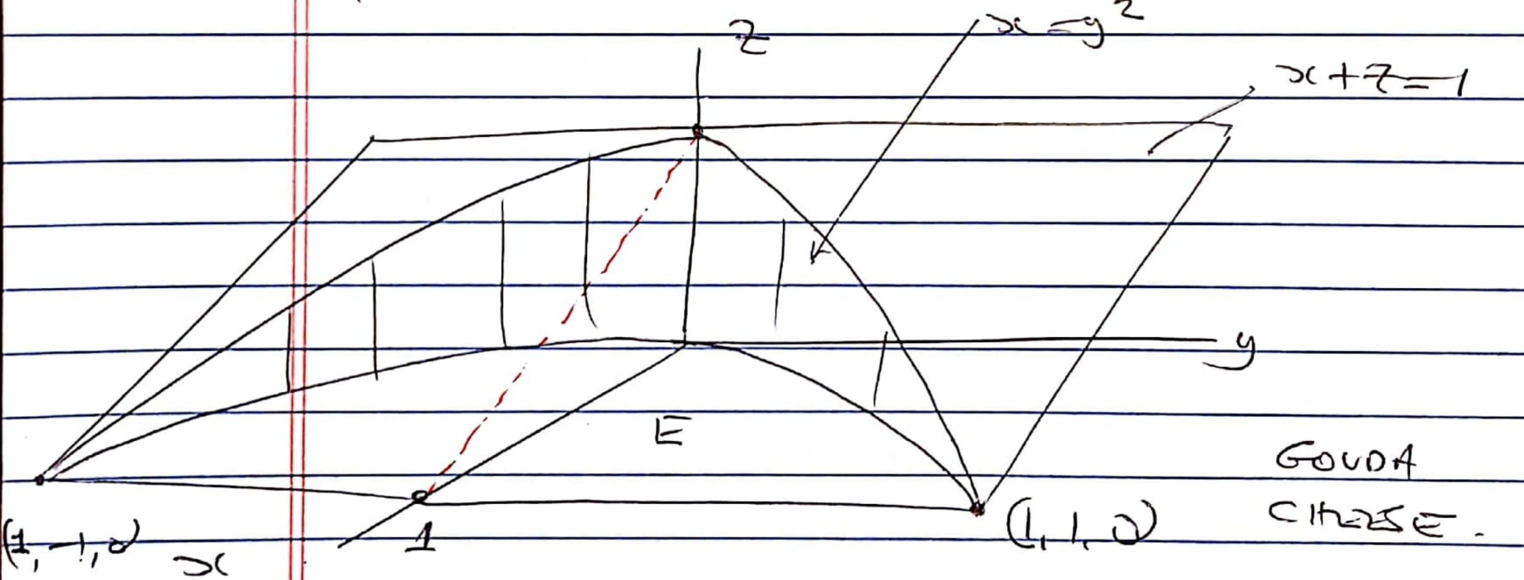
CORNER OF ROOF

$$\begin{cases} 0 \le x \le 1 \\ 0 \le z \le 1-x \\ 0 \le y \le 1-z \end{cases}$$

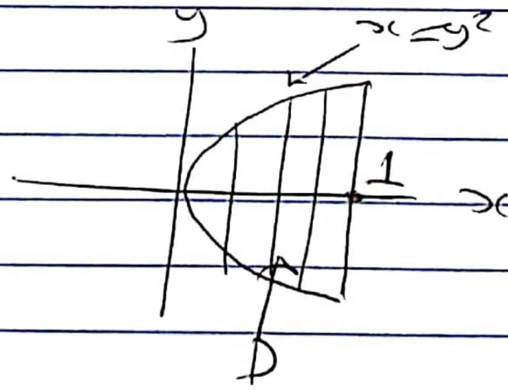
$$I = \int_{x=0}^{1} \int_{z=0}^{1-x} \int_{y=0}^{1-z} z \, dy \, dz \, dx$$ — ETC

④ Find VOL of solid bounded by

$z=0, \quad x+z=1, \quad x=y^2$



GOUDA CHEESE.



E is solid over D under $x+z=1$

E is $0 \le x \le 1$

$-\sqrt{x} \le y \le \sqrt{x}$

$0 \le z \le 1-x$

$$Vol(E) = \int_{x=0}^{1} \int_{y=-\sqrt{x}}^{y=+\sqrt{x}} \int_{z=0}^{z=1-x} 1 \, dz \, dy \, dx$$

$$= \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (1-x) \, dy \, dx$$

$$= 2 \int_0^1 (1-x)\sqrt{x} \, dx = ETC$$

CYL COORDS

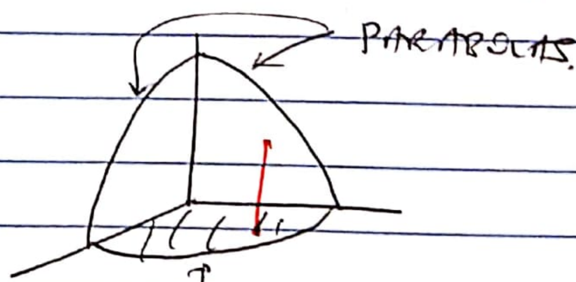
(5) $I = \iiint_E (x^3 + xy^2) \, dV$

E is solid in 1st octant below $z = 1 - x^2 - y^2$

$0 \le \theta \le \pi/2$

$0 \le r \le 1$

$0 \le z \le 1 - r^2$



$x^3 + xy^2 = x(x^2+y^2)$

$= r\cos\theta \, r^2 = r^3 \cos\theta.$

So
$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \int_{z=0}^{1-r^2} (r^3 \cos\theta) \boxed{r} \, dz \, dr \, d\theta$$

