15.10 CHANGE OF VARIABLES THM

TRANSFORMATIONS OF $\mathbb{R}^2$ (AKA CHANGE OF COORDS)

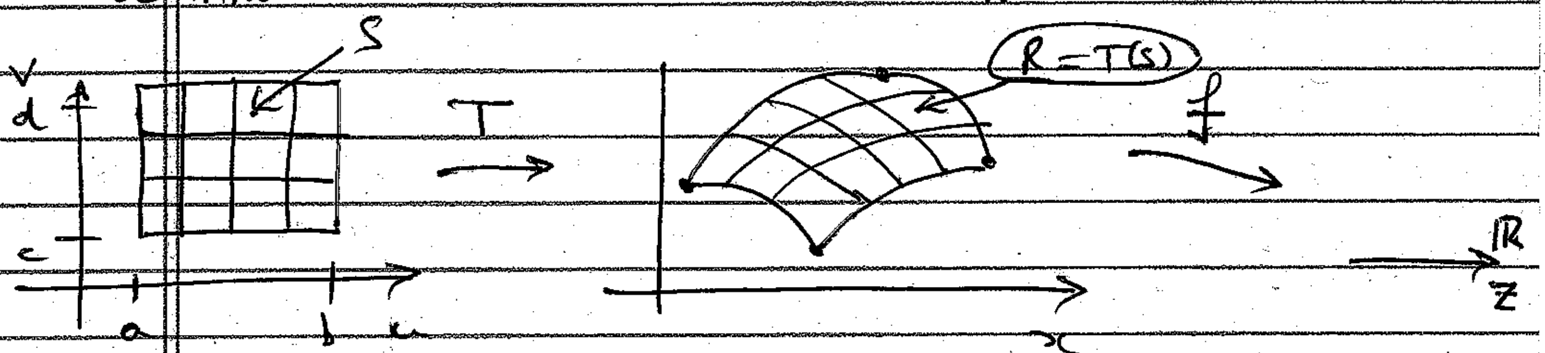
CONSIDER $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$(x,y) = T(u,v)$$

ALSO HAVE $z = f(x,y)$

SIMPLER DOMAIN

MORE COMPLEX DOMAIN

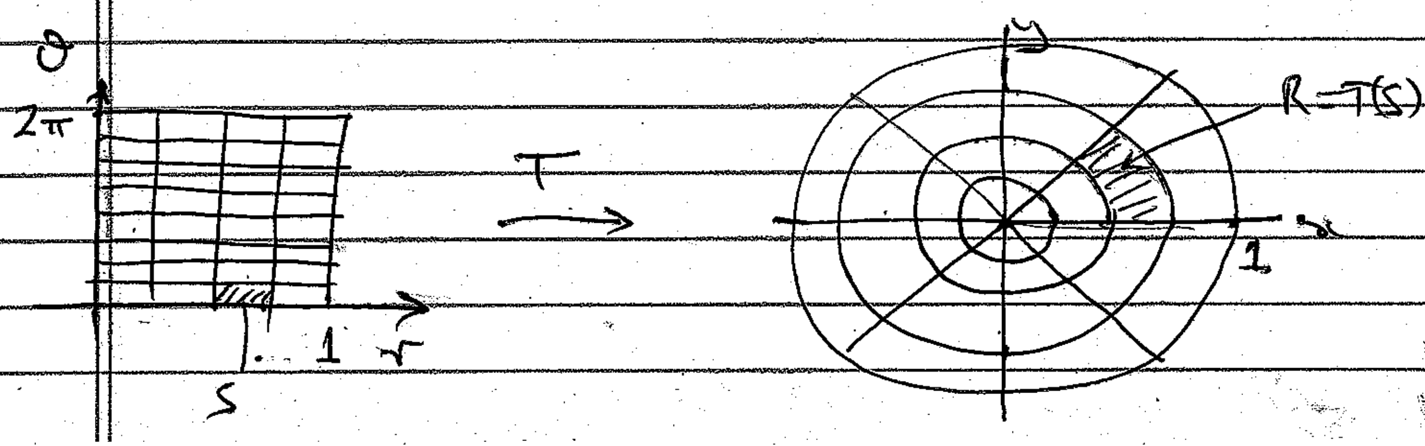


PARAMETER SPACE.

EX POLAR → RECT. COORD TRANSFN.

