16.2 LINE INTEGRALS

LINE INTEGRALS OF FUNCTIONS

RECALL FROM 13.3 : If C is a ~~parametrized~~ curve with parametrization $(x, y, z) = \vec{r}(t)$ for $a \leq t \leq b$, Then

$$\text{LENGTH}(C) = \int_{t=a}^{t=b} |\vec{r}\,'(t)|\,dt.$$

DEF If in addition we have a function $w = f(x, y, z)$ defined on C Then we defined

$$\int_C f\,ds = \int_{t=a}^{t=b} f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

ARCLENGTH ELEMENT: $ds = |\vec{r}\,'(t)|\,dt$

LENGTH = SPEED × TIME

SOME MEANINGS

① If C is a wire and

② If $w = f(x, y, z)$ = Density of Wire at a point (x, y, z) on C in UNITS of $\frac{\text{MASS}}{\text{LENGTH}}$

Then

$$\int_C f\,ds = \text{TOTAL MASS of Wire}$$

$\frac{\text{MASS}}{\text{LENGTH}} \times \text{LENGTH} = \text{MASS}$

(2) The Average Value of f along C

$$= \frac{\int_C f\,ds}{\text{LENGTH}(C)}.$$

EX If f = TEMPERATURE IN K

Then

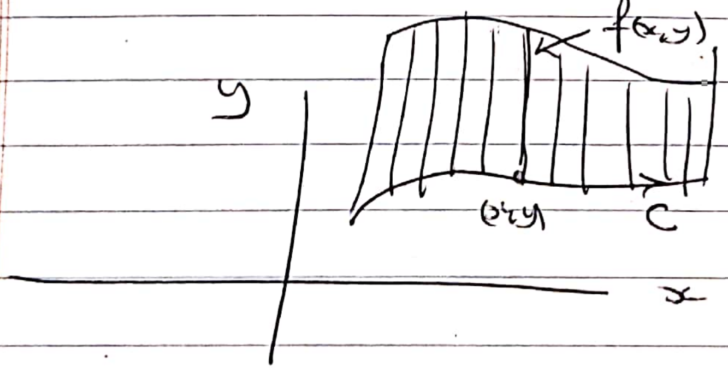
$$\frac{\int_C f\,ds}{\text{LENGTH}(C)} \text{ has UNITS } \frac{K.m}{m} = K.$$

(3) If C is a curve in the xy-plane that models the base of a fence and $z = f(x,y)$ = HEIGHT of fence at $(x,y) \in C$

Then

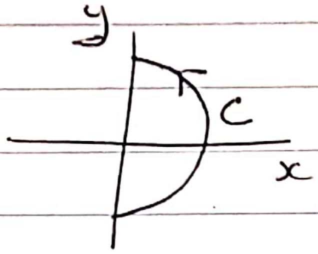
$$\int_C f\,ds = \text{AREA OF FENCE}$$

HEIGHT × LENGTH = AREA



<u>EXS</u>

1. $\int_C xy^4\,ds$



$x^2+y^2=16$

Choose $\vec{r}(t) = (4\cos t, 4\sin t)$ $\quad -\pi/2 \le t \le \pi/2$

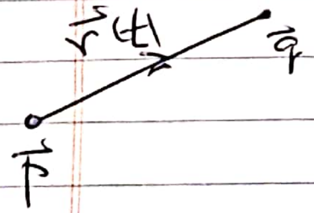
Then $|\vec{r}\,'(t)| = 4$ and so

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4\, 4\,ds$$

$$= 4^6 \int_{-\pi/2}^{\pi/2} \cos t \sin^4 t\,dt$$

$$= 4096 \left[\frac{u^5}{5}\right]_{-1}^{1} = \frac{8192}{5}.$$

2. $\int_C x^2 z\,ds$ where C = Line Segment from $\vec{p} = (0,6,-1)$ to $\vec{q} = (4,1,5)$



$$\vec{r}(t) = \vec{p} + t\vec{v} \qquad 0 \le t \le 1$$

$$= \vec{p} + t(\vec{q} - \vec{p})$$

$$= (0,6,-1) + t(4,-5,6)$$

$$\vec{r}\,'(t) = \vec{q} - \vec{p} = (4,-5,6)$$

$$|\vec{r}\,'(t)| = |\vec{q} - \vec{p}| = \sqrt{77}$$

$$\int_C x^2 z\,ds = \int_0^1 (4t)^2(-1+6t)\sqrt{77}\,dt = \sqrt{77}\,16\left(\frac{3}{2} - \frac{1}{3}\right).$$

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NOTE

BAD NEWS In general for curve C in $\mathbb{R}^2$

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

and so it is impossible to find a formula for the antiderivative of the integrand.

