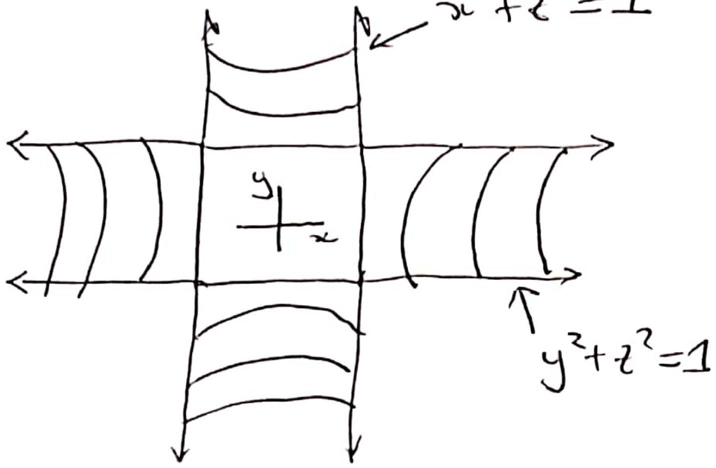
EX

Consider volume between floor and ceiling of part of a cathedral where the two barrel vaults intersect.

Model vaults as cylinders

$$x^2+z^2=1 \quad (1)$$

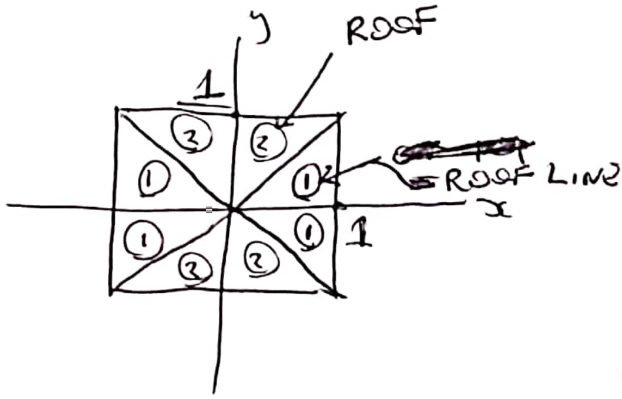
$$y^2+z^2=1 \quad (2)$$



The two cylinders intersect in

$$x^2 \overset{(1)}{=} 1-z^2 = y^2$$

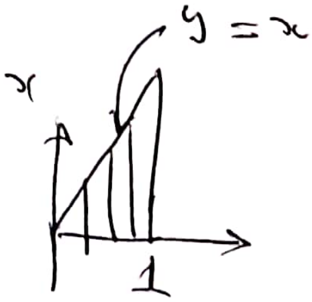
$y = \pm x$ Pair of Planes



Hard part of volume ~~to~~ calculate is solid over square D in xy-plane and under the roof.

$\boxed{\text{METHOD I}}$

$$\text{VOL} = 8 \iint_D \sqrt{1-y^2}\, dA$$



$$= 8 \int_{x=0}^{x=1} \int_{y=0}^{y=x} \sqrt{1-y^2}\, dy\, dx$$

$$= 8\int_{x=0}^{x=1} \left[\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}(y) \right]_{y=0}^{y=x} dx$$

ASIDE: SWITCHING to $\int dx\,dy$ is even worse!

$$= 4\int_0^1 x\sqrt{1-x^2} + \sin^{-1}(x)\,dx$$

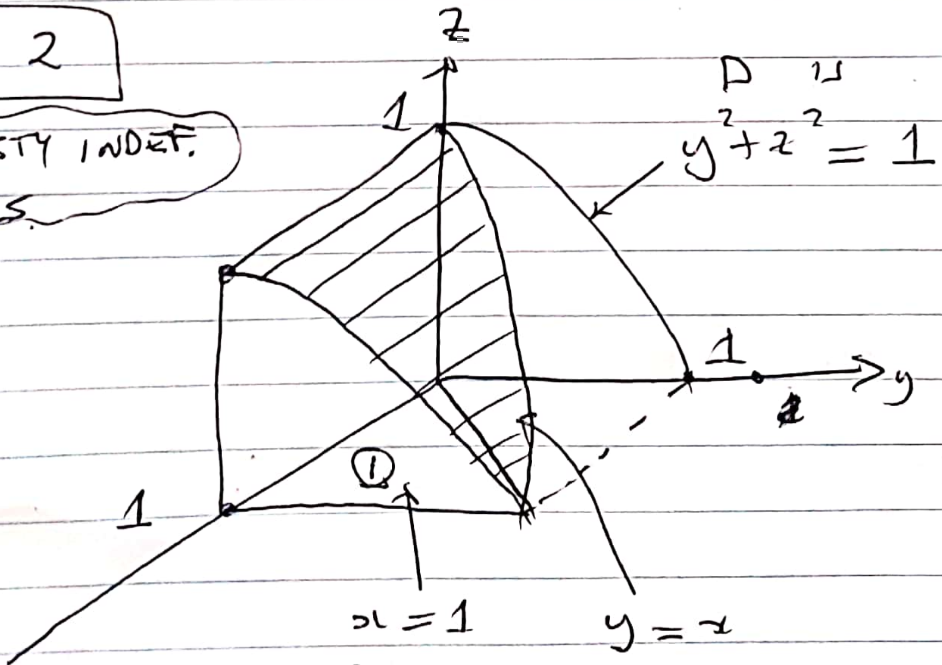
$$= 2\int_0^1 \sqrt{u}\,du + 4\left[x\sin^{-1}(x) + \sqrt{1-x^2}\right]_{x=0}^{x=1}$$

$$= \frac{4}{3} + 4\left(\frac{\pi}{2} - 1\right)$$

$$= 2\pi - 8/3$$

$\boxed{\text{METHOD 2}}$

AVOIDS NASTY INDEF. INTEGRALS.