USE $(u,v) = (r,\theta)$

$$(x,y) = T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$



C of V THM With $T: \mathbb{R}^2 \to \mathbb{R}^2$ and $f: \mathbb{R}^2 \to \mathbb{R}$ as above

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(T(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

MORE COMPLEX DOMAIN SIMPLER DOMAIN

SIMPLER INTEGRAND MORE COMPLEX INTEGRAND

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$ "JACOBIAN DETERMINANT"

and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$ = Abs Value of $\frac{\partial(x,y)}{\partial(u,v)}$

= AREA STRETCHING FACTOR (See Later)

EX POLAR → RECT COORDS

$$(x,y) = T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r$$

So C of V Thm becomes PC formula:

$$\iint_R f(x,y)\,dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

CALC I ANALOGY: INTEGRATION BY SUBSTITUTION

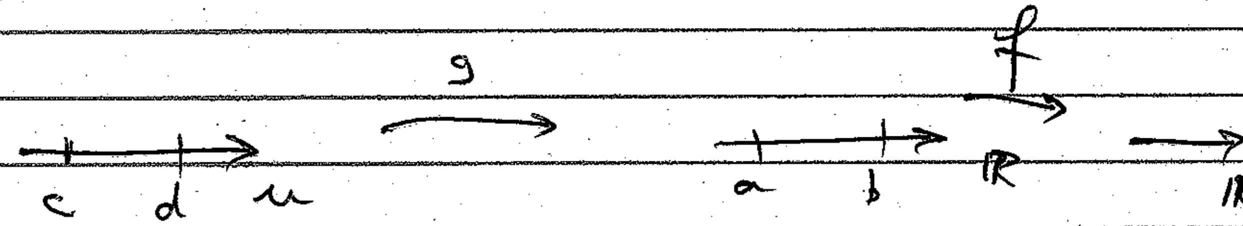
$$\int_{u=c}^{u=d} f(g(u))\,g'(u)\,du = \int_{x=a}^{x=b} f(x)\,dx$$

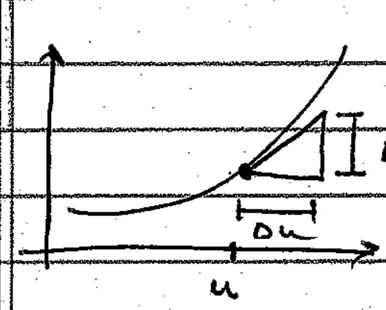
HARD INTEGRAND — EASY INTEGRAND

where $x = g(u)$
$dx = g'(u)\,du$

THINK $x = g(u)$, $g : \mathbb{R} \to \mathbb{R}$ as a TRANSFN OF $\mathbb{R}$

(Analogous to $T : \mathbb{R}^2 \to \mathbb{R}^2$)





$\Delta x = \frac{dx}{du}\,\Delta u = g'(u)\,\Delta u$

So $\frac{dx}{du}$ TELLS HOW MUCH g STRETCHES INTERVAL OF LENGTH Δu.

LENGTH STRETCHING FACTOR

SPECIAL CASE of C of V THM ($f \equiv 1$)

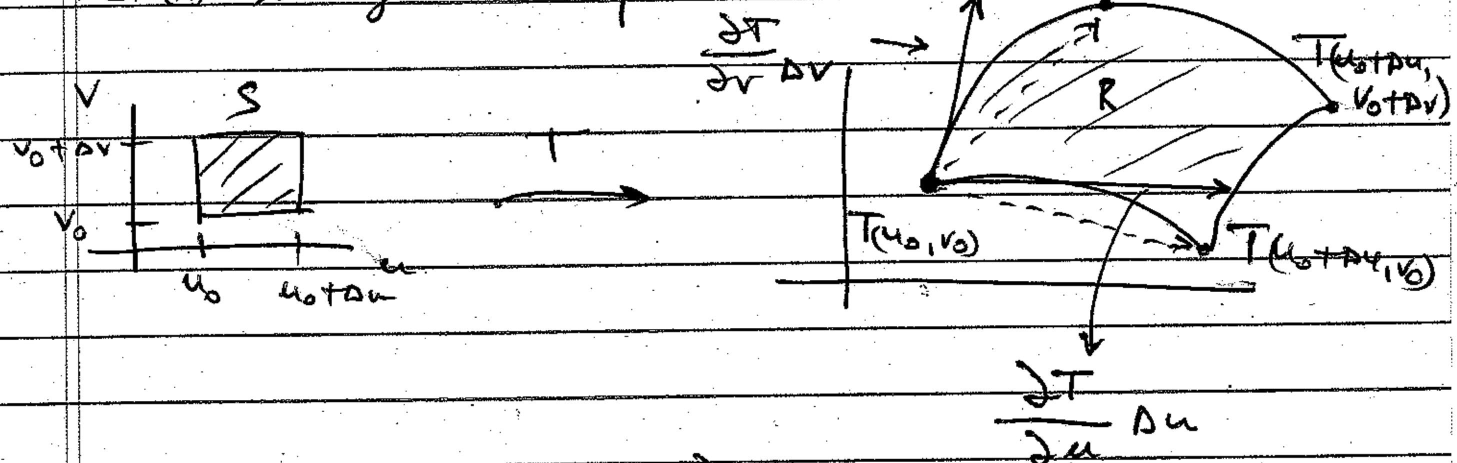
$$\text{AREA}(R) = \iint_R 1\, dA = \iint_S \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

WHY IS THIS TRUE?