GOOD NEWS Since we have a formula for the integrand and the integral is over an interval $[a,b]$ in $\mathbb{R}$ we can approximate ~~the~~ $\int_C f\,ds$ using numerical integration

<u>EG</u> – Riemann Sum
– Simpsons Rule

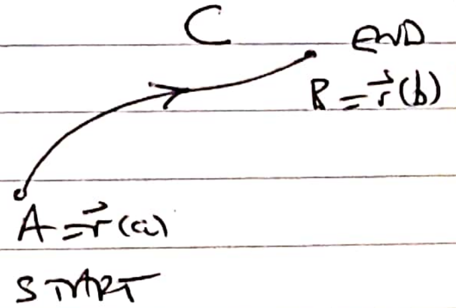
LINE INTEGRALS OF VECTOR FIELDS

<u>DEF</u>: Let C be an ~~ori~~ <u>ORIENTED</u> CURVE IN $\mathbb{R}^n$ $(n = 2 \text{ or } 3)$ with parametrization $\vec{r} : [a,b] \to \mathbb{R}^n$

and let $\vec{F}$ be a VF on $\mathbb{R}^n$.

Define

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$



DISPLACEMENT ELEMENT : $d\vec{r} = \vec{r}\,'(t)\,dt$

DISPLACEMENT = VELOCITY × TIME.

$\boxed{\text{FORMULA RELATING } \int_C \vec{F}\cdot d\vec{r} \text{ to } \int_C f\,ds}$

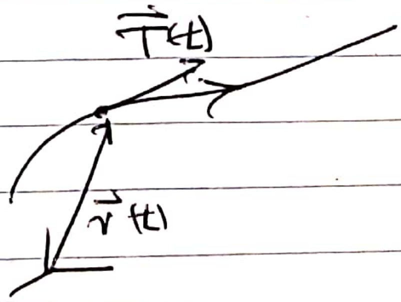
$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$= \int_a^b \left[\vec{F}(\vec{r}(t))\cdot\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}\right]|\vec{r}\,'(t)|\,dt$$

$$= \int_a^b (\vec{F}\cdot\vec{T})\,ds$$

where $ds = |\vec{r}(t)|\,dt$

and $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}(t)|}$ = UNIT TANGENT VF ALONG C



SUMMARY $\boxed{\int_C \vec{F}\cdot d\vec{r} = \int (\vec{F}\cdot\vec{T})\,ds}$

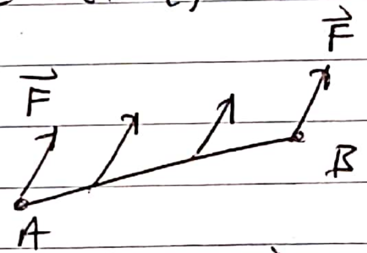
PHYSICS INTERPRETATION OF $\int_C \vec{F} \cdot d\vec{r}$ AS WORK

A force, $\vec{F}$, is said to do WORK if the force causes a test particle to move in the direction of the force.

For straight line motion ~~at constant speed~~ in a constant force field

WORK DONE to move particle from A to B

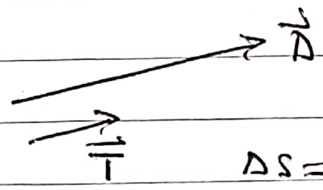
$$= (\text{COMPONENT of Force in Direction of Motion}) \times \text{DISTANCE TRAVELED}$$



$$= \left(\vec{F} \cdot \frac{\vec{D}}{|\vec{D}|}\right) |\vec{D}|$$

$\vec{D} = \overrightarrow{AB}$

$$= \vec{F} \cdot \vec{D}$$



$$= (\vec{F} \cdot \vec{T}) \, \Delta S$$

$\Delta S = |\vec{D}|$

In general, we approximate C by a ~~sequence~~ collection of linear segments such that $\vec{F}$ is approximately constant on each segment