THIS IS 1/8 OF VOL.

Use Cyl Coords (r, θ, x) :

$$x = x$$
$$y = r\cos\theta$$
$$z = r\sin\theta$$

D is $0 \le \theta \le \pi/2$
$0 \le r \le 1$

FRIES go from BACK to FRONT
$y \le x \le 1$

or $r\cos\theta \le x \le 1$

So

$$\text{VOL} = 8 \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \int_{x=r\cos\theta}^{x=1} r \, dx \, dr \, d\theta$$

$$= 8 \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r(1 - r\cos\theta) \, dr \, d\theta$$

$$= 8 \int_0^{\pi/2} \left[\frac{r^2}{2} - \frac{r^3}{3}\cos\theta \right]_{r=0}^{r=1} d\theta$$

$$= 8 \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{3}\cos\theta \right) d\theta$$

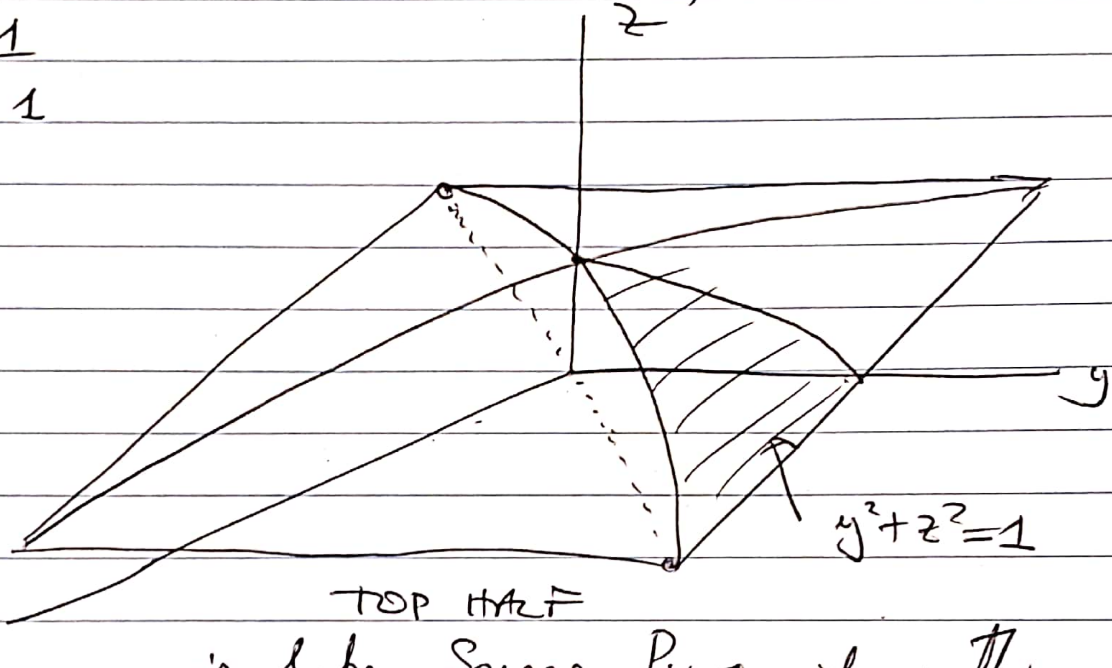
$$= 8 \left(\pi/4 - \frac{1}{3}\sin \pi/2 \right) = 2\pi - 8/3$$

as before!!

EX VOL enclosed by 2 Cylinders
Different from Cathedral/Vaults problem

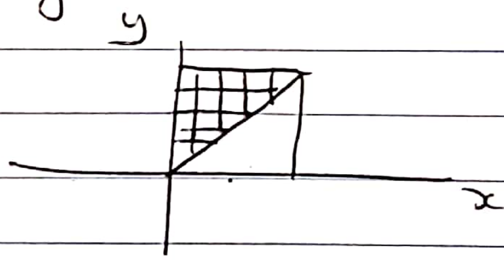
$x^2+z^2=1$

$y^2+z^2=1$



TOP HALF

is like Square Pyramid with sides that are parts of cylinders rather than triangles.



$$\text{VOL} = 16 \iint_D \sqrt{1-y^2}\, dA$$

$$= 16 \int_{y=0}^{1} \int_{x=0}^{x=y} \sqrt{1-y^2}\, dx\, dy$$

$$= 16 \int_{y=0}^{1} y\sqrt{1-y^2}\, dy$$

$$= 8 \int_0^1 \sqrt{u}\, du$$

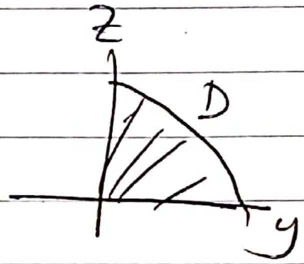
$$= 8 \tfrac{2}{3}\left[u^{3/2}\right]_0^1 = 16/3$$

OR Use CYL COORDS (x, r, θ)

$$x = x$$
$$y = r\cos\theta$$
$$z = r\sin\theta$$

$\frac{1}{16}$ Region is $\quad 0 < \theta < \pi/2$

$\quad\quad\quad\quad\quad\quad 0 < r < 1$



$$0 < x < y = r\cos\theta$$

$$\text{VOL} = 16 \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \int_{x=0}^{x=r\cos\theta} 1\, r\, dx\, dr\, d\theta$$

$$= 16 \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} r^2 \cos\theta\, dr\, d\theta$$

$$= 16 \left[\sin\theta\right]_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^1$$

$$= \frac{16}{3}$$ again.