Look at case S is small rectangle 

So
$$\iint_S \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \simeq \left| \frac{\partial(x,y)}{\partial(u,v)}(u_0,v_0) \right| \Delta u\, \Delta v.$$

AND AREA (R) is got as follows



AS

$$T(u_0+\Delta u, v_0) - T(u_0, v_0) \simeq \frac{\partial T}{\partial u} \Delta u$$

$$T(u_0, v_0+\Delta v) - T(u_0, v_0) \simeq \frac{\partial T}{\partial v} \Delta u$$

Area (R) $\simeq$ Area  $= \left| \frac{\partial T}{\partial u} \Delta u \times \frac{\partial T}{\partial v} \Delta v \right|$

$$= \left| \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} \right| \Delta u\, \Delta v$$

$$= \left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} \right\| \Delta u \Delta v = \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{matrix} \right\| \Delta u \Delta v.$$

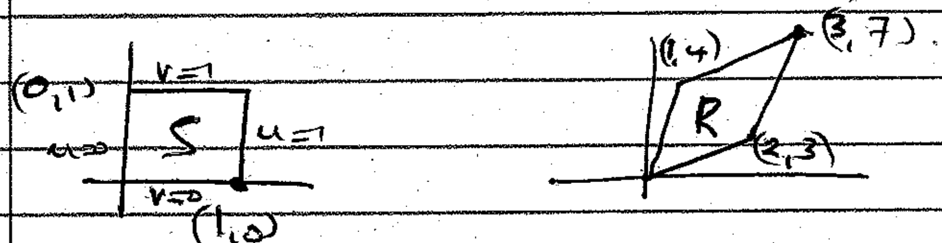
$$= \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v.$$

EX $T: \mathbb{R}^2 \to \mathbb{R}^2$ LINEAR TRANSFN. (LT)

GENERAL LT $(x,y) = T(u,v) = (au + bv,\ cu + dv)$

FACT If T is linear Then T maps lines to lines

So T maps rectangles to parallelograms.



EX $T(u,v) = (2u + v,\ 3u + 4v) = (x,y)$.

$T(1,0) = (2,3)$
$T(0,1) = (1,4)$
$T(1,1) = (3,7)$

If $u=0$ Then $(x,y) = (v, 4v) = v(1,4)$ for $v \in [0,1]$
par line from $(0,0)$ to $(1,4)$

If $u=1$ Then $(x,y) = (2+v, 3+4v) = (2,3) + v(1,4)$

par line from $(2,3)$ to $(3,7)$.

If $v=0$ Then $(x,y) = u(2,3)$ par line from
$(0,0)$ to $(2,3)$

If $v=1$ Then $(x,y) = (1,4) + u(2,3)$ par line
from $(1,4)$ to $(3,7)$.

UPSHOT T maps unit square to // gram with
vertices $(0,0)$, $(2,3)$, $(3,7)$, $(1,4)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = 8-3 = 5$$

$$\text{So } \iint_R \underbrace{xy}_{f(x,y)=xy}\, dx\, dy = \int_{u=0}^{u=1} \int_{v=0}^{v=1} (2u+v)(3u+4v)\, 5\, du\, dv$$

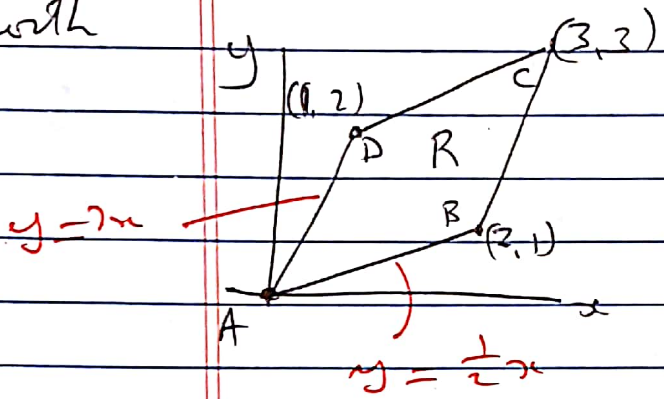
$= \text{ETC}$

EASY IF TEDIOUS

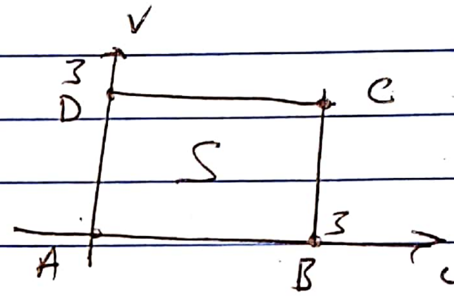
R NEITHER TYPE I / II S IS RECT.