Then Work done by $\vec{F}$ as particle moves along C

$\simeq$ SUM of Work Done on each line segment

$=$ " $\sum (\vec{F}.\vec{T})\Delta s$ "

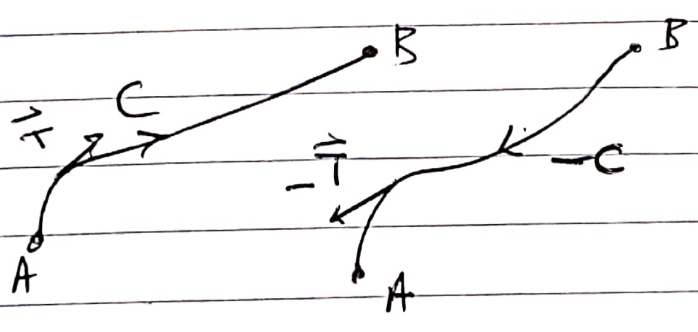
Taking the limit as $\Delta s \to 0$ it makes sense to DEFINE

WORK DONE By $\vec{F}$ along C $= \int_C (\vec{F}.\vec{T})\, ds = \int_C \vec{F}.d\vec{r}.$

NOTE

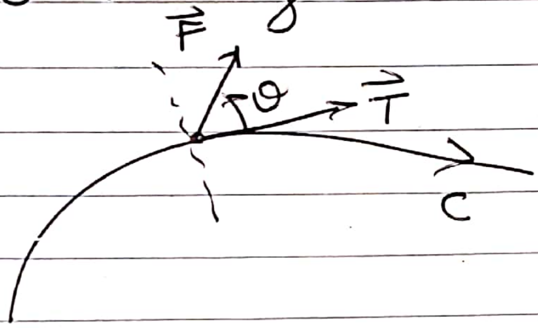
① Let $-C$ be the curve C traversed in opposite direction

Then
$$\int_{-C} \vec{F}.d\vec{r} = -\int_C \vec{F}.d\vec{r}$$



REASON $\vec{T}$ CHANGES SIGN.

② Let θ be angle between $\vec{F}$ and $\vec{T}$



Suppose $-\pi/2 \le \theta \le \pi/2$ everywhere on C

Then $\vec{F} \cdot \vec{T} = |\vec{F}|\,|\vec{T}| \cos\theta > 0$.

ie component of $\vec{F}$ in dirn $\vec{T}$ is positive

So $\int_C \vec{F} \cdot d\vec{r} > 0$. WORK IS +ve

EXS

① $\vec{F}(x,y,z) = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$
$$= t\,\vec{i} + t^2\,\vec{j} + t^3\,\vec{k} \qquad 0 \le t \le 2$$

$$\vec{r}\,'(t) = 1\,\vec{i} + 2t\,\vec{j} + 3t^2\,\vec{k}$$

$$\vec{F}(\vec{r}(t)) = t^2 . t^3\,\vec{i} + t t^3\,\vec{j} + t t^2\,\vec{k}$$
$$= t^5\,\vec{i} + t^4\,\vec{j} + t^3\,\vec{k}$$

So $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$

$= \int_0^2 (t^5\vec{i} + t^4\vec{j} + t^3\vec{k})\cdot(\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt$

$= \int_0^2 (t^5 + 2t^5 + 3t^5)\,dt = 64$

$\boxed{\text{ALTERNATE FORMULA}}$

$\boxed{\text{GENERALLY ON } \mathbb{R}^2}$

$\vec{F} = P\vec{i} + Q\vec{j}$

$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$

So

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \left[P(x(t), y(t))\,\vec{i} + Q(x(t), y(t))\,\vec{j}\right]\cdot \left[\frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}\right]dt$$

$$= \int_C \left[P(x(t), y(t))\frac{dx}{dt} + Q(x(t), y(t))\frac{dy}{dt}\right]dt$$

$$=: \int_C P\,dx + Q\,dy$$

UPSHOT $\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy$

where $\vec{F} = P\vec{i} + Q\vec{j}$

Ex $\int_C xy\,dx + x^2 dy$, C is $x = 2t$, $y = t^2$, $0 \le t \le 1$

$$= \int_C P\,dx + Q\,dy$$

$$= \int_0^1 \left[(2t)(t^2)\,2 + (2t)^2(2t)\right] dt$$

$$= \int_0^1 12t^3\,dt = 3.$$