$$= \left(\int_0^{\pi/2} \cos\theta \, d\theta\right) \left(\int_0^1 r^4 (1-r^2) \, dr\right)$$

$$= 2/35.$$

(6) $$I = \iiint_E \sqrt{x^2+y^2} \, dV$$

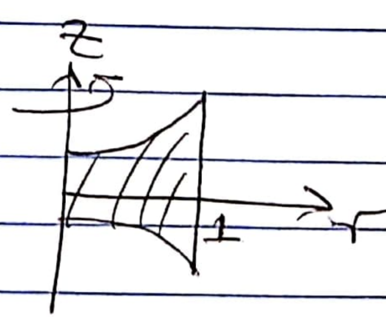
NOTE Since $r = \sqrt{x^2+y^2}$ = DIST of (x,y,z) from z axis

Average Distance from z axis of a point in E

$$= I / \text{VOL}(E)$$

Here E is bounded by

$$\begin{cases} x^2+y^2 = 1 & \text{CYL} \\ z = 1+x^2+y^2 \\ z = -1-x^2-y^2 \end{cases}$$



or

$r = 1$

$z = 1+r^2$

$z = -1-r^2$

$$I = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{z=1+r^2} r \; r \, dz\, dr\, d\theta = \text{ETC}$$

SPH COORDS

(7) E is $x^2+y^2+z^2 \leq 1$

$$\iiint_E z^2 \, dV = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1} (\rho\cos\phi)^2 \boxed{\rho^2 \sin\phi} \, d\rho\, d\phi\, d\theta$$

$$= 2\pi\left(\int_0^{\pi} \cos^2\phi \sin\phi \, d\phi\right)\left(\int_0^1 \rho^4 \, d\rho\right)$$

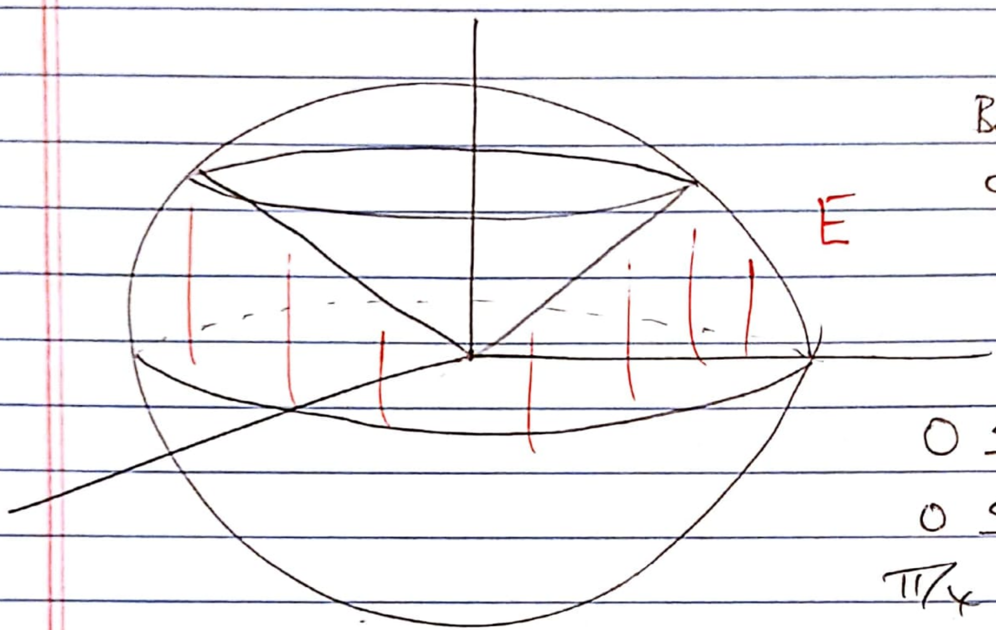
$$= \frac{4\pi}{15}.$$

(8) $I = \iiint_E z\, dV$

Average Height of Solid E $= \dfrac{I}{\text{VOL}(E)}$

E is solid

- Within Sphere $x^2+y^2+z^2 = 4$ $\quad \rho = 2$
- Above xy plane $\quad \phi = \pi/2$
- Below Cone $z = \sqrt{x^2+y^2}$ $\quad \phi = \pi/4$



BUNDT CAKE.

$0 \le \rho \le 2$

$0 \le \theta \le 2\pi$

$\pi/4 \le \phi \le \pi/2$

$$I = \int_{\theta=0}^{2\pi} \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{2} (\rho\cos\phi)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= 2\pi \int_{\pi/4}^{\pi/2} \left(\tfrac{1}{2}\sin 2\phi\right) d\phi \int_0^2 \rho^3\, d\rho$$

$$= \pi \left[-\tfrac{1}{2}\cos 2\phi\right]_{\pi/4}^{\pi/2} \frac{2^4}{4} = 2\pi$$