EX $I = \iint_R x \, dA$

with



R = ||gram



Set $\boxed{\begin{aligned} u &= 2x - y \\ v &= 2y - x. \end{aligned}}$

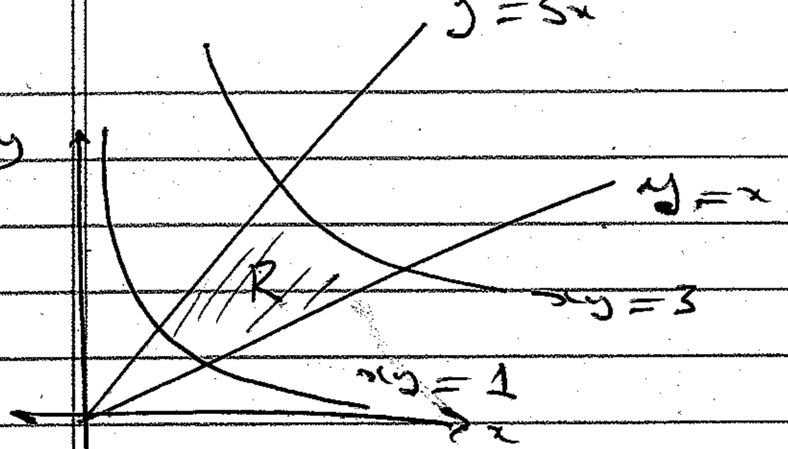
$$2u + v = 4x - 2y + 2y - x = 3x$$
$$u + 2v = 2x - y + 4y - 2x = 3y$$

So $\boxed{\begin{aligned} x &= \frac{2}{3}u + \frac{1}{3}v \\ y &= \frac{1}{3}u + \frac{2}{3}v \end{aligned}}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{vmatrix} = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}.$$

$$I = \int_{u=0}^{3}\int_{v=0}^{3} \left(\frac{2}{3}u + \frac{1}{3}v\right) dv\, du = \text{EASY.}$$

Ex



NEITHER TYPE I OR II

FIND $I = \iint_R xy \, dA$

SET $x = \frac{u}{v}, \quad y = v.$

Then

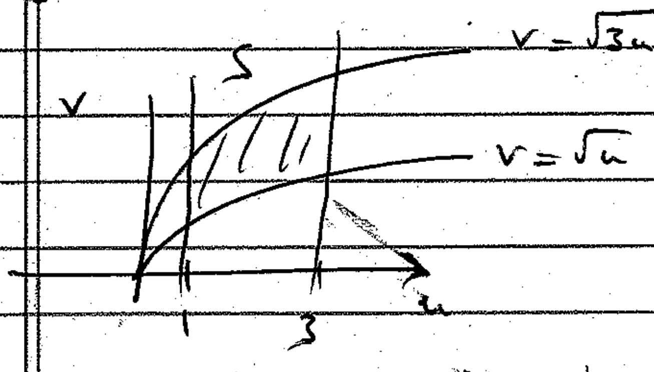
$$k = xy = \frac{u}{v} v = u$$

AND $k = \frac{y}{x} = v \cdot \frac{v}{u} = \frac{v^2}{u}$

So $y = x \iff v = \sqrt{u}$

$y = 3x \iff v = \sqrt{3u}$

So



TYPE I

ALSO $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$

AND $xy = f(x,y) = u = f(T(u,v))$

So

$$I = \iint_S f(T(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$= \int_{u=1}^{u=3} \int_{v=\sqrt{u}}^{v=\sqrt{3u}} u \cdot \frac{1}{v}\, dv\, du$$

$$= \int_{u=1}^{u=3} u \left[\ln|v| \right]_{v=\sqrt{u}}^{v=\sqrt{3u}} du.$$

$$= \frac{1}{2} \int_1^3 u \left(\ln|3u| - \ln|u| \right) du$$

$$= \frac{1}{2} \int_1^3 u \ln 3\, du = \frac{\ln 3}{2} \int_1^3 u\, du = 2 \ln 3.$$